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1. Introduction

In this note, we highlight the tension between stability and equality in non transferable utility matching. We consider many-to-one matchings and refer to the two sides of the market as students and schools. The latter have aligned preferences, as in Niederle and Yariv (2009), which in this context means that a school's utility is the sum of its students' utilities. A special case of aligned preferences, known as spatial allocation, arises when utilities are determined by commuting distance to school.

We show existence and uniqueness of a stable matching, as do Eeckhout (2000) and Niederle and Yariv (2009) under similar assumptions. This matching can be obtained with the Differed Acceptance Algorithm (DAA) of Gale and Shapley (1962). Stable matchings eliminate justifiable envy, hence are sometimes called fair. However, we show that this fairness comes at the cost of extreme forms of inequality of allocation[1]. In the spatial allocation case, this results in some students going to school across the street while other travel across the city. The intuition is that students and schools that are close to each other can block any allocation that involves a pair that is further away, and peripheral or marginal students get the long end of the subway ride.

We formalize this intuition by showing that the stable matching lexicographically maximizes the welfare of the matched pairs, starting with the best-off. We propose a simple algorithm that reflects this lexicographic ordering and makes the proof of our result transparent. We call this algorithm max-max-lex. Similarly, we propose an algorithm,

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[1]The inequality discussed here is between matched pairs, and within each side of the market, not between the two sides of the market as in in Gusfield and Irving (1989). In the latter, notions of equality and fairness relate to equalizing outcomes of both sides of the market while maintaining stability.

adapted from the bottleneck algorithm in Burkard et al. (2009), Section 6.2, that reverses the balance between stability and inequality and matches pairs in lexicographic order starting with the worse-off. We call this algorithm max-min-lex. The resulting matching is Rawlsian at the expense of stability.

2. Model

Consider a one-to-many matching problem with two sides $\mathcal{I}$ and $\mathcal{J}$. We will call the elements of $\mathcal{I}$ students, and the elements of $\mathcal{J}$ schools. Let $\mathcal{J}$ be a discrete set with cardinality weakly smaller than the cardinality of $\mathcal{I}$. Let each school $j \in \mathcal{J}$ have capacity q_j, which is the number of students it is equipped to serve. Finally, let u_{ij} be the utility of a student i when matched with j, and similarly let v_{jI} be the utility of a school j when matched with a set of students $I \subseteq \mathcal{I}$. We normalize the utility of unmatched students to $-\infty$. We assume that utilities are strictly positive, i.e., $u_{ij} > 0$ for every i and j; there are no indifferences, i.e., there are no pairs $i, i' \in \mathcal{I}$ and $j, j' \in \mathcal{J}$ such that $u_{ij} = u_{i'j}$ or $u_{ij} = u_{ij'}$, and preferences are strictly aligned, by which we mean that for all $j \in \mathcal{J}$ and $I \subseteq \mathcal{I}$, $v_{jI} = \sum_{i \in I} u_{ij}$. Strictly aligned preferences are so called because they require alignment between the utilities of the two sides of the market. They are a particular type of altruistic preference. When the matching is one-to-one, the definition of strictly aligned preferences coincides with the definition of aligned preferences in Niederle and Yariv (2009).

An allocation is a function $\mu : \mathcal{I} \cup \mathcal{J} \to 2^{\mathcal{I}} \cup \mathcal{J}$ such that $\mu(i) \in \mathcal{J} \cup \{i\}$ and $\mu(j) \subseteq \mathcal{I} \cup \{j\}$. The notation $\mu(i) = i$ indicates that student i is unassigned, and $j \in \mu(j)$ indicates that the number of students assigned to school j under μ is less than its capacity, that is $q_j > |\mu(j) \cap \mathcal{I}|$. An allocation is called feasible if each student is assigned to at most one school, and all school capacity constraints are respected, that is if $|\mu(i)| = 1$ for all $i \in \mathcal{I}$ and $|\mu(j)| \leq q_j$ for all $j \in \mathcal{J}$. An allocation is stable when there are no blocking pairs. In our context, this is equivalent to the following.

Definition 2.1. *The allocation* $\mu : \mathcal{I} \cup \mathcal{J} \to 2^{\mathcal{I}} \cup \mathcal{J}$ *is stable if* $\nexists\, i, j \in \mathcal{I} \times \mathcal{J}$ *such that* $u_{ij} > u_{i\mu(i)}$ *and* $[[|\mu(j))| < q_j]$ *or* $[|\mu(j))| = q_j$ *and* $\exists i' \in \mu(j),\ u_{i'j} < u_{ij}]]$.

The following algorithm will be shown to produce the unique stable matching.

(1) Match Step: select i and j such that the utility of their match is the highest in the set of students that are unassigned and schools that have some residual capacity.

(2) Update Step: reduce the capacity of the school found in the previous step by 1. Remove the assigned student from the set of unassigned students.

We call this algorithm the max-max-lex algorithm[2] because it iteratively pairs the students and schools that are each other's top choice among the schools and students that are still available. It does so in a lexicographic order, until there are no further students and schools to match. The max-max-lex algorithm is formally described below. It converges in a finite number of steps.

Algorithm 1: Max-max-lex Algorithm

```
Initialization:
Set I^0 = 𝓘, q^0 = q and t = 0
while I^t ≠ ∅ and q^t ≠ 0 do
    i^t, j^t = arg max_{i,j} u_ij
    s.t. i ∈ I^t and q^t_{j^t} ≠ 0
    Set μ(i^t) = j^t;
    if  j = j^t ; then
        q_j^{t+1} = q_j^t − 1;
    else
        q_j^{t+1} = q_j^t
    end
    I^{t+1} = I^t \ {i^t};
end
```

The following theorem shows three important results: first, the allocation resulting from the max-max-lex algorithm is the one that maximizes the vector of students' utilities in lexicographic order from higher to lower utility pairs. Second, it proves that the allocation

[2]The max-max-lex algorithm is lexicographic, starting from the top. This feature is shared with rank-maximal allocations, see Irving et al. (2006), where the number of agents receiving their first choice is maximized, subject to which a maximum number of remaining agents receive their second choice, etc...

is stable. Finally, it shows that the stable allocation is unique, therefore implying that the resulting matching outcome of the max-max-lex algorithm is identical to the matching outcome of the DAA.

Theorem 2.1. *(a) The max-max-lex algorithm maximizes (among all feasible allocations) the vector of ranked ordered utilities of student-school pairs in the lexicographic order, starting from the pair with the highest utility. (b) The assignment resulting from the max-max-lex algorithm is stable. (c) The stable allocation is unique.*

Proof. (a) Let $\mathcal{U} \subseteq \mathbb{R}^{|\mathcal{I}|}$ represent the set of utilities that are achievable in the economy in a feasible allocation. Formally, let $u = (u_i)_{i \in \mathcal{I}}$ be a vector in $\mathbb{R}^{|\mathcal{I}|}$. If $u \in \mathcal{U}$ then there exists a feasible allocation μ such that $u_{i\mu(i)} = u_i$. Let $u^{(k)}$ represent the k-th order statistic of vector u, with $u^{(|\mathcal{I}|)}$ being the highest component of vector u, and $u^{(1)}$ being its smallest. The first iteration of the max-max-lex algorithm selects among the vectors in $\mathcal{U}$ the ones with the highest value of $u^{(|\mathcal{I}|)}$. The n-th iteration of the max-max-lex algorithm selects among the vectors selected at the previous step, the ones with the highest value of $u^{(|\mathcal{I}|-n)}$, and so on. Therefore, the max-max-lex algorithm maximizes lexicographically the utility of students, starting from the pairs with the highest utility.

(b) Let μ^{MML} be the match resulting from the max-max-lex algorithm, and assume by contradiction that it is unstable. This means that there exists i and j such that $u_{ij} > u_{i\mu(i)}$ and for some $i' \in \mu(j)$, $u_{ij} > u_{i'j}$. However, this implies that the max-max-lex algorithm would have matched i and j, before matching i' and j, which leads to a contradiction.

(c) Let μ^S be a stable match and let μ^{MML} be the stable match arising from the max-max-lex algorithm. Suppose by contradiction that $\mu^S \neq \mu^{MML}$. This means that there exists $i \in \mathcal{I}$ such that $\mu^S(i) \neq \mu^{MML}(i)$. Since by Assumption 2 there are no indifferences, it must be that either (a) $u_{i\mu^{MML}(i)} < u_{i\mu^S(i)}$ or (b) $u_{i\mu^{MML}(i)} > u_{i\mu^S(i)}$. First suppose that (a) holds. Since i and $\mu^S(i)$ are not assigned through the max-max-lex algorithm, it must be that at the stage of the algorithm when i is assigned, school $\mu^S(i)$ is already at full capacity. This implies that $\exists I \subseteq \mathcal{I}$ s.t. $|I| \geq q_{\mu^S(i)}$ and $\min_{i' \in I} u_{i'\mu^S(i)} > u_{i\mu^S(i)}$. But this

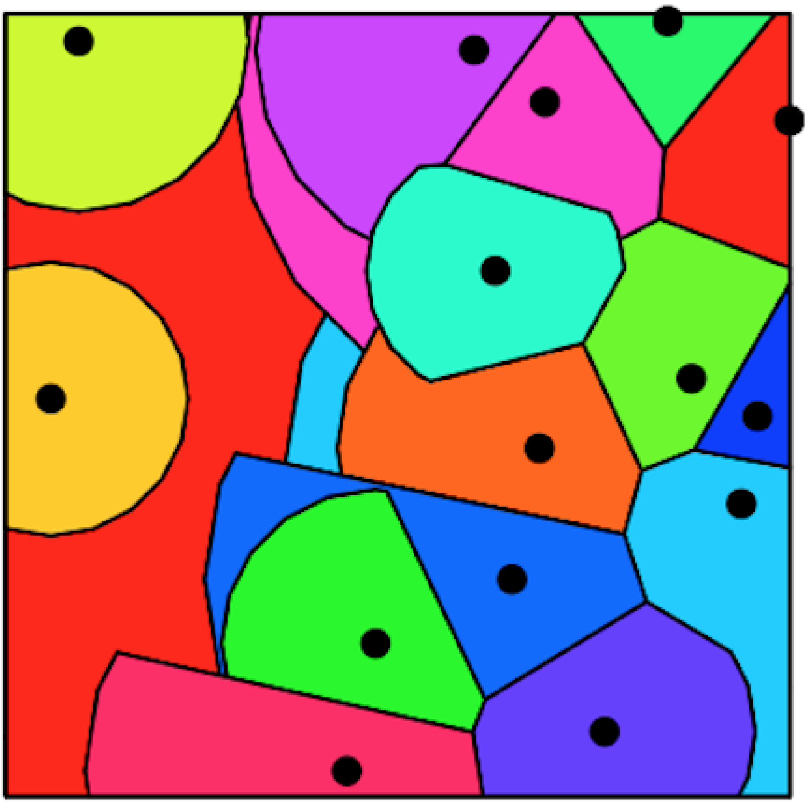

FIGURE 2.1. Stable allocation of students uniformly distributed on the unit square and 15 schools, represented by black dots. Student preferences are inversely proportional to distance traveled. Colored regions indicate sets of students attending the same school.

implies that any $i' \in I$ would form a blocking pair with $\mu^S(i)$ in μ^S. This contradicts that $\mu(s)$ is stable. Suppose then that (b) holds, i.e., $u_{i\mu^{MML}(i)} > u_{i\mu^S(i)}$. This implies that $\nexists I \subseteq \mathcal{I}$ s.t. $|I| \geq q_j$ and $\min_{i' \in I} u_{i'\mu^{MML}(i)} > u_{i\mu^{MML}(i)}$. But then $(i, \mu^{MML}(i))$ form a blocking pair in μ^S, which is a contradiction. Therefore $\mu^S = \mu^{MML}$. □

An illustration of the severe inequality displayed by the stable allocation in matching with aligned preferences is given in Figure 2.1. The latter shows the stable matching between a large number of students uniformly distributed on $[0,1]^2$ and 15 distinct schools in $[0,1]^2$ with heterogeneous capacities. Utilities are spatial, i.e., $u_{ij} = \sqrt{2} - d_{ij}$, where d_{ij} denotes Euclidean distance between i and j. For illustrative purposes, Figure 2.1 actually represents the limit allocation when $\mathcal{I} = [0,1]^2$. See Hoffman et al. (2006) for details. Dots in the figure represent schools, and territories of the same color represent students who attend the same school. One characteristic of this assignment is that all schools lie in the territory that they serve. As one can see from the figure, some students in the red territory have to travel almost the maximum distance that can be traveled in the square, while others travel no distance at all. This results in very dispersed utilities in the stable allocation.

The lexicographic nature of the stable allocation suggests a Rawlsian alternative, where pairs are matched in lexicographic order, starting with the lowest utility pair. The corresponding algorithm we propose below is adapted from the bottleneck algorithm in, for instance, Burkard et al. (2009), Section 6.2. It converges in finite time and produces an allocation that maximizes the utility of the worse-off student, then at each step maximizes the utility of the worse off students among those remaining. We thus call this algorithm max-min-lex. In the formal description of the algorithm below, we let $\mathcal{U} = (u_{ij})_{i\in\mathcal{I},j\in\mathcal{J}}$. Recall that the utility for an unassigned student is $-\infty$.

Algorithm 2: Max-min-lex Algorithm

Initialization:

Set $U^0 = \mathcal{U}$

while $|U^t| \geq 1$ **do**

  Let $u^\star$ the median of U^t;

  Let $U^{t-} = \{u \in U^t : u \leq u^\star\}$ and $U^{t+} = \{u \in U^t : u \geq u^\star\}$;

  if *there exists a feasible match such that no assigned student has a utility below* $u^\star$ **then**

    $U^{t+1} = U^t \setminus U^{t-}$;

  else

    $U^{t+1} = U^t \setminus U^{t+}$;

  end

end

The equalitarian nature of max-min-lex allocations come at the expense of stability. This is straightforward, given uniqueness of the stable allocation. It also stems from the logic of the max-min-lex algorithm, which creates blocking pairs. It is most easily seen in a 2 students, 2 schools example, with $u_{ij} > u_{ij'} > u_{i'j} > u_{i'j'}$. Max-max-lex matches (i, j) and $(i'j')$, whereas max-min-lex matches (i, j') and $(i'j)$, thereby decreasing inequality and creating a blocking pair.

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