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Symmetric Smooth Consumption Externalities*

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Based on Arrow's model of a pure exchange economy with smooth consumption externalities, this paper studies how the internalization of external effects through a network of markets between agents introduces symmetry breakings in the set of equilibria. It shows indeed how identical agents can be treated asymmetrically by complete markets. This work emphasizes that equilibrium allocations may be very sensitive to the way Coase-type rights are distributed: *Journal of Economic Literature* Classification Numbers: D50, D62, H23, K11. © 1996 Academic Press, Inc.

1. INTRODUCTION

It is often asserted that all models in which equilibria are not necessarily Pareto efficient give rise to sunspot equilibria, more precisely, to equilibria that are neither walrasian nor Pareto efficient, as those exhibited by Cass and Shell [5]. As examples of such models, those dealing with externalities are often cited (see Shell [15] and Spear [17]), although few of them are studied in the literature. It is indeed well known that the presence of externalities in the economy generally prevents equilibria from being Pareto efficient for the simple reason that there is no trade on external effects. In other words, it is due to the fact that external effects are not completely internalized. Claiming that such inefficient equilibria are sunspots would not be satisfactory for a lot of reasons. One could argue that the concept of Pareto efficiency is not appropriate in such a framework, or that the market fails to implement efficient allocations because of its incompleteness, to the extent that all the costs entailed by the economic behavior of an agent are not taken into account by this agent, and then the sunspots observed could belong to the large and well studied category of sunspots due to incomplete markets. Then it may be helpful for the understanding of that problem to define clearly some models of economies with externalities

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in which the concept of sunspot equilibria appears to be as usual a powerful one. The existence of sunspot phenomena should imply strong and distinctive properties of the structure of the equilibrium set. The model presented in this paper exhibits one kind of sunspot equilibria which is not due to uncertainty, time, or market incompleteness but only to agents inflicting external effects on each other.

Balasko [3] makes a thorough study of the mathematical structure of models in which sunspot equilibria are likely to appear. He shows that those phenomena consist essentially in symmetry breakings in the equation system and are ruled out when the first welfare theorem holds true. Then he asserts that the only two ways to obtain symmetry breakings are either to consider non-convex preferences or to investigate models which rule out the first welfare theorem, for example, by imposing multiple budget constraints on some agents. This paper is based on these remarks. It tries to give a convincing example of how externalities can give rise to "sunspot equilibria" in Balasko's sense of asymmetric equilibria (asymmetric solutions of a symmetric system of equations). This reading of the concept of sunspot equilibrium seems to be common sense in the framework of this paper. It is indeed Arrow's [1] static model of a pure exchange economy without uncertainty, where the markets are totally complete and all the externalities are internalized through a network of markets.[1] One could expect under such strong conditions to prevent phenomena like sunspots from happening; however, they do not (even when preferences are assumed to be convex, in which case the equilibrium is optimal) and that is why the model enlightens the links between non-convexity of preferences, conditions under which the first welfare theorem holds true and existence of asymmetric equilibria.

Another point is raised by the occurrence of symmetry breakings under such conditions. It is the very distinctive feature of Arrow's model that allow for those phenomena, and for which the standard symmetrization argument fails. It is often argued that this model allows to reproduce by analogy all the theoretical treatment performed for the standard Arrow–Debreu economy. This assertion deserves to be checked. That is the object of Section 2, where the mathematical structure of the model is studied through the accomplishment of the main stages developed by Balasko [2] in his research program.

The idea of symmetry developed here is natural and has already been introduced in studies about public goods (see, e.g., Champsaur [7] and more recently Goenka [11] which focuses on fiscal policies). It consists in asserting that agents with identical characteristics (i.e., with the same

[1] An illustration could be the market for polluting rights in the United States.

preferences and endowments) should be treated identically by the market and then come out with identical allocations. The issue of symmetry is at the center of this article which focuses on resource allocation. Only consumption externalities will be investigated here through Arrow's model. A set of new parameters (legal entitlements) are introduced. They characterize the initial juridical situation from which the agents trade on external effects. It is shown that, for a given bundle of initial endowments and legal entitlements, an equilibrium obtains, which may be Pareto efficient. The introduction of new dimensions of endowments—through this network of legal entitlements—increases the number of parameters of the model. It increases sufficiently to give enough new degrees of freedom to an economy where the agents are identical, even in a full sense, for asymmetric equilibria to appear. What can be underlined is that internalizing externalities may create some particular phenomena such as inequalities of treatment by the market, bifurcations in the equilibrium manifold and instability. The creation of markets to complete the economy is likely to provoke cycles or chaos. The bottom line is that the final allocations seem to be very sensitive to the distribution of liability rules. This non-neutrality result sounds like an anti-Coase-theorem argument. There could then be another hint, in addition to the presence of high transaction costs, to be careful, when creating such markets, concerning the way to rule them: the distribution of rights is not immune to giving rise to "undesirable" phenomena.

This paper is organized as follows. The basic model is described and its mathematical structure is studied in Section 2. Then the concepts of identity between the agents and of symmetric versus asymmetric equilibria are developed and commented on in Section 3, first through the simple example of an economy with two agents and then in a general framework. In Section 4, the existence of asymmetric equilibria is proved for the general framework, and is more specifically commented on in the case where all the utility functions, proper or external, are concave; in particular, the problem of the optimality of the allocations will be discussed. Then, Section 5 investigates some conditions under which an equilibrium is automatically symmetric; it tries to enlighten the distinctive features of the model that are responsible for the asymmetries to appear. Section 6 concludes with some remarks and comments. The two heavy proofs are contained in the appendices.

2. The Basic Model

The study of the basic model will facilitate the communication of the symmetric case. Another incentive is to clear up the accepted idea that

Arrow's [1] formalization allows to reproduce all the theoretical treatment of the standard Arrow–Debreu model. This section is self-contained.

The Agents and the Market

Consider a pure exchange economy with l commodities and m consumers. The preferences of agent i are described by a utility function which depends on the consumptions of all consumers. This implies that external effects are associated with measurable commodities. The consumptions and initial endowments are assumed to be strictly positive. Furthermore the utility functions are assumed to be separable in proper and external consumptions; they thus take the form

$$U_i(X) = u_i(x_i) + v_i(\bar{x}_i),$$

where $x_i = (x_i^{(1)}, ..., x_i^{(l)}) \in \mathbf{R}^l_{++}$ denotes consumer i's consumption, $\bar{x}_i = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_m)$ the consumptions of agents other than i, and where $X = (x_1, ..., x_m)$ is the whole consumption vector.

The fundamental property of Arrow's model is the *individualization* of external effects. The vector $\bar{x}_i$ is considered to be proper to agent i: it represents how consumer i sees the consumptions of the other consumers, other than i. Then $\bar{x}_i$ is replaced by $\tilde{x}_i = (x_i^1, ..., x_i^{i-1}, x_i^{i+1}, ..., x_i^m)$—where x_i^k denotes agent k's consumption seen from agent i—is a variable appearing in the utility function of consumer i, and in no one else's. X_i denotes the vector $(x_i^1, ..., x_i^m) \in \mathbf{R}^{lm}_{++}$.

There are lm commodities for each agent. A huge number of new commodities are thus artificially created: it is at this expense that the consumption sets of the agents become independent. Agent i expresses a demand for his external consumption, just as he does for his proper consumption. This implies that there exists markets for external consumption goods and that the agents behave on these markets just as they do on the market for proper consumption goods, i.e., as price-takers.

The proper utility functions u_i are assumed to be smooth, strictly increasing, strictly concave and the closure of their indifference surfaces are assumed to be contained in the strictly positive orthant; the first order derivatives are supposed to tend to infinity on the frontier of the positive orthant. On the other hand, the external utility functions v_i are smooth but are not assumed to be concave.[2] We impose the two following assumptions ((C) and (N)) on the v_i's.

[2] This assumption is not economically relevant. One can indeed easily imagine threshold effects with respect to the consumption of external goods. Take, for example, the music your neighbor listens to: externalities may be positive with modest consumptions and become negative if he keeps listening all day long to the same kind of music that you are not particularly fond of.

(C)-*Assumption.* The first-order derivatives of v_i are uniformly bounded in a neighborhood of 0.

To be illustrated, this assumption merely states that even if you can feel very upset when you see someone starving, you cannot feel as badly as if you were starving yourself. As a matter of fact, most of consumption externalities come from abundant consumptions, which translates in first-order derivatives that tend toward zero when the corresponding consumption becomes very low.

It will also be assumed that there is at least one commodity, suppose it is commodity l, for which the external effects inflicted by each agent on the others are almost zero.[3] More precisely, the marginal utility, for agent i, following the consumption of an extra unit of commodity l by agent k is negligible compared to the marginal utility following his own consumption of this extra unit.

(N)-*Assumption.* For each i, there exists a real $\varepsilon_i > 0$ close to zero[4] such that uniformly on $\mathbf{R}^{lm}_{++}$ and for all $k \neq i$, one has

$$\left|\frac{\partial v_i(\tilde{x}_i)}{\partial x_k^{(l)}}\right| < \varepsilon_i \frac{\partial u_i(x_i)}{\partial x_i^{(l)}}.$$

Let $\tilde{U}$ denote the set of utility functions fulfilling the preceding assumptions.

The Budget Constraints

There is a market between each couple of agents (i, k). Consumer i faces a market price $p \in \mathbf{R}^l$ for his proper consumption x_i, and a market price $p_i^k \in \mathbf{R}^l$ for his external consumption x_i^k regarding agent k (the latter can be positive or negative whether agent i endures a positive or a negative externality). The quantity $p_i^k \cdot x_i^k$ is paid directly to consumer k.

Reciprocally, consumer i receives the quantity $p_k^i \cdot x_i$ directly from consumer k. In this last quantity, x_i appears instead of x_k^i since everything is seen here from agent i's point of view, and he does not know how the other agents see his proper consumption. Then agent i faces an individualized price $p_i^i = p_i = p - \sum_{h=1;\, h\neq i}^{h=m} p_h^i \in \mathbf{R}^l_{++}$ for his proper consumption.

[3] This assumption has already been made in the literature on externalities. Starrett [18] notes that to rule out points of local Pareto satiation, a sufficient condition is the existence of a commodity which every one finds desirable and with which no one associates any consumption externalities; then the proposes labor, under the condition that no one is jealous of his industrious neighbor.

[4] See in Appendix 2.1 why $\varepsilon_i < (m!)^{-1}$ is largely sufficient.

The following price compatibility conditions[5] follow from the definition of $p_i^i = p_i$:

$$\sum_{h=1}^{h=m} p_h^i = p, \qquad \forall i, \quad 1 \leqslant i \leqslant m. \tag{1}$$

Let $\tilde{S}$ denote the set of prices $(p, (p_i^k)_{1 \leqslant k \leqslant m;\, 1 \leqslant i \leqslant m})$ where $p_i \in \mathbf{R}^l_{++}$ and $\tilde{X}$ the set of allocations $(X_i)_{1 \leqslant i \leqslant m} \in \mathbf{R}^{lm^2}_{++}$.

Let $\omega_i = (\omega_i^{(1)}, ..., \omega_i^{(l)}) \in \mathbf{R}^l_{++}$ denote the initial endowments of consumer i in proper goods. Other parameters are introduced which define the initial situation from which the agents trade on external consumptions. They are legal entitlements, i.e., the *rights* given (by the juridical or political authorities) to agent i on agent k, and reciprocally to k on i, in case of external effects involving both k and i. A straightforward category of these rights could be the liability rules for the damages caused by an agent on another. It could be for example a ban issued by the director of a firm, on smoking in offices unless there is a deal between all the persons working in an office; in such a situation, it is clear that the non-smokers have a right on the smokers: not to smoke is the situation of reference.

The legal entitlements of consumer k with respect to consumer i's consumption, ω_k^i, do not have to be equal to ω_i: what I have the right to obtain from (resp. the duty to give to) someone else does not need to be what he initially has (resp. I initially have). Since they do not correspond to any real consumption, they are not assumed to be positive. Each agent is endowed with a vector of initial endowments and legal entitlements $((\omega_i), (\omega_i^k)_{1 \leqslant k \leqslant m,\, k \neq i}) \in \mathbf{R}^l_{++} \times \mathbf{R}^{l(m-1)}$.

Consumer i's budget constraint is

$$\sum_{k=1}^{k=m} p_i^k \cdot x_i^k = p \cdot \omega_i + \sum_{k=1;\, k \neq i}^{k=m} p_i^k \cdot \omega_i^k - \sum_{h=1;\, h \neq i}^{h=m} p_h^i \cdot \omega_h^i, \tag{2}$$

where $x_i^i = x_i$. Let $\tilde{\Omega}$ denote the set of initial endowments $((\omega_i)_{1 \leqslant i \leqslant m}, (\omega_i^k)_{1 \leqslant i,\, k \leqslant m,\, k \neq i})$ which are elements of $\mathbf{R}^{lm}_{++} \times \mathbf{R}^{lm(m-1)}$.

The Equilibrium of an Economy

The external utility functions v_i are not concave. It prevents us from considering that the first-order conditions of the maximization of utility

[5] These conditions can be interpreted another way. To buy a unit of commodity j, agent i has to buy it for himself (and pay $p_i^{(j)}$) and he has also to "buy" it for the other agents (and pay $p_i^{k(j)}$ $k \neq i$) in the sense that he has to fix damages for an external deseconomy (and then $p_i^{k(j)}$ is positive), or sees his consumption subsidized if it entails an external economy (then $p_i^{k(j)}$ is negative). Eventually, the resulting price, $\sum_{h=1}^{h=m} p_h^{i(j)}$, must be the same for everybody, i.e., the market price. David Cass pointed out to me that in Arrow [1] three relationships come from profit maximization when agents are considered as producers of external effects.

functions under budget constraints are sufficient. That is why two different concepts of equilibrium are necessary, the classical one and an extended one (Smale's [16] terminology is used).

DEFINITION 1. An economy is a vector $((\omega_i), (\omega_i^k)) \in \tilde{\Omega}$. A *state* of the economy $((\omega_i), (\omega_i^k))$ is a vector $Z = ((p, (p_i^k)); (x_i^k); ((\omega_i), (\omega_i^k))) \in \tilde{S} \times \tilde{X} \times \tilde{\Omega}$. It is said to be attainable if it fulfills equations (1) and:[6]

$$\sum_{i=1}^{i=m} x_i = \sum_{i=1}^{i=m} \omega_i \tag{3}$$

$$x_i^k = x_k \qquad (\forall k, i, \quad 1 \leqslant k, i \leqslant m, \quad k \neq i). \tag{4}$$

DEFINITION 2. A *classical equilibrium* of the economy $((\omega_i), (\omega_i^k))$ is a vector $((\lambda_i)_{1 \leqslant i \leqslant m}, Z)$ where Z is an attainable state such that X_i maximizes U_i under the budget constraint, and λ_i is the inverse of the associated Lagrange multiplier. An *extended equilibrium* of the economy $((\omega_i), (\omega_i^k))$ is a vector $((\lambda_i)_{1 \leqslant i \leqslant m}, Z)$ fulfilling the first-order conditions of the former maximization problem.

Denote by Λ the m-simplex: the set of vectors $(\lambda_i)_{1 \leqslant i \leqslant m}$ such that $\lambda_i > 0$ and

$$\sum_{i=1}^{i=m} \lambda_i = 1. \tag{5}$$

A classical equilibrium is obviously an extended equilibrium. Equation (5) is the normalization condition chosen (considering (1), (3), and (4), the budget constraint of consumer m is derived from those of the others); it allows us to see the inverse Lagrange multipliers as welfare weights, a point of view that becomes clearer through Eq. (7) below.

An extended equilibrium is defined by Eqs. (1), (3), (4), (5), the budget constraints (2), and

$$\begin{cases} \lambda_i Du_i(x_i) = p_i & (6.1) \\ \lambda_i Dv_i^k(\tilde{x}_i) = p_i^k & (6.2) \end{cases} \tag{6}$$

where $Dv_i^k(\tilde{x}_i)$ denotes the derivatives of v_i with respect to the variables x_i^k, $k \neq i$,

$$Dv_i^k(\tilde{x}_i) = \frac{Dv_i}{Dx_i^k}(\tilde{x}_i).$$

[6] Equations (3) are the market clearing conditions on the standard market for proper consumptions, Eqs. (4) are the market clearing conditions on the markets for external consumptions. It states that what consumer i demands in terms of consumer k's consumption must eventually be equal to what consumer k effectively consumes.

The system of equilibrium equations (1) to (6) is too disegregated. One can easily give an equivalent system of equilibrium equations that exhibits more obviously the structure of the set of equilibria. It is obtained by removing all the variables related to the process of individualization of the external effects, i.e., the individualized prices and the individualized external commodities.

LEMMA 3. *An equivalent system of equilibrium equations is given by equations* (3) *and* (5), *along with*

$$\lambda_i Du_i(x_i) + \sum_{h=1,\, h\neq i}^{h=m} \lambda_h\, Dv_h^i(\bar{x}_h) = p, \qquad 1 \leqslant i \leqslant m \tag{7}$$

$$\lambda_i Du_i(x_i)\cdot x_i + \sum_{k=1,\, k\neq i}^{k=m} \lambda_i\, Dv_i^k(\bar{x}_i)\cdot x_k = p\cdot\omega_i + \sum_{k=1,\, k\neq i}^{k=m} \lambda_i\, Dv_i^k(\bar{x}_i)\cdot\omega_i^k - \sum_{h=1,\, h\neq i}^{h=m} \lambda_h\, Dv_h^i(\bar{x}_h)\cdot\omega_h^i,$$

$$1 \leqslant i \leqslant m-1. \tag{8}$$

The present framework differs from the classical pure exchange economy in three ways at least—so that the line of reasoning reproducing analogically all the results of the latter should be investigated carefully. First, the market clearing conditions for external consumptions are different from those for proper consumptions. Second, the assumption of concavity is dropped for the external utility functions. Third, beside the initial endowments in proper consumption goods, legal entitlements are introduced, which represent the initial juridical situation from which the agents trade on external consumptions.

Uniqueness of the Price Supporting an Equilibrium Allocation

With the convention that $Dv_i^i(X) = Du_i(x_i)$ and $Dv_h^i(X) = Dv_h^i(\bar{x}_h)$, define the following vector $D_i^{(j)}(X) = (Dv_h^{i(j)}(X))_{1\leqslant h\leqslant m} \in \mathbf{R}^m$. Equations (7) of Lemma 3 can be rewritten

$$\lambda\cdot(D_i^{(j)} - D_1^{(j)})(X) = 0, \qquad 2 \leqslant i \leqslant m, \quad 1 \leqslant j \leqslant l \tag{7'}$$

This means that, at an extended equilibrium, λ is an element of norm 1 of the orthogonal space (in $\mathbf{R}^m$) of the space $D(X)$ engendered by the $l(m-1)$ vectors: $\{(D_i^{(j)} - D_1^{(j)})(X),\, 2 \leqslant i \leqslant m,\, 1 \leqslant j \leqslant l\}$.

LEMMA 4. *Each matrix with* $(m-1)$ *rows extracted from the* $m \times l(m-1)$ *matrix* DV

$$((D_2^{(1)} - D_1^{(1)}) \cdots (D_2^{(l)} - D_1^{(l)}) \cdots (D_m^{(1)} - D_1^{(1)}) \cdots (D_m^{(l)} - D_1^{(l)}))$$

has full rank at each allocation reached by an extended equilibrium.

Proof. This comes essentially from Assumption (N); see Appendix 2.1. ∎

This lemma entails that the fundamental property of uniqueness of the price supporting an equilibrium allocation holds in this framework:

PROPOSITION 5. *If* X *is an allocation such that* $(p, (p_i^k); \lambda; X)$ *is a vector price-allocation reached by an extended equilibrium—*$(p, (p_i^k); \lambda)$ *is a supporting price of* X*—there does not exist another price supporting allocation* X*.*

Proof. Obvious considering that $\lambda = (\lambda_i)_{1 \leqslant i \leqslant m}$ is the normalized vector of the orthogonal space of $D(X)$ which has dimension $(m-1)$ (in $\mathbf{R}^m$). ∎

The Extended Equilibrium Manifold

Define the smooth map $F_{(U_i)}: \mathbf{R}^l \times \mathbf{R}_{++}^m \times \mathrm{Diag}(\tilde{X}) \times \tilde{\Omega} \to \mathbf{R}^{lm+l+m}$ by

$$F_{(U_i)}(p, \lambda, X, \Omega) = \begin{pmatrix} \sum_{i=1}^{i=m} (x_i - \omega_i) \\ \lambda_i Du_i(x_i) + \sum_{h1,\, h \neq i}^{h=m} \lambda_h \, Dv_h^i(\bar{x}_h) - p \\ \left(\sum_{i=1}^{i=m} \lambda_i \right) - 1 \\ \lambda_i Du_i(x_i) \cdot x_i + \sum_{k=1,\, k \neq i}^{k=m} \lambda_i Dv_i^k(\bar{x}_i) \cdot (x_k - \omega_i^k) \\ + \sum_{h=1,\, h \neq i}^{h=m} \lambda_h \, Dv_h^i(\bar{x}_h) \cdot \omega_h^i - p \cdot \omega_i \end{pmatrix}.$$

LEMMA 6. *The set* $\tilde{V}_1$ *of utility functions* $v_1 \in C^\infty(\mathbf{R}_{++}^{l(m-1)}, \mathbf{R})$ *such that* 0 *is a regular value of* $F_{(U_i)}$*, is an open and dense subset of* $C^\infty(\mathbf{R}_{++}^{l(m-1)}, \mathbf{R})$.

Proof. See Appendix 2.2. ∎

This lemma yields straightforwardly, with the help of the regular value theorem, the following result:

THEOREM 7. *For a generic set of external utility functions of the first consumer,* $\tilde{V}_1$*, the extended equilibrium set,* $\tilde{E}_{(U_i)}$*, is a submanifold of* $\tilde{S} \times \mathbf{R}^m \times \tilde{X} \times \tilde{\Omega}$ *of dimension* lm^2.

The equivalent system of equations given in Lemma 3 gives much information about the structure of the extended equilibrium manifold. In such a framework the solutions $(p, \lambda, X) \in \mathbf{R}^l \times \mathbf{R}^m \times \mathbf{R}^{lm}_{++}$ of Eqs. (5) and (7) play the role of the no-trade equilibria, i.e., compose the "non-linear" part of the manifold. However, the set they define does not have the close links with the Pareto optima that it has in the classical framework without externalities. Although it is not difficult to prove that it is generically a manifold of dimension $l+m-1$, there is no reason for it to be connected, unlike in Balasko [2], and then the equilibrium manifold could be disconnected too. However, still a point deserves to be stressed: fix such a vector (p, λ, X), then the set of $\Omega \in \tilde{\Omega}$ satisfying the remaining equations (3) and (8) is a linear manifold of dimension $lm^2 - l - m + 1$ which is a fiber of the equilibrium manifold. The presence of fibers of such a high relative dimension is a clear sign that the model investigated has a wealthy structure.[7]

Existence and Optimality of Equilibria

The technics followed here is essentially inspired from Balasko [2] and Smale [16]. It consists in building a homotopy between the economy under scope and the classical one of pure exchange without externalities defined by the u_i's and the ω_i's, which will be called the 0-economy. This will be done by considering the t-economies defined by the utility functions

$$U_{it}(X) = u_i(x_i) + tv_i(\tilde{x}_i).$$

The natural projection for the original economy: $\Pi_{(U_i)}: \tilde{E}_{(U_i)} \to \tilde{\Omega}$ by $\Pi_{(U_i)}((p, \lambda, X, \Omega)) = ((\omega_i), (\omega_i^k))$ is smooth. The economy $\Omega = ((\omega_i), (\omega_i^k)) \in \tilde{\Omega}$ is regular (resp. singular) if it is a regular (resp. singular) value of the natural projection. The set of singular economies Σ is closed and of Lebesgue measure zero in $\tilde{\Omega}$. All these definitions and results are proved and commented on in Balasko [2].

[7] The so-called Coase theorem [6] was sometimes read as stating the neutrality of the distribution of legal entitlements with respect to the equilibrium allocations. We know it is false since Buchanan and Stubblebine [4], and that it can merely be translated in a first welfare theorem (see, e.g., Theorem 11 below and Cooter [8]). But the fact is that a redistribution of legal entitlements within a fiber does not change the equilibrium allocations. Introducing those markets and legal entitlements does not change the dimension of the "non-linear" part of the manifold (it is $l+m-1$ as well in the standard case), although the dimension of the manifold itself increases a lot (it is multiplied by the number of agents): it only contributes in increasing the dimension of the "linear part." Especially, introducing those legal entitlements increases a lot the *relative* dimension of the fibers in the equilibrium manifold, and that is definitely the strongest interpretation we can give to the Coase theorem.

PROPOSITION 8. *The natural projection Π is proper.*

Proof. This is entailed by condition (C); see Appendix 2.3. ∎

Fix now a regular economy $\Omega = ((\omega_i), (\omega_i^k)) \in \tilde{\Omega}$. Let $\omega_0 = (\omega_{i0})_{1 \leqslant i \leqslant m}$ be a regular economy of the 0-economy, in a neighborhood of $\omega = (\omega_i)$ (it is ω if ω is regular). Denote $\omega_t = (\omega_{it})_{1 \leqslant i \leqslant m}$ where $\omega_{it} = t\omega_i + (1-t)\,\omega_{i0}$. The smooth mapping $G_{[(U_i), \Omega]}: [0, 1] \times \mathbf{R}^l \times \mathbf{R}^m_{++} \times \mathbf{R}^{lm}_{++} \to \mathbf{R}^{l+m+lm}$ is defined the same way as $F_{(U_i)}$, but with U_{it} and ω_{it} instead of U_i and ω_i. Notice that $G_{[(U_i), \Omega]}(1, (p, \lambda, X)) = F_{[(U_i)]}(p, \lambda, X, \Omega)$ and

$$G_{[(U_i), \Omega]}(0, (p, \lambda, X)) = f_{[(u_i), (\omega_{i0})]}(p, (\lambda_i), (x_i)) = \begin{pmatrix} \sum_{i=1}^{i=m} (x_i - \omega_{i0}) \\ \lambda_i Du_i(x_i) - p \\ \left(\sum_{i=1}^{i=m} \lambda_i\right) - 1 \\ p \cdot (x_i - \omega_{i0}) \end{pmatrix}.$$

Thanks to Proposition 8: [The following proposition obtains]

PROPOSITION 9. *The mapping $G_{[(U_i), \Omega]}$ is a smooth proper mapping.*

One then obtains that, since (ω_{i0}) is a regular economy of the 0-economy, the degree of $G_{[(U_i), \Omega]}(1, \cdot)$ is defined and is the same as the one of $f_{[(u_i), (\omega_{i0})]}$ which is known to be one (see Balasko [2, Math. 2.11]). The last argument is to notice that if Ω (resp. ω_0) is a regular economy then 0 is a regular value of $G_{[(U_i), \Omega]}(1, \cdot)$ (resp. $f_{[(u_i), (\omega_{i0})]}$); note also that if $[G_{(U_i), \Omega]}(1, \cdot)]^{-1}(0) = \varnothing$ then Ω is regular. Then the following existence result[8] obtains.

THEOREM 10. *For all $\Omega \in \tilde{\Omega}$, there exists an extended equilibrium.*

The two following welfare theorems hold as well.

THEOREM 11. *Every classical equilibrium is a Pareto optimum. Reciprocally, let X be a Pareto optimum; there exists a price vector supporting X (and corresponding distributions of initial endowments and legal entitlements).*

Proof. The proofs are standard and can be found in Crès [9]. ∎

[8] Counter-examples can be found where there exists a unique extended equilibrium which is not classical; then there is no classical equilibrium.

3. The Model with Identical Consumers

The emphasis of this paper is on the possibility of symmetry breakings when external effects are internalized through a complete system of markets. The issue of symmetry breaking consists essentially in stating that symmetric systems of equations can give rise to asymmetric solutions. A means to create a symmetric system of equations is to study a market composed with "identical" agents with respect to their fundamentals: utility functions, initial endowments and legal entitlements (see, for example, Malinvaud's model for individual risks [13]). The agents being identical *ex ante*, it is natural to expect that the market implement symmetric equilibrium allocations. However, in fact, it is in general not always the case. For this idea to appear clearer, let us first investigate the simple case of an economy composed of two identical consumers.

3.1. *The Leading Example*

There are l commodities and two consumers 1 and 2 with identical utility functions $u+v$, i.e., $u(x_1)+v(x_2)$ for consumer 1 and $u(x_2)+v(x_1)$ for consumer 2 (x_i represents agent i's consumption). They both have $\omega \in \mathbf{R}^l_{++}$ as initial endowments in proper consumption goods, and $\omega' \in \mathbf{R}^l$ as legal entitlements with respect to the consumption of the other. Under these conditions the system of equilibrium equations (7), (5), (3), (8) becomes

$$\begin{cases} \lambda_1 Du(x_1) + \lambda_2 Dv(x_1) = p \\ \lambda_2 Du(x_2) + \lambda_1 Dv(x_2) = p \end{cases} \tag{9}$$

$$\lambda_1 + \lambda_2 = 1 \tag{10}$$

$$x_1 + x_2 = 2\omega \tag{11}$$

$$\lambda_1 Du(x_1)\cdot x_1 + \lambda_1 Dv(x_2)\cdot x_2 = p\cdot\omega + (\lambda_1 Dv(x_2) - \lambda_2 Dv(x_1))\cdot\omega'. \tag{12}$$

Definition 12. The set $\{(p(\omega) = 1/2(Du(\omega) + Dv(\omega)),\ (1/2, 1/2), (\omega, \omega),\ \omega, \omega'),\ (\omega, \omega') \in \tilde{\Omega}\}$ is a $2l$ dimensional manifold of extended equilibria. It is the set of symmetric equilibria; for each economy (ω, ω') there exists one and only one symmetric equilibrium.

The central issue is the existence of asymmetric equilibria, i.e., equilibria such that $x_1 \neq x_2$. A sufficient condition is that $\lambda_1 \neq \lambda_2$.[9] Indeed

[9] If there exist p such that the equation $(Du + Dv)(x) = 2p$ has at least two distinct solutions x_1 and x_2, an asymmetric equilibrium obtains for *even* welfare weights $\lambda_1 = \lambda_2 = 1/2$ (one indeed has $Dv(x_1) \neq Dv(x_2)$ otherwise $Du(x_1) = Du(x_2)$ and then $x_1 = x_2$); this condition is not necessary.

Proposition 5 states that to each equilibrium allocation corresponds a unique supporting price (p, λ); and to $x_1 = x_2$ $(= \omega)$ corresponds $(p(\omega), (1/2, 1/2))$.

We split the system of equilibrium equations into two pieces; Eqs. (9) and (10) on the one hand, (11) and (12) on the other. Pick an asymmetric solution $y = (p, (\lambda_1, \lambda_2), (x_1, x_2))$ of (9) and (10). The system of equations (11) and (12) yields the endowments and entitlements (ω, ω') compatible with y, i.e., which make y feasible; it is a *linear* system of equations. By (11), compatible initial endowments $\omega \gg 0$ are obtained. Then if $\lambda_1 Dv(x_2) - \lambda_2 Dv(x_1) \neq 0$ the set of compatible legal entitlements ω', is defined by equation (12). Finally, there exists a $(l-1)$-dimensional affine space[10] of $(\omega, \omega') \in \Omega$ compatible with the asymmetric allocation (x_1, x_2).

It is already clear at this point in the line of reasoning that the presence of some new parameters (legal entitlements) gives enough new degrees of freedom to the model to compensate for the small dimension of the space of initial endowments due to the identical characteristics of agents. In the case developed here, the budget hyperplane, at a fixed allocation, remains a big enough linear manifold to fulfill the equilibrium conditions for some parameters.

Thus the issue reduces to a very simple analysis. Let V_{f_2} be the set of elements y fulfilling Eqs. (9) and (10) and such that $\lambda_1 \neq 1/2$. Let V_{g_2} be the subset of V_{f_2} composed with the elements fulfilling in addition $\lambda_1 Dv(x_2) - \lambda_2 Dv(x_1) = 0$. Does there exist some y's in $V_{f_2} \backslash V_{g_2}$? For we know that these y's are associated to asymmetric equilibria. A stronger result obtains.

PROPOSITION 13. *Generically*[11] *with respect to the external utility function* v, V_{f_2} *is a smooth manifold of dimension* $(l+1)$ *whereas* V_{g_2} *is a submanifold of* V_{f_2} *of dimension* 1.

Moreover, for each fixed $\lambda_1 \in]0, 1[\backslash \{1/2\}$ *and* $\omega \in \mathbf{R}^2_{++}$ *there exists in* V_{f_2} *an element associated to* λ_1 *such that* $x_2 + x_1 = 2\omega$; *then* V_{f_2} *is non empty.*

Proof. See Appendix 3.1. ∎

The argument can go further. An identical line of reasoning shows that $V_{f_2}(\omega)$—i.e., the subspace of $y \in V_{f_2}$ such that $x_1 + x_2 = 2\omega$—is generically in utility functions a submanifolds of V_{f_2} of dimension 1. If it lies entirely in V_{g_2}, it cannot be stated that there exists an asymmetric equilibrium for this ω. However, $V_{f_2}(\omega)$ and V_{g_2} both being manifolds of dimension 1,

[10] The problem has then the same dimension as the equivalent problem for an economy without externalities, with two agents who have the same preferences but not necessarily the same initial endowments, since it is the dimension of a fiber for fixed total resources.

[11] Throughout the paper, by generically, we mean for an open and dense set.

there cannot exist an open set U of initial endowments ω such that $V_{f_2}(\omega)$ lies entirely in V_{g_2} for all $\omega \in U$. Then, generically in initial endowments ω, there exist in $V_{f_2}(\omega)$ some open one-dimensional subsets of asymmetric allocations, each of them associated with a $(l-1)$-dimensional fiber of legal entitlements.

THEOREM 14. *Generically with respect to the external utility function v and the initial endowments ω, there exist an l-dimensional manifold of asymmetric equilibria.*

The following section will prove that asymmetric equilibria also appear when the external utility function v is concave. A point needs to be stressed that underlines the very distinctive feature of the model. Unlike in the standard case, Balasko's symmetrization argument fails, although the symmetrized allocation is feasible whatever the specification of the legal entitlements (it is the symmetric allocation: $1/2(x_1 + x_2) = \omega$). The point is merely that there is no reason why $u(\omega) + v(\omega)$ should be bigger than both $u(x_1) + v(x_2)$ and $u(x_2) + v(x_1)$. The only property that can be stated is that, when v is concave, if $u(x_1) + v(x_2) \geqslant u(\omega) + v(\omega)$, then $u(x_2) + v(x_1) \leqslant u(\omega) + v(\omega)$, i.e., if one agent is better off with the asymmetric allocation, then the other agent is better off with the symmetric one.

The interest of the paper will not increase much by studying how this line of reasoning generalizes to a bigger economy. There is nevertheless one point in this generalization that deserves to be developed and commented on. It is to show that the number of parameters in the economy remains big enough, despite the identical characteristics of the agents, to yield enough degrees of freedom for the equilibrium conditions to hold at almost every point of V_{f_2}. We shall see that the concept of identity defined in this framework, although it is a full identity, preserves the usual number of parameters, i.e., the number of agents times the number of goods.

3.2. *The General Case*

There are m agents and l commodities. The agents have the same utility function $u + v$, i.e., $u(x_i) + v(\cdot)$ for agent i. They all have the same initial endowments in physical consumption goods: $\omega \in \mathbf{R}^l_{++}$. As far as legal entitlements are concerned, the identity between consumers does not require that each couple of agents have exactly the same rights each on the other. Take the example of three agents, let $\omega_1^2 = \omega(1)$ and $\omega_1^3 = \omega(2)$ be the rights of agent 1 on agents 2 and 3 respectively. Then the utility function of agent 1 is $u(x_1) + v(x_2, x_3)$: agent 2 (resp. agent 3) is said to be the agent of type (1) (resp. type (2)) of agent 1.

To be identical to agent 1, agent 2 must be globally endowed with the same legal entitlements on agents 1 and 3; i.e.; he must also have his agent

of type (1) and his agent of type (2): he is endowed either with $\omega(1)$ on agent 1 and $\omega(2)$ on agent 3 or with $\omega(2)$ on agent 1 and $\omega(1)$ on agent 3. The same for agent 3. However, reciprocally, if agent 1 ends up being the agent of type (2) of both agent 2 and agent 3, whereas agent 2 ends up being the agent of type (1) of both agent 1 and agent 3, then agent 1 and 2 cannot be said to be identical with respect to their "duties," i.e., the rights given to the others on their own consumptions.

Take two different agents i and k, for the concept of identity to be full, the rights given on agent i to agents other than i must be globally similar to the rights given on agent k to agents other than k. In our example with three agents, this idea translates in the following way: the rights given to agents 2 and 3 on agent 1 must be either $(\omega(1), \omega(2))$ or $(\omega(2), \omega(1))$.

It is useful to draw the following table. The ith row describes on whom agent i is given the rights $\omega(h)$, $h = 1, 2$, and the jth column defines who is given the rights $\omega(h)$ on agent j (here agent 1 is given the rights $\omega(1)$ on agent 2 and $\omega(2)$ on agent 3 and agent 2 (resp. agent 3) is given the rights $\omega(2)$ (resp. $\omega(1)$) on agent 1):

	1	2	3
1	*	$\omega(1)$	$\omega(2)$
2	$\omega(2)$	*	$\omega(1)$
3	$\omega(1)$	$\omega(2)$	*

What is important for the concept of identity to be fully completed, is that, in such a table, *on each row and each column each vector of legal entitlements appears one and only once*. Then for an economy with three agents, there are only two ways to distribute properly the rights, the second one is described by the following table:

	1	2	3
1	*	$\omega(2)$	$\omega(1)$
2	$\omega(1)$	*	$\omega(2)$
3	$\omega(2)$	$\omega(1)$	*

In the general case, there is a set $\mathscr{L}$ of $(m-1)$ vectors of legal entitlements given in the economy. Each agent must be endowed globally with the whole set; i.e., he is endowed once and only once with each element of this set on somebody else, whoever this person is. Reciprocally, if one considers the set of rights the others have on him, it must cover the whole set $\mathscr{L}$ too. And the same tables with the same properties can be drawn as in the case with three agents.

Let us formalize this idea for an economy with a finite number of agents. Let M and M_i be respectively the sets $\{1, ..., m\}$ and $\{1, ..., i-1, i+1, ..., m\}$ for all i; let $\mathscr{L} = \{\omega(1), ..., \omega(m-1)\} \in \mathbf{R}_{++}^{l(m-1)}$ be the set of legal entitlements in the economy. Let L denote the set $\{1, ..., m-1\}$ of types of agents. To agent i is associated a bijection $\mu_i: L \to M_i$, such that the legal entitlements of i on $j \in M_i$ be $\omega(\mu_i^{-1}(j)) \in \mathscr{L}$. In other words, $\mu_i(k)$ is the agent on which consumer i is given the rights $\omega(k)$; he is the agent of type (k) of agent i and his consumption bundle: $x_{\mu_i(k)}$ will appear in the k th position as an argument of i's external utility function:

$$u(x_i) + v(x_{\mu_i(1)}, ..., x_{\mu_i(m-1)}).$$

DEFINITION 15. The m-tuple $(\mu_i)_{1 \leqslant i \leqslant m}$ is *acceptable* if and only if for all i, $\{(\mu_h^{-1}(i))_{h \in M_i}\} = L$; i.e., the set of legal entitlements given to all agents other than i on agent i himself covers the whole set of legal entitlements.

When there are two agents, there is only one way to allocate the legal entitlements ω'. When there are more than two agents, the number of *acceptable* ways to allocate the $(m-1)$ legal entitlements (i.e., the number of acceptable m-tuples $(\mu_i)_{1 \leqslant i \leqslant m}$) increases very fast. It is 2 when $m = 3$, 24 when $m = 4$ and 1344 when $m = 5$. More generally, it is the number of Latin squares of size m with the integer m uniformly on the diagonal. It is proved to be bigger than the number of Latin squares of size $m-1$ (see, e.g., Van Lint and Wilson [19, p. 170]). The number L_n of Latin square of size n is given by the formula $n!(n-1)!\, l_n$ where only the eight first values of l_n are known, l_7 being already equal to 16, 942, 080. Another result that speaks for itself is a "poor" lower bound of $L(n)$: $L(n) \geqslant (n!)!$.

An acceptable m-tuple of bijections from L into M_i defines the situation described a few lines above. The agents indeed have identical characteristics: the same utility functions, the same initial endowments and the same network of legal entitlements on the others (as far as their rights are concerned) and "against" them (as far as their "duties" are concerned). In fact, it is possible to draw a diagram for each m-tuple: each agent is represented by a point; from each point, $(m-1)$ arrows start toward all the other points, one for each vector of legal entitlements in $\mathscr{L}$. The m-tuple is acceptable when in this diagram, $(m-1)$ arrows arrive at each point, one for each vector of legal entitlements. Such a diagram will be called a *network*. The agents are meant to be identical in the sense that the points in this network cannot be distinguished because the same bundle of arrows start from each of them and arrive at each of them.

DEFINITION 16. A *symmetric economy* $\mathscr{E}_l$ is defined by a couple of proper and external utility functions u and v, an acceptable network

$(\mu_i)_{1 \leqslant i \leqslant m}$ and a vector of initial endowments and legal entitlements $(\omega, (\omega(k))_{k \in L} \in \tilde{\Omega}$.

The number of parameters in this economy is lm (the same as in an economy free from external effects where the agents have the same preferences but not necessarily the same initial endowments). It keeps in the mathematical treatment enough degrees of freedom to allow for a lot of asymmetric equilibria to appear.

It is necessary to define from μ_i a kind of inverse bijection v_i from L into M_i the following way: let k be an element of L, $v_i(k) = j$, where j is the agent such that $\mu_j(k) = i$. In other words, $v_i(k)$ is the agent whose rights on i are $\omega(k)$. The acceptability of the m-tuple of bijections μ entails the existence of the bijections v, and the following properties hold: $\forall (i, j, k) \in M^2 \times L$, $v_i(k) = j$ if and only if $\mu_j(k) = i$; and $\mu_{v_i(k)}(k) = i$.

The system of equilibrium equations defining the equilibria of a symmetric economy is, along with the normalization condition (5):

$$\lambda_i Du(x_i) + \sum_{h \in L} \lambda_{v_i(h)} D_h v[(x_{\mu_{v_i(h)}(k)})_{k \in L}] = p \tag{13}$$

$$\sum_{i=1}^{i=m} x_i = m\omega \tag{14}$$

$$\begin{aligned} \lambda_i Du(x_i) \cdot x_i + \sum_{k \in L} \lambda_i D_k v[(x_{\mu_i(k')})_{k' \in L}] \cdot x_{\mu_i(k)} \\ = p \cdot \omega + \sum_{k \in L} \{\lambda_i D_k v[(x_{\mu_i(k')})_{k' \in L}] \\ - \lambda_{v_i(k)} D_k v[(x_{\mu_{v_i(h)}(k')})_{k' \in L}]\} \cdot \omega(k) \end{aligned} \tag{15}$$

PROPOSITION 17. *For an economy* $(\omega, (\omega(k))_{k \in L})$ *there exists a unique* symmetric *equilibrium, defined by* $p_s = 1/m(Du(\omega) + \sum_{h=1}^{h=m-1} D_h v(\omega, ..., \omega)$, $\lambda_i = 1/m$ *and* $x_i = \omega$ *for all i. The set of symmetric equilibria is a smooth manifold of dimension lm globally parameterized by the parameters*: $(\omega, (\omega(k))_{1 \leqslant k \leqslant m-1})$.

Proof. Uniqueness comes from Proposition 5. ∎

DEFINITION 18. An equilibrium $(p, (\lambda_i)_{1 \leqslant i \leqslant m}, (x_i)_{1 \leqslant i \leqslant m}, (\omega, (\omega(k))_{k \in L}))$ is *asymmetric* if there exist two different agents i and k such that $x_i \neq x_k$.

PROPOSITION 19. *An equilibrium* $(p, (\lambda_i), (x_i), (\omega, (\omega(k)))$ *such that there exist two different agents i and k with* $\lambda_i \neq \lambda_k$ *is asymmetric.*

Proof. See Proposition 5. ∎

Since we deal here with extended equilibria, there is no way to compare the Pareto efficiency of symmetric and asymmetric equilibria. However, at the end of Section 4, the case where the external utility function is concave is investigated: both kinds of equilibrium (symmetric or asymmetric) are Pareto efficient.

First Remark. In the leading example, if (x_1, x_2) is an asymmetric equilibrium allocation associated with the economy (ω, ω'), then the vector (x_2, x_1) is too. This property can be generalized in the following way: if there exists a bijection σ of the set of agents M such that for all i, $v_{\sigma(i)} = \sigma \circ v_i$, then if $(x_i)_{1 \leqslant i \leqslant m}$ is an equilibrium allocation (symmetric or not) of the economy $\{u, v, (\mu_i), (\omega, (\omega(k))_{k \in L})\}$, $(x_{\sigma(i)})_{1 \leqslant i \leqslant m}$ is too.

Second Remark. Fix u, v, and $(\omega, (\omega(k))_{k \in L})$. For any equilibrium $(x_i)_{1 \leqslant i \leqslant m}$ for the network (μ_i) and any bijection σ of M, there exists an acceptable network (μ_i^σ), defined by: $\forall i, \mu_i^\sigma = \sigma^{-1} \circ \mu_{\sigma(i)}$, such that $(x_{\sigma(i)})$ is an equilibrium for the network (μ_i^σ). The network (μ_i^σ) defines the same network of responsibilities as (μ_i), after having permuted the places of the agents through the bijection σ. These two networks are said to be *equivalent*.

The two preceding remarks convey the impression that if (x_i) is an asymmetric equilibrium for a specified network, it is in general not one for another network, even an equivalent one. An interesting issue is the study of those equilibria which are equilibria for all the networks. One knows that the symmetric equilibria have this property. The question to address is whether there exist a reciprocal link. It will be addressed in Section 5.

4. The Asymmetric Equilibria

This section shows that the set of asymmetric equilibria can be very big, even in the case where the external utility functions are concave. The line of reasoning is close from the one followed in the leading example. There is no chance here to prove the existence of asymmetric equilibria by showing that they add some complement dimensions to the symmetric equilibrium manifold in the set of equilibria (as was the case in Balasko [3]) for the mere reason that if there were a manifold of asymmetric equilibria, it would be of the same dimension as the symmetric one. The fact is that in this framework the equilibrium set is in general not even a manifold: there are, in the manifold of symmetric equilibria, bifurcation points from which branches of asymmetric equilibria start. It would be sufficient to prove the existence of at least one of those bifurcation points. However, sufficient conditions are hard to obtain in such a general

framework where few assumptions are made on utility functions. The existence will be proven here through a simpler analysis, by "computing" the system of equilibrium equations, and splitting it into two pieces—to exploit the fiber-bundle structure: Eqs. (5) and (13) on the one hand, which yield a vector of endogenous variables, and on the other hand, this latter vector being fixed, Eqs. (14) and (15) which yield the compatible parameters through a system of linear equations.

Let V_f be the set of vectors $(p, (\lambda_i), (x_i)) \in \mathbf{R}^l \times \mathbf{R}^m_{++} \times \mathbf{R}^{lm}_{++}$ fulfilling Eqs. (5) and (13) and such that the $m \times m$ matrix

$$\tilde{\Lambda} = \begin{pmatrix} \lambda_1 & \lambda_{v_1(1)} & \cdots & \lambda_{v_1(m-1)} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \lambda_m & \lambda_{v_m(1)} & \cdots & \lambda_{v_m(m-1)} \end{pmatrix}$$

is invertible (condition (S)). It excludes obviously the *even* welfare distribution $\lambda_i = 1/m$, $\forall i$. Denote $\bar{\Lambda}$ the set of weights (λ_i) fulfilling (5) and condition (S).

Let V_g be the set of elements of V_f fulfilling in addition the condition that the following $(m-1) \times l(m-1)$ matrix $M(p, (\lambda_i), (x_i))$ does not have full rank;

$$(\lambda_i D_k v[(x_{\mu_i(k')})_{k' \in L}] - \lambda_{v_i(k)} D_k v[(x_{\mu_{v_i(h)}(k')})_{k' \in L}])_{(1 \leqslant i \leqslant m-1) \times (1 \leqslant k \leqslant m-1)},$$

the gradients being written horizontally.

LEMMA 20. *$\bar{\Lambda}$ is an open dense subset of the set Λ of $(\lambda_i) \in \mathbf{R}^m_{++}$ fulfilling* (5).

Proof. The acceptability of the network (μ_i) entails that the weight λ_1 appears once and only once in each row and each column of $\tilde{\Lambda}$. Then the matrix A_1 built from $\tilde{\Lambda}$ putting 1 instead of λ_1 and zero everywhere else is invertible. For any small real ε but a finite number—related to the eigenvalues of the matrix $(A_1)^{-1}\tilde{\Lambda}$—the matrix $(1/1+\varepsilon)(\tilde{\Lambda} + \varepsilon A_1)$ is invertible, can be taken arbitrarily close to $\tilde{\Lambda}$ and $((1/1+\varepsilon)(\lambda_1 + \varepsilon), (1/1+\varepsilon)\lambda_2, ..., (1/1+\varepsilon)\lambda_m)$ fulfills (5). ∎

The issue reduces here to show that the two sets V_f and V_g do not coincide. Take an element y in V_f which is not in V_g. From (14), compatible initial endowments ω are found. Then Eqs. (15) can be read as a full rank (since it is not in V_g) linear system of $(m-1)$ equations with $l(m-1)$ unknowns (the legal entitlements $(\omega(k))_{k \in L}$). It yields an $(l-1)(m-1)$-dimensional affine space of legal entitlements compatible with y. Since,

thanks to condition (S), the welfare weights cannot be the even ones, Proposition 5 entails that each point y in $V_f \backslash V_g$ supports an $(l-1)(m-1)$-dimensional affine space of asymmetric equilibria.

PROPOSITION 21. *Generically with respect to the external utility functions v, V_f is a smooth manifold of dimension $l+m-1$.*

Moreover, for any $\Lambda \in \bar{\Lambda}$ and any $\omega \in \mathbf{R}^l_{++}$, there exists at least one element in V_f associated with (Λ, ω).

Proof. See Appendix 4.1. ∎

The aim is now to prove that, at least generically with respect to the utility functions v and u, the set V_g is "small" in V_f.

PROPOSITION 22. *For a residual subset of proper and external utility functions u and v, V_g is the complement of an open and dense subset of the manifold V_f.*

Proof. See Appendix 4.2. ∎

THEOREM 23. *For a residual subset of proper and external utility functions u and v, there exists an open and dense subset of the $(l+m-1)$-dimensional manifold of asymmetric price-allocation vectors, each point of which supports an $(l-1)(m-1)$-dimensional linear manifold of initial endowments and legal entitlements.*

These equilibria are not necessarily solutions of the maximization programs of the consumers; they are only describing the first-order conditions of these programs. Then the first welfare theorem does not hold for all of them. But in the case where the external utility function v is strictly concave, the line of reasoning followed in this section still holds (V_f is always a smooth manifold and the perturbation technics used in the proof of Propositions 21 and 22 keep the strict concavity of v, as it is argued for u in the same proof). However, for a concave external utility function v, the two concepts of equilibrium (classical and extended) coincide. This entails in particular that all the equilibria are Pareto efficient and all the Pareto efficient allocations can be described as equilibrium allocations. However, there still exist many asymmetric equilibria.

That so many asymmetric equilibria may appear seems at first sight surprising. Especially if one considers, as it was often argued, that Arrow's model for externalities allows to reproduce by analogy for an economy presenting external effects the theoretical treatment performed for economies without externalities. The fact that there exist symmetry breakings in a totally complete and statics economy without uncertainty,

where the agents have strictly concave utility functions and where the two welfare theorems hold stresses the distinctive feature of this model, for which the standard symmetrization argument fails.

Finally, it may be useful to recall from the leading example that in this framework, asymmetric equilibria may also appear for even distributions of welfare weights ($\lambda_i = 1/m, \forall i$) as well as when there is not any redistribution of wealth through the legal entitlements (i.e., $\omega(k) = \omega, \forall k$).

5. The Equivariant Equilibria

At the end of Section 3, the concept of equivariant equilibrium allocations was introduced.

Definition 24. Let u, v, and $(\omega, (\omega(k))_{k \in L})$ be fixed. The equilibrium $(p, (\lambda_i), (x_i))$ (associated with the triple $\{u, v, (\omega, (\omega(k))_{k \in L}\})$ is *equivariant* if it is an equilibrium allocation of the economy $\{u, v, (\mu_i)_{1 \leqslant i \leqslant m}, (\omega, (\omega(k))_{k \in L}\}$ for *all* acceptable networks (μ_i).

We investigate here the links between those allocations and the symmetric ones. We know that a symmetric allocation is equivariant. A simplified version of the model where the utility functions are completely separable is set up for which an equivariant allocation is automatically symmetric.

Take $m \geqslant 3$. The external utility function v is assumed to be totally separable in the following sense: let $x = (x_1, ..., x_{m-1}) \in \mathbf{R}_{++}^{l(m-1)}$ then $v(x) = v_1(x_1) + \cdots + v_{m-1}(x_{m-1})$. The v_k's are moreover assumed to be concave. The interest of this model is twofold. First of all, the structure of the set V_f is wealthier:

Proposition 25. *V_f is then a smooth manifold which can be parameterized globally by the $l + m - 1$ variables $(p, \lambda_1, ..., \lambda_{m-1})$.*

Proof. See Appendix 5.1. ∎

Second, the shape of the utility functions is reminiscent of the classical models dealing with uncertainty (see, for example, Balasko [3] or Malinvaud [13]). There is in this formulation a formal identity between types of agents and states of nature or individual states. Consider agent i's utility function:

$$u(x_i) + v_1(x_{\mu_i(1)}) + \cdots + v_{m-1}(x_{\mu_i(m-1)}).$$

To each agent (i himself included) corresponds a utility function and a vector of legal entitlements. We will refer to this economy as the *separable* one.

Another assumption is made:

(R)-*Assumption.* There is one commodity (assume it is commodity l) and two external utility functions (assume they are v_1 and v_2), such that for all $x \in \mathbf{R}^l_{++}$

$$\left|\frac{\partial v_1(x)}{\partial x^{(l)}}\right| \leqslant \left|\frac{\partial v_2(x)}{\partial x^{(l)}}\right|.$$

It means that an agent is always more sensitive to the consumption of commodity l by his agent of type (2) than by his agent of type (1). An illustration could be a firm with identical employees: one is always more sensitive to the consumption of cigarettes by his office mate than by an agent at the end of the corridor.

We know that asymmetric equilibria exist for this model where the utility functions v_i's are strictly concave. In this section it is only needed that they be simply concave.

PROPOSITION 26. *In this separable economy, an equivariant equilibrium is symmetric.*

Proof. Let $(p, (\lambda_i), (x_i)_{1 \leqslant i \leqslant m}$ be an equivariant equilibrium. In this simpler framework, Eqs. (13) become

$$\lambda_i Du(x_i) + \sum_{h \in L} \lambda_{v_i(h)} Dv_h(x_i) = p.$$

Since it is independent from the network (μ_i), for all i and every couple of bijections v_i and v'_i:

$$\sum_{h \in L} (\lambda_{v_i(h)} - \lambda_{v'_i(h)}) Dv_h(x_i) = 0.$$

This means that for all i, j the vector $(\partial v_h(x_i)/\partial x_i^{(j)})_{h \in L}$ is orthogonal to the vector $w_{v_i, v'_i} = (\lambda_{v_i(h)} - \lambda_{v'_i(h)})_{h \in L}$ for each couple (v_i, v'_i).

LEMMA 27. *Take $(\lambda_i)_{1 \leqslant i \leqslant m} \neq (1/m)_{1 \leqslant i \leqslant m}$ then the orthogonal space of the linear space engendered by the vectors $w_{v, v'}$ for all (v, v') is contained in* Vect(1, ..., 1).

Proof. Suppose, without any loss of generality, that $\{\lambda_1, ..., \lambda_{m-1}\}$ reaches at least two values. Among all the values reached, choose the one

which is reached the least. Suppose it is reached by $n+1$ values of i (for example by $i=m-n-1, ..., m-1$). The matrix composed first with the vectors $(w_{\delta, \tau_{j, m-n-1}})_{1 \leqslant i \leqslant m-n-2}$ ($\tau_{j,j'}$ is the transposition between indices j and j' and δ is such that for all $k \in L$, $\delta(k)=k \in M_n$), then with the vectors $(w_{\delta, \tau_{1, m-n-1+j'}})_{1 \leqslant j' \leqslant n}$. It is equivalent to the $(m-1) \times (m-2)$ matrix

$$\begin{pmatrix} I_{m-n-2} & 1 \cdots 1 \\ -1 \cdots -1 & 0_{1 \times n} \\ 0_{n \times (m-n-2)} & -I_n \end{pmatrix},$$

which clearly has full rank. On top of that a vector orthogonal to all the columns of this matrix has all its coefficient equal. ▌

This lemma proves that the gradients $(Dv_h(x_m))_{h \in L}$ are all equal. Such a property is excluded by Assumption (R). Then all the weights λ_i are equal. The global parametrization then entails that all the allocations are the same. This proof still holds to show that an allocation which is an equilibrium to a whole class of equivalent networks is symmetric. It holds too in much more general frameworks but a priori fails each time there can be asymmetric allocations associated with even welfare weights. ▌

6. Concluding Comments

The results obtained in this paper are essentially on the sensitiveness of the equilibrium allocations to distributions of Coase-type rights on the markets opened to trade on external effects. Symmetric agents may end up being treated asymmetrically by a complete network of markets. Developing an intuition on the symmetry-breaking result seems to be a difficult task. The fact that the standard argument of symmetrization for complete market fails in this setup (cf. comments following Theorem 14) underlines the distinctive features of this model. Dropping the assumption of concavity on the external utility function, although it may multiply the symmetry breakings, is not responsible for their occurrence. The idea to implement market processes to internalize externalities aimed at improving the allocative efficiency of the market. It does so. But the way the legal entitlements are endowed has real effects on the economy and is not immune to giving rise to phenomena such as inequality of treatment, instability and so on, which are often considered undesirable. This statement is a major criticism to an "all-market remedy" to the inefficiency entailed by the presence of external effects in the economy.

APPENDIX 2.1

Proof of Lemma 4. This property is insured by the (N)-Assumption. By extraction of the $(m-1)$ columns related to good 1, one obtains the following $m \times (m-1)$ matrix:

$$M_l = \begin{pmatrix} \frac{\partial v_1(\tilde{x}_1)}{\partial x_2^{(l)}} - \frac{\partial u_1(x_1)}{\partial x_1^{(l)}} & \frac{\partial v_1(\tilde{x}_1)}{\partial x_3^{(l)}} - \frac{\partial u_1(x_1)}{\partial x_1^{(l)}} & \cdots & \frac{\partial v_1(\tilde{x}_1)}{\partial x_m^{(l)}} - \frac{\partial u_1(x_1)}{\partial x_1^{(l)}} \\ \frac{\partial u_2(x_2)}{\partial x_2^{(l)}} - \frac{\partial v_2(\tilde{x}_2)}{\partial x_1^{(l)}} & \frac{\partial v_2(\tilde{x}_2)}{\partial x_3^{(l)}} - \frac{\partial v_2(\tilde{x}_2)}{\partial x_1^{(l)}} & \cdots & \frac{\partial v_2(\tilde{x}_2)}{\partial x_m^{(l)}} - \frac{\partial v_2(\tilde{x}_2)}{\partial x_1^{(l)}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial v_m(\tilde{x}_m)}{\partial x_2^{(l)}} - \frac{\partial v_m(\tilde{x}_m)}{\partial x_1^{(l)}} & \frac{\partial v_m(\tilde{x}_m)}{\partial x_3^{(l)}} - \frac{\partial v_m(\tilde{x}_m)}{\partial x_1^{(l)}} & \cdots & \frac{\partial u_m(x_m)}{\partial x_m^{(l)}} - \frac{\partial v_m(\tilde{x}_m)}{\partial x_1^{(l)}} \end{pmatrix}.$$

If there were no external effects with respect to commodity 1, this matrix would become

$$M_l^o = \begin{pmatrix} -\frac{\partial u_1(x_1)}{\partial x_1^{(l)}} & -\frac{\partial u_1(x_1)}{\partial x_1^{(l)}} & \cdots & \cdot & -\frac{\partial u_1(x)}{\partial x_1^{(l)}} \\ \frac{\partial u_2(x_2)}{\partial x_2^{(l)}} & 0 & \cdots & \cdot & 0 \\ 0 & * & \cdot & & \cdot \\ \cdot & \cdot & * & \cdot & \cdot \\ \cdot & & \cdot & * & 0 \\ 0 & 0 & \cdots & 0 & \frac{\partial u_m(x_m)}{\partial x_m^{(l)}} \end{pmatrix}.$$

which clearly has all its extracted $(m-1)$-square matrices invertible (by the strict monotonicity of the u_i). Let us show that the (N)-Assumption guarantees this result to hold also for M_l.

Let us prove first that the matrix M_{lm} obtained by elimination of the last row of M_l is invertible. In each row i, except in the first one, there is only one term which is "big" in the sense that it dominates the others; it is the one expressed with a partial derivative of u_i. The determinant of M_{lm} can then be decomposed into two terms. The first one is obtained, as for M_{lm}^o, by the multiplication of all these dominant terms, and its absolute value is bigger than $\prod_{i=1}^{i=m}(1-\varepsilon_i)\,\partial u_i(x_i)/\partial x_i^{(l)}$, and then bigger than $(1-E)^m \prod_{i=1}^{i=m} \partial u_i(x_i)/\partial x_i^{(l)}$, if we denote $E = \max_{1 \leqslant i \leqslant m} \varepsilon_i$. The second one is the remaining of the determinant; it is formed by the multiplication

of $(m-1)$ terms of the matrix, less than $(m-2)$ being dominant terms; and so their absolute values are smaller than $E^p(1+E)^{m-p}\prod_{i=1}^{i=m}\partial u_i(x_i)/\partial x_i^{(l)}$, if there are p dominant terms in the product. The only thing that remains to be proved is that

$$(1-E)^m - \sum_{1\leqslant p\leqslant m-2}\alpha(p)\,E^p(1+E)^{m-p} > 0$$

if $\alpha(p)$ denotes the number of dominant terms in the product defining an element in the sum expressing the determinant. Since $\sum_{1\leqslant p\leqslant m-2}\alpha(p) = -1+(m-1)!$ it is sufficient to check that $(1-E)^m > (m-1)!\,E(1+E)^{m-1}$. It is right for $E<(m!)^{-1}$. M_l is a full-rank $(m-1)$-square matrix extracted from DV, then the result holds. ▮

Appendix 2.2

Proof of Lemma 6. Take $\tilde{K}_N$ a compact subset of $\tilde{Z} = \mathbf{R}^l \times \mathbf{R}_{++}^m \times \mathrm{Diag}(\tilde{X}) \times \tilde{\Omega}$. Denote K_N (resp. $\bar{K}_N$) the projection of $\tilde{K}_N$ on the last $l(m-1)$ components of $\mathrm{Diag}(\tilde{X})$ (resp. on $\mathrm{Diag}(\tilde{X})$); one can choose $\tilde{K}_N$ such that $\bigcup_{N\in\mathbf{N}}\tilde{K}_N = \mathbf{R}_{++}^{lm}$. Let B_N be an open subset containing K_N. It is possible to build a smooth characteristic function α_N of K_N such that

$$\alpha_N(K_N) = 1 \qquad \text{and} \qquad \alpha_N((\mathrm{Diag}(\tilde{X}))_{2\leqslant i\leqslant m}\backslash B_N) = 0.$$

Let $\tilde{S}_1$ be the unit ball of $\mathbf{R}_{++}^{l(m-1)}$ and define, for $\tilde{s}_1\in\tilde{S}_1$, the utility function $v_{1,\tilde{s}_1}\in C^\infty(\mathbf{R}^{l(m-1)},\mathbf{R})$ by

$$v_{1,\tilde{s}_1}(\tilde{x}_1) = v_1(\tilde{x}_1) + \varepsilon\alpha_n(\tilde{x}_1)\,\tilde{s}_1\cdot\tilde{x}_1.$$

One can obviously choose ε such that, for the topology of uniform convergence on compact sets, $v_{1,\tilde{s}_1}$ is arbitrarily close to v_1, and $u_1+v_{1,\tilde{s}_1}$ still is in $\tilde{U}$ (recall that v_1 has not to be concave).

Write down the Jacobian matrix of $\bar{F}_{(U_i)}(Z,\tilde{s}_1)$, which is the map $F_{(U_i)}$ written for $v_1 = v_{1,\tilde{s}_1}$ on $\tilde{K}_N\times\tilde{S}_1$. One obtains a $(lm+l+m)\times(2lm+l+m+l(m-1))$ matrix of the following shape (derivation resp. of the blocks of Eqs. (4), (7), (3), (8), with respect to X, p, λ, (ω_i), $(\omega_i^k)_{k\neq i}$ and $\tilde{s}_1$),

$$D\bar{F}_{(U_i)} = \begin{pmatrix} I_l\cdots I_l & 0 & 0 & -I_l\cdots -I_l & 0 & 0 & 0\\ * & J_\alpha & * & 0 & 0 & 0 & J_\beta\\ 0 & 0 & 1\cdots 1 & 0 & 0 & 0 & 0\\ * & * & * & A & A & B & * \end{pmatrix},$$

where

$$J_\alpha = \begin{pmatrix} -I_t \\ \vdots \\ -I_l \end{pmatrix}_{lm \times l} \quad \text{and} \quad J_\beta = \begin{pmatrix} 0_{l \times l(m-1)} \\ \lambda_1 \varepsilon I_{l(m-1)} \end{pmatrix}.$$

The $(m-1) \times lm$ matrix A, obtained by derivation of the $(m-1)$ first budget constraints with respect to $(\omega_i)_{1 \leqslant i \leqslant m}$ (so that ω_m does not appear), can be written, with $p = (p^{(1)}, ..., p^{(l)})$,

$$A = \begin{pmatrix} -p & 0 & \cdots & 0 & 0 \\ 0 & -p & \cdots & 0 & 0 \\ \cdot & \cdot & & \cdot & 0 \\ \cdot & \cdot & & \cdot & 0 \\ 0 & 0 & \cdots & -p & 0 \end{pmatrix}.$$

We extract the columns of $D\bar{F}_{(U_i)}$ of the derivations with respect to ω_m, p, $\tilde{s}_1$, λ_1, and $(\omega_i^{(l)})_{1 \leqslant i \leqslant m-1}$. Notice that the (N)-Assumption prevents $p^{(l)}$ from being zero; it is indeed strictly positive: one has $p^{(l)} = \sum_h p_h^{i(l)}\ \forall i$, then $p^{(l)} = (1/m) \sum_h (\sum_i p_h^{i(l)})$. But

$$\sum_i p_h^{i(l)} = \lambda_h \left\{ \frac{\partial u_h}{\partial x_h^{(l)}} (x_h) + \sum_{i \neq h} \frac{\partial v_h}{\partial x_i^{(l)}} (\tilde{X}_h) \right\},$$

which is strictly positive for ε_h small enough.

Then one obtains a square matrix of $lm + l + m$ rows,

$$\begin{pmatrix} -I_l & 0 & 0 & 0 & * \\ 0 & J_\alpha & J_\beta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & * & 0 & * & -p^{(l)} I_{(m-1)} \end{pmatrix},$$

where

$$(J_\alpha J_\beta) = \begin{pmatrix} -I_t & 0 \\ * & \lambda_1 \varepsilon I_{l(m-1)} \end{pmatrix}$$

is invertible. Then the Jacobian matrix of $\bar{F}_{(U_i)}$ defined on $\tilde{K}_N \times \tilde{S}_1$ has always full-rank.

A straightforward application of a version of the transversality theorem (cf. Golubitsky and Guillemin [12, Chap. II, Lemma 4.6]) allows one to assume that for a dense open subset of $\tilde{S}_1$, and then trivially for one $\tilde{s}_1$, 0 is a regular value of the map $\bar{F}_{(U_i)}(\cdot, \tilde{s}_1)$ defined on $\tilde{K}_N$.

It has been proved then that the set $\tilde{V}_{1N}$ of utility functions v_1 such that 0 is a regular value of $F_{(U_i)}$ on $\tilde{K}_N$ is dense. $\tilde{V}_{1N}$ is obviously open thanks

to the continuity of the determinant and the fact that $\tilde{K}_N$ is compact. Denote $\tilde{V}_1 = \bigcap_{N \in \mathbf{N}} \tilde{V}_{1N}$; $\tilde{V}_1$ is obviously residual and hence dense in $C^\infty(\mathbf{R}^{l(m-1)}, \mathbf{R})$, since the Baire property holds for the topology used.

Finally the openness of $\tilde{V}_1$ follows from Lemma 4.5 of Golubitsky and Guillemin [12] and the fact that the map $F_{(U_i)}$ varies continuously with respect to the utility functions. ∎

Appendix 2.3

Proof of Proposition 8. Let K be a compact subset of $\tilde{\Omega}$; then there exist $(\omega_i^{k\prime})$ and $(\omega_i^{k\prime\prime})$ such that for all $(\omega_i^k) \in K$, $\omega_i^{k\prime} \leqslant \omega_i^k \leqslant \omega_i^{k\prime\prime}$ $\forall k, i$. All one has to prove is that $\prod^{-1}(K)$ is compact.

Let $X = (x_i^k)$ be an allocation (which then fulfills (4)) associated with $(\omega_i^k) \in K$ and $(p, (p_i^k)) \in \tilde{S}$.

We have: $x_i^k = x_k \gg 0 \forall i, k$ and $\sum_{i=1}^{i=m} x_i = \sum_{i=1}^{i=m} \omega_i \leqslant \sum_{i=1}^{i=m} \omega_i''$ which makes sure that the allocation X is uniformly bounded on K. It remains to check that it is uniformly "far away" from the boundaries of $\tilde{X}$.

Let $k_{(q)} = (p_{(q)}, (p_{i(q)}^k); \lambda_{(q)}; X_{(q)}; \Omega_{(q)})$ be a sequence of $\prod^{-1}(K)$. We already know that $\Omega_{(q)}$ bounded entails $X_{(q)}$ bounded in $\mathbf{R}_+^{lm}$. Moreover $0 < \lambda_{i(q)} \leqslant 1$. One can then consider that $(\lambda_{(q)}; X_{(q)}; \Omega_{(q)})$ converges to $(\lambda; X; \Omega)$. The equations (6) then make sure that $(p_{i(q))}^k)_{k \neq i}$ converges to a bounded value $(p_i^k)_{k \neq i}$.

What remains to show is:

1. first that $\forall i, j, x_i^{(j)}$ doesn't tend to 0, which secures that $p_i^{(j)} < \infty$;
2. then that $p_i \gg 0$ $\forall i$;
3. and finally that $\lambda \gg 0$.

Suppose there is an i for which there exists a non-empty subset $J_i \subseteq \{1, ..., m\}$ such that, for all $j \in J_i$, $x_{i(q)}^{(j)} \to 0$. Then the assumptions on the utility functions u_i imply that $p_{i(q)}^{(j)} \to +\infty$ for $j \in J_i$ and, through equations (1) (especially because of $\sum_{h=1}^{h=m} p_{h(q)}^{i(j)} = p_{(q)}^{(j)}$, where $p_{h(q)}^{i(j)}$ remains bounded for $k \neq i$ thanks to (C)-Assumption), one has $p_{(q)}^{(j)} \to +\infty$; moreover it does so in a way "equivalent" to the sequence $p_{i(q)}^{(j)}$.

Consider now i's budget constraint (2); since $(\omega_i^k) \in K$ and $\omega_i \gg 0$, the right side of the equation (income part) contains a term $\sum_{j \in J_i} p^{(j)} \cdot \omega_i^{(j)}$ tending toward infinity, the others remaining bounded because of the (C)-Assumption. On the left side of the constraint, the only terms which are likely not to be bounded are the following: $\sum_{j \in J_i} p_i^{(j)} \cdot x_i^{(j)}$. But $p_{(q)}^{(j)}$ tends toward infinity in a way equivalent to the way that $p_{i(q)}^{(j)}$ tend toward infinity, $x_{i(q)}^{(j)}$ tends toward zero, and $\omega_{i(q)}^{(j)} \gg 0$ $\forall j \in J_i$; all these arguments lead to a contradiction, which is that the right member becomes greater than the left member in the budget constraint when q tends toward ∞.

Then $x^{(j)}_{i(q)}$ does not tend toward 0.

Now suppose there exist j, $1 \leqslant j \leqslant l$, and i such that $p_i = (p_i^{(1)}, ..., p_i^{(l)})$ with $p_i^{(j)} = 0$. From (6.1), one extracts $\lambda_i Du_i^{(j)}(x_i) = p_i^{(j)} = 0$ where $Du_i^{(j)}(x_i) \gg 0$ ($Du_i^{(j)}$ reaches its strictly positive minimum on K_X), thus $\lambda_i = 0$ and $p_i^k = 0$, $\forall k$. So, if we prove that $\lambda \gg 0$ (point 3 above), then the proof is over since point 3 entails point 2.

Without any loss of generality, suppose $i = 1$ and $\lambda_1 = 0$. Consider the vectors $\overline{D_i^{(j)}(X)} = (Dv_2^{i(j)}(X), ..., Dv_m^{i(j)}(X)) \in \mathbf{R}^{m-1}$. One knows that the space engendered by the family $\{(\overline{D_i^{(j)}} - \overline{D_1^{(j)}})(X), 2 \leqslant i \leqslant m, 1 \leqslant j \leqslant l\}$ has dimension exactly $(m-1)$ since they constitute a matrix $\overline{DV}$, which is obtained from DV by elimination of the first row (see Lemma 4). Note $\bar{\lambda} = (\lambda_2, ..., \lambda_m) \in \mathbf{R}^{(m-1)}$. One has $\bar{\lambda} \cdot (\overline{D_i^{(j)}} - \overline{D_1^{(j)}})(X) = 0$ ($\forall i$ $1 \leqslant i \leqslant m$ $\forall j$ $1 \leqslant j \leqslant l$). A necessary condition is then $\bar{\lambda} = 0$ which is impossible since in this case $\sum_{i=1}^{i=m} \lambda_i = \sum_{i=2}^{i=m} \lambda_i = 1$. ∎

Appendix 3.1

Proof of Proposition 13. Let V_2 be the set of vectors $((\lambda_1, \lambda_2), (x_1, x_2)) \in \mathbf{R}^{2l+2}_{++}$ such that $\lambda_1 \neq \lambda_2$. It is straightforward to prove that it is a manifold of dimension $2l+2$. Define the two smooth mappings f_2 and g_2, respectively from V_2 into $\mathbf{R}^{l+1}$ and from V_2 into $\mathbf{R}^{2l+1}$,

$$f_2 = \begin{pmatrix} \lambda_1 Du(x_1) + \lambda_2 Dv(x_1) - \lambda_2 Du(x_2) - \lambda_1 Dv(x_2) \\ \lambda_1 + \lambda_2 - 1 \end{pmatrix}$$

and

$$g_2 = \begin{pmatrix} \lambda_1 D_(x_1) - \lambda_2 Du(x_2) \\ \lambda_2 Dv(x_1) - \lambda_1 Dv(x_2) \\ \lambda_1 + \lambda_2 - 1 \end{pmatrix}.$$

V_{f_2} (resp. V_{g_2}) is the set $f_2^{-1}(0)$ (resp. $g_2^{-1}(0)$). It is straightforward to see that V_{g_2} defines the elements of V_{f_2} such that $\lambda_1 Dv(x_2) - \lambda_2 Dv(x_1) = 0$. The proposition obtains by simple applying the regular value theorem to the maps f_2 and g_2. One only needs the following lemma:

Lemma 28. *The set of external utility functions* $v \in \mathscr{C}^\infty(\mathbf{R}^l_{++}, \mathbf{R})$ *such that* 0 *is a regular value of* f_2 *and* g_2 *is a open and dense subset of* $\mathscr{C}^\infty(\mathbf{R}^l_{++}, \mathbf{R})$.

Proof. The proof follows the same path as in Appendix 2.2. Take K_n a sequence of compact subsets of V_2 such that $\bigcup_{n \in \mathscr{N}} K_n = V_2$. Let B_n be an open subset containing K_n. Consider a smooth characteristic function α_n on

K_n such that $\alpha_n = 1$ on K_n and $\alpha_n = 0$ on $V_2 \backslash B_n$. Let $\tilde{S}_2$ be the unit ball of $\mathbf{R}^l{}_{++}$, and define, for $s \in \tilde{S}_2$, the utility function v_s by

$$v_s(x) = v(x) + \varepsilon \alpha_n(x)\, s \cdot x,$$

it is possible to choose ε sufficiently small such that, for the topology of the uniform convergence on compact sets, v_s is arbitrarily close to v and still admits all the properties required. Denote $\overline{f_2}$ the map defined from f_2 on $K_n \times \tilde{S}_2$; the same for $\overline{g_2}$. The Jacobian matrices of $\overline{f_2}$ and $\overline{g_2}$ are

$$J_{\overline{f_2}} = \begin{pmatrix} * & * & * & * & \varepsilon(\lambda_2 - \lambda_1)\, I \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$J_{\overline{g_2}} = \begin{pmatrix} * & * & \lambda_1 D^2 u(x_1) & * & 0 \\ * & * & * & * & \varepsilon(\lambda_2 - \lambda_1) I \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

They clearly have full rank, since in V_2, $\lambda_2 \neq \lambda_1$. Then, by the transversality theorem (in its simple version: Chap. II, Lemma 4.6 in Golubitsky and Guillemin [12]), for an open and dense subset of $\tilde{S}$, 0 is a regular value of $\overline{f_2}$ defined on K_n; the same argument holds true for $\overline{g_2}$. Take an s in the intersection of these two sets. The utility function v_s thus built proves that the set W_n of utility functions such that 0 is a regular value of both f_2 and g_2 on K_n is dense; it is also open since the determinant map is continuous and K_n is compact. Take $W_2 = \bigcap_{n \in \mathcal{N}} W_{2n}$; it is residual in $\mathscr{C}^\infty(\mathbf{R}^l{}_{++}, \mathbf{R})$ and hence dense since the Baire property holds for the topology used here on the space of utility functions. It is also open; this follows from Lemma 4.5 of Golubitsky and Guillemin and the fact that the maps considered varies continuously with respect to the utility functions. ∎

The second part of the proposition remains to be proved. Fix $\lambda_1 \neq 1/2 \in]0, 1[$, $\lambda_2 = 1 - \lambda_1$ and $r = 2\omega \in \mathbf{R}^l{}_{++}$. Consider Φ: $[0, 1] \times (]0, r^1[\times \cdots \times]0, r^l[) \to \mathbf{R}^l$

$$\Phi(t, x_1) = \lambda_1 Du(x_1) - \lambda_2 Du(r - x_1) + t(\lambda_2 Dv(x_1) - \lambda_1 Dv(r - x_1)).$$

It is clearly a smooth proper mapping. Indeed, if $\lambda_1 Du(x_1) - \lambda_2 Du(r - x_1) + t(\lambda_2 Dv(x_1) - \lambda_1 Dv(r - x_1))$ remains bounded, since Assumption (C) ensures us that $\lambda_2 Dv(x_1) - \lambda_1 Dv(r - x_1)$ is uniformly bounded on $]0, r^l[\times \cdots \times]0, r^l[$, then x_1 must remain far away from the boundaries of $]0, r^1[\times \cdots \times]0, r^l[$; because of the assumptions on u, whose derivatives with respect to a variable tending toward zero tend toward infinity.

Then $\phi = \Phi(0, \cdot)$ and $\psi = \Phi(1, \cdot)$ are defined on $]0, r^1[\times \cdots \times]0, r^l[$ (smooth manifold of dimension l) to $\mathbf{R}^l$, have the same degree modulo 2. And ϕ is known to be of degree 1. Finally, 0 is a regular value of both ϕ and ψ (for a convenient v). Hence the proof. ∎

Appendix 4.1

Proof of Proposition 21. Let us follow the same line of reasoning as for Proposition 13. Denote V the set of vectors $(p, (\lambda_i), (x_i)) \in \mathbf{R}^l \times \bar{\Lambda} \times \mathbf{R}_{++}^{lm}$. It is a manifold of dimension $l + m - 1 + lm$. Define the smooth mapping f from V into $\mathbf{R}^{lm}$:

$$f = \left(\lambda_i Du(x_i) + \sum_{h \in L} \lambda_{v_i(h)} D_h v[(x_{\mu_{vi}(h)(k)})_{k \in L}] - p \right)_{1 \leqslant i \leqslant m}.$$

Lemma 29. *The set of external utility functions* $v \in \mathscr{C}^\infty(\mathbf{R}_{++}^{l(m-1)}, \mathbf{R})$ *such that* 0 *is a regular value of* f *and is an open and dense subset of* $\mathscr{C}^\infty(\mathbf{R}_{++}^{l(m-1)}, \mathbf{R})$.

Proof. Following the same principle of perturbation as in the proof of Lemma 28 in the preceding appendix, the Jacobian matrix of $\bar{f}$, smooth mapping from $K_n \times \tilde{S}$, where $\tilde{S}$ is the unit ball of $\mathbf{R}_{++}^{l(m-1)}$, is (derivatives with respect to (λ_i), (x_i), p, s_1, ..., s_{m-1}, in this order):

$$J_{\bar{f}} = \begin{pmatrix} * & * & -I & \varepsilon\lambda_{v_1(1)} I & \cdots & \varepsilon\lambda_{v_1(m-1)} I \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ * & * & -I & \varepsilon\lambda_{v_m(1)} I & \cdots & \varepsilon\lambda_{v_m(m-1)} I \end{pmatrix}.$$

Consider the square matrix composed with the lm last columns. By elementary computations (subtract the first l-block-rows to the $(m-1)$ others), it has the same rank as

$$\begin{pmatrix} I & \lambda_{v_1(1)} I & \cdots & \lambda_{v_1(m-1)} I \\ 0 & (\lambda_{v_2(1)} - \lambda_{v_1(1)}) I & \cdots & \lambda_{v_2(m-1)} - \lambda_{v_1(m-1)}) I \\ \cdot & \cdot & \cdots & \cdot \\ 0 & (\lambda_{v_m(1)} - \lambda_{v_1(1)}) I & \cdots & \lambda_{v_m(m-1)} - \lambda_{v_1(m-1)}) I \end{pmatrix},$$

which is invertible thanks to condition (S). Indeed, $\tilde{\Lambda}$ is equivalent to (add to the first column all the others to obtain 1 everywhere in the first column and then subtract the first row to all the others):

$$\begin{pmatrix} 1 & \lambda_{v_1(1)} & \cdots & \lambda_{v_1(m-1)} \\ 0 & \lambda_{v_2(1)} - \lambda_{v_1(1)} & \cdots & \lambda_{v_2(m-1)} - \lambda_{v_1(m-1)} \\ . & . & \cdots & . \\ 0 & \lambda_{v_m(1)} - \lambda_{v_1(1)} & \cdots & \lambda_{v_m(m-1)} - \lambda_{v_1(m-1)} \end{pmatrix}.$$

The same argument as in Appendix 3.1, Lemma 28, ensures us that the set of compatible external utility functions v: $W = \bigcap_{n \in \mathcal{N}} W_n$ is open and dense in $\mathscr{C}^\infty(\mathbf{R}_{++}^{l(m-1)}, \mathbf{R})$. ▮

The regular value theorem allows one to conclude that generically, V_f is a smooth manifold of dimension $l+m-1$. ▮

Appendix 4.2

Proof of Proposition 22. Take a point $Z = (p, \Lambda, (x_i))$ in V_f. The idea is to perturb the utility functions so that Z remains in V_f but is not in V_g.

Consider the linear perturbations on v used in the preceding proof and similar linear perturbations of the proper utility function u;

$$u_t(x) = u(x) + \varepsilon\beta_Z(x)\, t \cdot x, \qquad v_s(X) = v(X) + \varepsilon\beta_Z(x)\, s \cdot X,$$

where t is in $\mathbf{R}^l$, $s = (s_i)_{1 \leqslant i \leqslant m-1}$ in $\mathbf{R}^{l(m-1)}$ and β_Z is the characteristic function of a compact neighborhood of Z. For well chosen reals ε and characteristic functions β_Z, the perturbed proper utility function u_t still fulfills the standard condition of strict monotonicity and strict concavity (first the perturbation occurs on a compact, second it can be as small as required). For details, see Rojas [14].

In this proof, the perturbation will concern only the variables related to the last commodity l. Take then $t = (0, ..., 0, t^l)$ and $s_i = (0, ..., 0, s_i^l)$. For Z to remain a point of V_f, it is necessary and sufficient that it still fulfills Eqs. (13) written with u_t and v_s. Then the only condition is:

$$(\lambda_i - \lambda_1)\, t^l + \sum_{h \in L} (\lambda_{v_i(h)} - \lambda_{v_1(h)})\, s_h^l = 0, \qquad \forall i.$$

It is a linear system of m unknowns and $m-1$ independent equations (the extracted matrix composed with the coefficient of the variables s_h^l is invertible); the set of solutions is a linear space of dimension 1.

Take $M^l_{s,t}(Z)$, the $m-1$ square matrix extracted from $M(Z)$ (written for u_t and v_s) by cancelling all the columns but those related to commodity l. It can be decomposed into two parts,

$$M^l_{s,t}(Z) = M^l(Z) + \varepsilon[\Lambda'_1 s_1^l \cdots \Lambda'_{m-1} s_{m-1}^l],$$

where Λ'_i is the $(m-1)$-dimensional vector $(\lambda_i - \lambda_{v_i(k)})_{(1 \leqslant k \leqslant m-1)}$. Then the $m-1$ square matrix $\tilde{\Lambda}(s) = [\Lambda'_1 s'_1 \cdots \Lambda'_{m-1} s^l_{m-1}]$ is invertible (and for all the values of ε but a finite number, related to the eigenvalues of $\tilde{\Lambda}(s)^{-1} M^l(Z)$, $M^l_{s,t}$ is invertible too). Take indeed $\tilde{\Lambda}$, which is invertible, add the $m-1$ first rows to the last one and then subtract the first column to all the $m-1$ others, then one obtains

$$\begin{pmatrix} \lambda_1 & & & \\ \cdot & & -\tilde{\Lambda}' & \\ \lambda_{m-1} & & & \\ 1 & 0 & \cdots & 0 \end{pmatrix},$$

where $\tilde{\Lambda}'$ is the matrix $[\Lambda'_1 \cdots \Lambda'_{m-1}]$.

What is obtained at this point of the proof is that for a dense (and of course open, the determinant being a continuous mapping and one deals here with compact sets) set $Y_Z \times W_Z$ of utility functions u and v fulfilling all the assumptions of this model, $M(Z)$ has full rank.

Take a point $\Lambda \in \bar{\Lambda}_{\mathbf{Q}} = \bar{\Lambda} \cap \mathbf{Q}$ a vector of rational welfare weights. Take also $\omega \in \mathbf{Q}^l_{++}$. Consider the set $X(\lambda, \omega)$ of vectors $(p, (x_i)_{1 \leqslant i \leqslant m})$ such that (13) and (14) hold for this couple (λ, ω). It is a finite set. Indeed, the mapping defining this set is a submersion between two manifolds of same dimension: $l(m+1)$, at least for $v \in W$ (see the proof of Lemma 29); then $X(\lambda, \omega)$ is a discrete set. However, it is furthermore in a compact since all the consumptions are strictly positive and bounded by Eqs. (14). All these results entail that the set $\mathscr{X}_{\mathbf{Q}} = \bigcup_{(\lambda, \omega) \in \bar{\Lambda}_{\mathbf{Q}} \times \mathbf{Q}^l_{++}} X(\lambda, \omega)$ is countable.

The mapping f defining V_f being smooth, $\mathbf{Q}^l_{++}$ being dense in $\mathbf{R}^l_{++}$ and $\bar{\Lambda}_{\mathbf{Q}}$ being dense in $\bar{\Lambda}$, $\mathscr{X}_{\mathbf{Q}}$ is dense in V_f. Take $(u, v) \in [\bigcap_{Z \in \mathscr{X}_{\mathbf{Q}}} Y_Z] \times [\bigcap_{Z \in \mathscr{X}_{\mathbf{Q}}} W_Z \bigcap_{n \in \mathscr{N}} W_n] = \tilde{Y} \times \tilde{W}$, which is residual in $\mathscr{C}^\infty(\mathbf{R}^{l(m-1)}_{++}, \mathbf{R}) \times \mathscr{C}^\infty(\mathbf{R}^l_{++}, \mathbf{R})$, hence dense. Then $M^l(\Lambda, (x_i))$ has full rank on $\mathscr{X}_{\mathbf{Q}}$ and the continuity entails that it has full rank for a dense subset of the manifold V_f. Hence V_g is the complement of an open and dense subset of V_f. ∎

Appendix 5.1

Proof of Proposition 25. In the framework of this section, Eqs. (17) become

$$\left(\lambda_i Du + \sum_{h \in L} \lambda_{v_i(h)} Dv_h\right)(x_i) = p. \tag{16}$$

The proper utility function u is strictly concave and the external one (v_h) are concave. Then the map $(\lambda_i Du + \sum_{h \in L} \lambda_{v_i(h)} Dv_h)$ is strictly concave.

Since it is defined on a rectangle ($\mathbf{R}^l_{++}$) the theorem of Gale and Nikkaido [10] holds here and entails that this map is univalent. The allocation x_i is so uniquely defined if p and the welfare weights are fixed. A simple application of Lemma 3.2.1 in Balasko [2] allows one to conclude. ▮

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