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Variational representations for N-cyclically monotone vector fields

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Abstract

Given a convex bounded domain Ω in $\mathbb{R}^d$ and an integer $N \geq 2$, we associate to any *jointly N-monotone* $(N-1)$-tuplet $(u_1, u_2, ..., u_{N-1})$ of vector fields from Ω into $\mathbb{R}^d$, a Hamiltonian H on $\mathbb{R}^d \times \mathbb{R}^d ... \times \mathbb{R}^d$, that is concave in the first variable, jointly convex in the last $(N-1)$ variables such that for almost all $x \in \Omega$,

$$(u_1(x), u_2(x), ..., u_{N-1}(x)) = \nabla_{2,...,N} H(x, x, ..., x).$$

Moreover, H is N-sub-antisymmetric, meaning that $\sum_{i=0}^{N-1} H(\sigma^i(\mathbf{x})) \leq 0$ for all $\mathbf{x} = (x_1, ..., x_N) \in \Omega^N$, σ being the cyclic permutation on $\mathbb{R}^d$ defined by $\sigma(x_1, x_2, ..., x_N) = (x_2, x_3, ..., x_N, x_1)$. Furthermore, H is N-antisymmetric in a sense to be defined below. This can be seen as an extension of a theorem of E. Krauss, which associates to any monotone operator, a concave-convex antisymmetric saddle function. We also give various variational characterizations of vector fields that are almost everywhere N-monotone, showing that they are dual to the class of measure preserving N-involutions on Ω. .

1 Introduction

Given a domain Ω in $\mathbb{R}^d$, recall that a single-valued map u from Ω to $\mathbb{R}^d$ is said to be *N-cyclically monotone* if for every cycle $x_1, ..., x_N, x_{N+1} = x_1$ of points in Ω, one has

$$\sum_{i=1}^{N} \langle u(x_i), x_i - x_{i+1} \rangle \geq 0. \tag{1}$$

A classical theorem of Rockafellar [10] states that a map u from Ω to $\mathbb{R}^d$ is *N-cyclically monotone for every* $N \geq 2$ if and only if

$$u(x) \in \partial\phi(x) \text{ for all } x \in \Omega, \tag{2}$$

where $\phi : \mathbb{R}^d \to \mathbb{R}$ is a convex function. On the other hand, a result of E. Krauss [9] yields that u is a monotone map, i.e., a 2-cyclically monotone map, if and only if

$$u(x) \in \partial_2 H(x, x) \text{ for all } x \in \Omega, \tag{3}$$

where H is a concave-convex antisymmetric Hamiltonian on $\mathbb{R}^d \times \mathbb{R}^d$, and $\partial_2 H$ is the subdifferential of H as a convex function in the second variable.

In this paper, we extend the result of Krauss to the class of N-cyclically monotone vector fields, where $N \geq 3$. We shall give a representation for a family of $(N-1)$ vector fields, which may or may not be individually N-cyclically monotone. Here is the needed concept.

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Definition 1 *Let $u_1, ..., u_{N-1}$ be bounded vector fields from a domain $\Omega \subset \mathbb{R}^d$ into $\mathbb{R}^d$. We shall say that the $(N-1)$-tuple $(u_1, u_2, ..., u_{N-1})$ is jointly N-monotone, if for every cycle $x_1, ..., x_{N+\ell}$ of points in Ω such that $x_{N+i} = x_i$ for $1 \leq i \leq \ell$, one has*

$$\sum_{i=1}^{N} \sum_{\ell=1}^{N-1} \langle u_l(x_i), x_i - x_{l+i} \rangle \geq 0. \tag{4}$$

Examples of jointly N-monotone families of vector fields:

- It is clear that $(u, 0, 0, ..., 0)$ is jointly N-monotone if and only if u is N-monotone.
- More generally, if each u_ℓ is N-monotone, then the family $(u_1, u_2, ..., u_{N-1})$ is jointly N-monotone. Actually, one only needs that for $1 \leq \ell \leq N-1$, the vector field u_ℓ be (N, ℓ)-*monotone*, in the following sense: for every cycle $x_1, ..., x_{N+\ell}$ of points in Ω such that $x_{N+i} = x_i$ for $1 \leq i \leq \ell$, we have

$$\sum_{i=1}^{N} \langle u_\ell(x_i), x_i - x_{\ell+i} \rangle \geq 0. \tag{5}$$

  This notion is sometimes weaker than N-monotonicity since if ℓ divides N, then it suffices for u to be $\frac{N}{\ell}$-monotone in order to be an (N, ℓ)-monotone vector field. For example, if u_1 and u_3 are 4-monotone operators and u_2 is 2-monotone, then the triplet (u_1, u_2, u_3) is jointly 4-monotone.
- Another example is when (u_1, u_2, u_3) are vector fields such that u_2 is 2-monotone and

$$\langle u_1(x) - u_3(y), x - y \rangle \geq 0 \text{ for every } x, y \in \mathbb{R}^d.$$

  In this case, the triplet (u_1, u_2, u_3) is jointly 4-monotone. In particular, if u_1 and u_2 are both 2-monotone, then the triplet (u_1, u_2, u_1) is jointly 4-monotone.
- More generally, it is easy to show that $(u, u, ..., u)$ is jointly N-monotone if and only if u is 2-cyclically monotone.

In the sequel, we shall denote by σ the cyclic permutation on $\mathbb{R}^d \times ... \times \mathbb{R}^d$, defined by

$$\sigma(x_1, x_2, ..., x_{N-1}, x_N) = (x_2, x_3, ..., x_N, x_1),$$

and consider the family of continuous N-*antisymmetric* Hamiltonians on Ω^N, that is

$$\mathcal{H}_N(\Omega) = \{H \in C(\Omega^N);\ \sum_{i=0}^{N-1} H(\sigma^i(x_1, ..., x_N)) = 0\} \tag{6}$$

We shall say that H is N-*sub-antisymmetric on* Ω if

$$\sum_{i=0}^{N-1} H(\sigma^i(x_1, ..., x_N)) \leq 0 \text{ on } \Omega^N \quad \text{and} \quad H(x, x, ..., x) = 0 \text{ on the diagonal.} \tag{7}$$

We shall also say that a function F of two variables is N-*cyclically sub-antisymmetric on* Ω, if

$$F(x, x) = 0 \text{ and } \sum_{i=1}^{N} F(x_i, x_{i+1}) \leq 0 \text{ for all cyclic families } x_1, ..., x_N, x_{N+1} = x_1 \text{ in } \Omega. \tag{8}$$

Note that if a function $H(x_1, ..., x_N)$ N-sub-antisymmetric and if it only depends on the first two variables, then the function $F(x_1, x_2) := H(x_1, x_2, ..., x_N)$ is N-cyclically sub-antisymmetric.

We associate to any function H on Ω^N, the following functional on $\Omega \times (\mathbb{R}^d)^{N-1}$,

$$L_H(x, p_1, ..., p_{N-1}) = \sup \left\{ \sum_{i=1}^{N-1} \langle p_i, y_i \rangle - H(x, y_1, ..., y_{N-1}); y_i \in \Omega \right\}. \tag{9}$$

Note that if Ω is convex and if H is convex in the last $(N-1)$ variables, then L_H is nothing but the Legendre transform of $\tilde{H}$ with respect to the last $(N-1)$ variables, where $\tilde{H}$ is the extension of H over $(\mathbb{R}^d)^N$, defined as: $\tilde{H} = H$ on Ω^N and $\tilde{H} = +\infty$ outside of Ω^N. Since $H(x,...,x) = 0$ for any $H \in \mathcal{H}_N(\Omega)$, then for any such H, we have for $x \in \Omega$ and $p_1,...,p_{N-1} \in \mathbb{R}^d$,

$$L_H(x, p_1, ..., p_{N-1}) \geq \sum_{i=1}^{N-1} \langle x, p_i \rangle. \tag{10}$$

To formulate variational principles for such vector fields, we shall consider the class of σ-invariant probability measures on Ω^N, which are those $\pi \in \mathcal{P}(\Omega^N)$ such that for all $h \in L^1(\Omega^N, d\pi)$, we have

$$\int_{\Omega^N} h(x_1,...,x_N) d\pi = \int_{\Omega^N} h(\sigma(x_1,...,x_N)) d\pi. \tag{11}$$

We denote

$$\mathcal{P}_{\rm sym}(\Omega^N) = \{\pi \in \mathcal{P}(\Omega^N);\ \pi\ \sigma\text{-invariant probability on } \Omega^N\}. \tag{12}$$

For a given probability measure μ on Ω, we also consider the class

$$\mathcal{P}^{\mu}_{\rm sym}(\Omega^N) = \{\pi \in \mathcal{P}_{\rm sym}(\Omega^N);\ {\rm proj}_1 \pi = \mu\}, \tag{13}$$

i.e., the set of all $\pi \in \mathcal{P}_{\rm sym}(\Omega^N)$ with a given first marginal μ, meaning that

$$\textstyle\int_{\Omega^N} f(x_1)\, d\pi(x_1,...,x_N) = \int_\Omega f(x_1)\, d\mu(x_1) \text{ for every } f \in L^1(\Omega, \mu). \tag{14}$$

Consider now the set $\mathcal{S}(\Omega, \mu)$ of μ-measure preserving transformations on Ω, which can be identified with a closed subset of the sphere of $L^2(\Omega, \mathbb{R}^d)$. We shall also consider the subset of $\mathcal{S}(\Omega, \mu)$ consisting of N-involutions, that is

$$\mathcal{S}_N(\Omega, \mu) = \{S \in \mathcal{S}(\Omega, \mu);\ S^N = I\ \mu \text{ a.e.}\}.$$

2 Monotone vector fields and N-antisymmetric Hamiltonians

In this section, we establish the following extension of a theorem of Krauss.

Theorem 2 *Let $N \geq 2$ be an integer, and consider $u_1,...,u_{N-1}$ to be bounded vector fields from a convex domain $\Omega \subset \mathbb{R}^d$ into $\mathbb{R}^d$.*

1. *If the $(N-1)$-tuple $(u_1,...,u_{N-1})$ is jointly N-monotone, then there exists an N-sub-antisymmetric Hamiltonian H that is concave in the first variable, convex in the other $(N-1)$ variables such that*

$$(u_1(x), ..., u_{N-1}(x)) = \nabla_{2,...,N} H(x, x, ..., x) \quad \text{for a.e. } x \in \Omega. \tag{15}$$

   Moreover, H is N-antisymmetric in the following sense

$$H(x_1, x_2, ..., x_N) + H_{2,...,N}(x_1, x_2, ..., x_N) = 0, \tag{16}$$

   where $H_{2,...,N}$ is the concavification of the function $K(\mathbf{x}) = \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x}))$ with respect to the last $(N-1)$ variables.

   Furthermore, there exists a continuous N-antisymmetric Hamiltonian $\bar{H}$ on Ω^N, such that

$$L_{\bar{H}}(x, u_1(x), u_2(x), ..., u_{N-1}(x)) = \sum_{i=1}^{N-1} \langle u_i(x), x \rangle \text{ for all } x \in \Omega. \tag{17}$$

2. *Conversely, if $(u_1,...,u_{N-1})$ satisfy (15) for some N-sub-antisymmetric Hamiltonian H that is concave in the first variable, convex in the other variables, then the $(N-1)$-tuple $(u_1,...,u_{N-1})$ is jointly N-monotone.*

Remark 3 *Note that in the case* $N = 2$*,* $K(\mathbf{x}) = H(x_2, x_1)$ *is concave with respect to* x_2*, hence* $H_2(x_1, x_2) = H(x_2, x_1)$*, and (16) becomes*

$$H(x_1, x_2) + H(x_2, x_1) = 0,$$

thus H *is antisymmetric, recovering well-known results [9], [4], [7], [8].*

We start with the following lemma.

Lemma 4 *Assume the* $(N-1)$*-tuple of bounded vector fields* $(u_1, ..., u_{N-1})$ *on* Ω *is jointly* N*-monotone. Let* $f(x_1, ..., x_n) := \sum_{l=1}^{N-1} \langle u_l(x_1), x_1 - x_{l+1} \rangle$ *and consider the function* $\tilde{f}(x_1, ..., x_n)$ *to be the convexification of* f *with respect to the first variable, that is*

$$\tilde{f}(x_1, x_2, ..., x_N) = \inf \left\{ \sum_{k=1}^{n} \lambda_k f\left(x_1^k, x_2, ..., x_N\right) : n \in \mathbb{N}, \lambda_k \geq 0, \sum_{k=1}^{n} \lambda_k = 1, \sum_{k=1}^{n} \lambda_k x_1^k = x_1 \right\}. \tag{18}$$

Then, $\tilde{f}$ *satisfies the following properties:*

1. $f \geq \tilde{f}$ *on* Ω^N*;*
2. $\tilde{f}$ *is convex in the first variable and concave with respect to the other variables;*
3. $\tilde{f}(x, x, ..., x) = 0$ *for each* $x \in \Omega$*,*
4. $\tilde{f}$ *satisfies*

$$\textstyle\sum_{i=0}^{N-1} \tilde{f}(\sigma^i(x_1, ..., x_N)) \geq 0 \text{ on } \Omega^N. \tag{19}$$

Proof: Since the $(N-1)$-tuple $(u_1, ..., u_{N-1})$ is jointly N-monotone, it is easy to see that the function

$$f(x_1, ..., x_n) := \sum_{l=1}^{N-1} \langle u_l(x_1), x_1 - x_{l+1} \rangle$$

is linear in the last $(N-1)$ variables, that $f(x, x, ..., x) = 0$, and that

$$\textstyle\sum_{i=0}^{N-1} f(\sigma^i(x_1, ..., x_N)) \geq 0 \text{ on } \Omega^N. \tag{20}$$

It is also clear that $f \geq \tilde{f}$, that $\tilde{f}$ is convex with respect to the first variable x_1, and that it is concave with respect to the other variables $x_2, ..., x_N$, since f itself is concave (actually linear) with respect to $x_2, ..., x_n$. We now show that $\tilde{f}$ satisfies (19).

For that, we fix $x_1, x_2, ..., x_N$ in Ω and consider $(x_1^k)_{k=1}^n$ in Ω, and $(\lambda_k)_k$ in $\mathbb{R}$ such that $\lambda_k \geq 0$ such that $\sum_{k=1}^n \lambda_k = 1$ and $\sum_{k=1}^n \lambda_k x_1^k = x_1$. For each k, we have

$$f(x_1^k, x_2, ..., x_N) + f(x_2, ..., x_N, x_1^k) + ... + f(x_N, x_1^k, x_2, ..., x_{N-1}) \geq 0.$$

Multiplying by λ_k, summing over k, and using that f is linear in the last $(N-1)$-variables, we have

$$\sum_{k=1}^{n} \lambda_k f(x_1^k, x_2, ..., x_N) + f(x_2, ..., x_N, x_1) + ... + f(x_N, x_1, x_2, ..., x_{N-1}) \geq 0.$$

By taking the infimum, we obtain

$$\tilde{f}(x_1, x_2, ..., x_N) + \sum_{i=1}^{N-1} f(\sigma^i(x_1, x_2, ..., x_N)) \geq 0.$$

Let now $n \in \mathbb{N}$, $\lambda_k \geq 0$, $x_N^k \in \Omega$ be such that $\sum_{k=1}^n \lambda_k = 1$ and $\sum\limits_{k=1}^{n} \lambda_k x_2^k = x_2$. We have for every $1 \leq k \leq n$,

$$\tilde{f}\left(x_1, x_2^k, x_3, ..., x_N\right) + f\left(x_2^k, x_3, , ..., x_1\right) + ... + f\left(x_N, x_1, x_2^k, x_3, ..., x_{N-1}\right) \geq 0.$$

Multiplying by λ_k, summing over k and using that $\tilde{f}$ is convex in the first variable and f is linear in the last $(N-1)$-variables, we obtain

$$\tilde{f}\left(x_1, x_2, x_3, ..., x_N\right) + \sum_{k=1}^{n} \lambda_k f\left(x_2^k, x_3, , ..., x_1\right) + ... + f\left(x_N, x_1, x_2, x_3, ..., x_{N-1}\right)$$

$$\geq \sum_{k=1}^{n} \lambda_k \tilde{f}\left(x_1, x_2^k, x_3, ..., x_N\right) + \sum_{k=1}^{n} \lambda_k f\left(x_2^k, x_3, , ..., x_1\right) + ... + \sum_{k=1}^{n} \lambda_k f\left(x_N, x_1, x_2^k, x_3, ..., x_{N-1}\right) \geq 0.$$

By taking the infimum over all possible such choices, we get

$$\tilde{f}\left(x_1, x_2, x_3, ..., x_N\right) + \tilde{f}\left(x_2, x_3, , ..., x_1\right) + ... + f\left(x_N, x_1, x_2, x_3, ..., x_{N-1}\right) \geq 0.$$

By repeating this procedure with $x_3, ..., x_{N-1}$, we get

$$\sum_{i=0}^{N-2} \tilde{f}\left(\sigma^i(x_1, x_2, , ..., x_N)\right) + f\left(x_N, x_1, x_2, x_3, ..., x_{N-1}\right) \geq 0.$$

Finally, since

$$f\left(x_N, x_1, x_2, x_3, ..., x_{N-1}\right) \geq -\sum_{i=0}^{N-2} \tilde{f}\left(\sigma^i(x_1, x_2, , ..., x_N)\right).$$

and since $\tilde{f}$ is concave in the last $(N-1)$ variables, we have for fixed $x_1, x_2, ..., x_{N-1}$, that the function

$$x_N \to -\sum_{i=0}^{N-2} \tilde{f}\left(\sigma^i(x_1, x_2, , ..., x_N)\right)$$

is a convex minorant of $x_N \to f\left(x_N, x_1, x_2, x_3, ..., x_{N-1}\right)$. It follows that

$$f\left(x_N, x_1, x_2, x_3, ..., x_{N-1}\right) \geq \tilde{f}\left(x_N, x_1, x_2, x_3, ..., x_{N-1}\right) \geq -\sum_{i=0}^{N-2} \tilde{f}\left(\sigma^i(x_1, x_2, , ..., x_N)\right),$$

which finally implies that $\sum_{i=0}^{N-1} \tilde{f}(\sigma^i(x_1, x_2, ..., x_N)) \geq 0$.

This clearly implies that $\tilde{f}(x, x, ..., x) \geq 0$ for any $x \in \Omega$. On the other hand, since $\tilde{f}(x, x, ..., x) \leq f(x, x, ..., x) = 0$, we get that $\tilde{f}(x, x, ..., x) = 0$ for all $x \in \Omega$. □

Proof of Theorem 2: Assume the $(N-1)$-tuple of vector fields $(u_1, ..., u_{N-1})$ is jointly N-monotone on Ω, and consider the function $f(x_1, ..., x_n) := \sum_{l=1}^{N-1} \langle u_l(x_1), x_1 - x_{l+1}\rangle$ as well as its convexification with respect to the first variable $\tilde{f}(x_1, ..., x_n)$.

By Lemma 4, the function $\psi(x_1, ..., x_n) := -\tilde{f}(x_1, ..., x_n)$ satisfies the following properties

(i) $x_1 \to \psi(x_1, ..., x_n)$ is concave;

(ii) $(x_2, x_3, ..., x_N) \to \psi(x_1, ..., x_n)$ is convex;

(iii) $\psi(x_1, ..., x_n) \geq -f(x_1, ..., x_n) = \sum_{l=1}^{N-1} \langle u_l(x_1), x_{l+1} - x_1\rangle$;

(iv) ψ is N-sub-antisymmetric.

Consider now the family $\mathcal{H}$ of functions $H : \Omega^N \to \mathbb{R}$ such that

1. $H(x_1, x_2, ..., x_N) \geq \sum_{l=1}^{N-1} \langle u_l(x_1), x_{l+1} - x_1\rangle$ for every N-tuple $(x_1, ..., x_N)$ in Ω^N;
2. H is concave in the first variable;
3. H is jointly convex in the last $(N-1)$ variables;
4. H is N-sub-antisymmetric.

Note that $\mathcal{H} \neq \emptyset$ since ψ belongs to $\mathcal{H}$. Moreover, by N-subsymmetry, any $H \in \mathcal{H}$ satisfies for all $\mathbf{x} = (x_1, ..., x_N) \in \Omega^N$,

$$H(\mathbf{x}) \leq -\sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x})) \leq -\sum_{i=1}^{N-1} \psi(\sigma^i(\mathbf{x})). \tag{21}$$

This also yields that

$$\sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_{\ell+1} - x_1 \rangle \leq H(\mathbf{x}) \leq -\sum_{i=2}^{N} \sum_{\ell=1}^{N-1} \langle u_\ell(x_i), x_i - x_{i+\ell} \rangle, \tag{22}$$

where we denote $x_{i+N} := x_i$ for $i = 1, ..., \ell$. This yields that $H(x, x, ..., x) = 0$ for every $H \in \mathcal{H}$ and any $x \in \Omega$.

On the other hand, it is easy to see that every directed family $(H_i)_i$ in $\mathcal{H}$ has a supremum $H_\infty \in \mathcal{H}$, meaning that $\mathcal{H}$ is a Zorn family, and therefore it has a maximal element H.

Consider now the function

$$\bar{H}(\mathbf{x}) = \frac{(N-1)H(\mathbf{x}) - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x}))}{N},$$

and note that

(i) $\bar{H}$ is N-antisymmetric, since

$$\bar{H}(\mathbf{x}) = \frac{1}{N} \sum_{i=1}^{N-1} [H(\mathbf{x}) - H(\sigma^i(\mathbf{x}))],$$

and each $K_i(\mathbf{x}) := H(\mathbf{x}) - H(\sigma^i(\mathbf{x}))$ is N-antisymmetric.

(ii) $\bar{H} \geq H$ on Ω^N, since

$$N[\bar{H}(\mathbf{x}) - H(\mathbf{x})] = -\sum_{i=0}^{N-1} H(\sigma^i(\mathbf{x})) \geq 0,$$

because H itself is N-sub-antisymmetric.

The maximality of H would have implied that $H = \bar{H}$ is N-antisymmetric if only $\bar{H}$ was jointly convex in the last $(N-1)$-variables, but since this is not necessarily the case, we consider for $\mathbf{x} = (x_1, x_2, ..., x_N)$, the function

$$K(x_1, x_2, ..., x_N) = K(\mathbf{x}) := -\sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x})),$$

which is already concave in the first variable x_1. Its convexification in the last $(N-1)$-variables, that is

$$K^{2,...,N}(\mathbf{x}) = \inf \left\{ \sum_{i=1}^{n} \lambda_i K(x_1, x_2^i, ..., x_N^i);\ \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i (x_2^i, ..., x_N^i, 1) = (x_2, ..., x_N, 1) \right\},$$

is still concave in the first variable, but is now convex in the last $(N-1)$ variables. Moreover,

$$H \leq K^{2,...,N} \leq K = -\sum_{i=1}^{N-1} H \circ \sigma^i. \tag{23}$$

Indeed, $K^{2,...,N} \leq K$ from the definition of $K^{2,...,N}$, while $H \leq K^{2,...,N}$ because $H \leq K$ and H is already convex in the last $(N-1)$-variables. It follows that

$$H \leq \frac{(N-1)H + K^{2,...,N}}{N} \leq \frac{(N-1)H + K}{N} = \frac{(N-1)H - \sum\limits_{i=1}^{N-1} H \circ \sigma^i}{N} = \bar{H}.$$

The function $H' = \frac{(N-1)H + K^{2,...,N}}{N}$ belongs to the family $\mathcal{H}$ and therefore $H = H'$ by the maximality of H.

This finally yields that H is N-sub-antisymmetric, that $H(x, x, x) = 0$ for all $x \in \Omega$ and that

$$H(\mathbf{x}) + H_{2,...,N}(\mathbf{x}) = 0 \text{ for every } \mathbf{x} \in \Omega^N,$$

where $H_{2,...,N} = -K^{2,...,N}$, which for a fixed x_1, is nothing but the concavification of $(x_2, ..., x_N) \to \sum_{i=1}^{N-1} H(\sigma^i(x_1, x_2, ..., x_N))$.

Note now that since for any $x_1, ..., x_N$ in Ω,

$$H(x_1, x_2, ...x_N) \geq \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_{\ell+1} - x_1 \rangle, \tag{24}$$

and

$$H(x_1, x_1, ..., x_1) = 0, \tag{25}$$

we have

$$H(x_1, x_2, ..., x_N) - H(x_1, ..., x_1) \geq \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_{\ell+1} - x_1 \rangle. \tag{26}$$

Since H is convex in the last $(N-1)$ variables, this means that for all $x \in \Omega$, we have

$$(u_1(x), u_2(x), ..., u_{N-1}(x)) \in \partial_{2,...,N} H(x, x, ..., x). \tag{27}$$

as claimed in (15). Note that this also yield that

$$L_H(x, u_1(x), ..., u_{N-1}(x)) + H(x, x, ..., x) = \textstyle\sum_{\ell=1}^{N-1} \langle u_\ell(x), x \rangle \text{ for all } x \in \Omega.$$

In other words, $L_H(x, u_1(x), ..., u_{N-1}(x)) = \sum_{\ell=1}^{N-1} \int_\Omega \langle u_\ell(x), x \rangle$ for all $x \in \Omega$. As above, consider

$$\bar{H}(\mathbf{x}) = \frac{(N-1)H(\mathbf{x}) - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x}))}{N}.$$

We have that $\bar{H} \in \mathcal{H}_N(\Omega)$ and $\bar{H} \geq H$, and therefore $L_{\bar{H}} \leq L_H$. On the other hand, we have for all $x \in \Omega$,

$$L_{\bar{H}}(x, u_1(x), ..., u_{N-1}(x)) = L_{\bar{H}}(x, u_1(x), ..., u_{N-1}(x)) + \bar{H}(x, x, ..., x) \geq \sum_{\ell=1}^{N-1} \langle u_\ell(x), x \rangle.$$

To prove (17), we use the appendix in [6] to deduce that for $i = 2, ..., N$, the gradients $\nabla_i H(x, x, ..., x)$ actually exist for a.e. x in Ω.

The converse is straightforward since if (27) holds, then (26) does, and since we also have (25), then the property that $(u_1, ..., u_{N-1})$ is jointly N-monotone follows from (24) and the sub-antisymmetry of H. □

In the case of a single N-monotone vector field, we can obviously apply the above theorem to the $(N-1)$-tuple $(u, 0, ..., 0)$ which is then N-monotone to find a N-sub-antisymmetric Hamiltonian H, which is concave in the first variable, convex in the last $(N-1)$ variables such that

$$(-u(x), u(x), 0, ..., 0) = \nabla H(x, x, ..., x) \quad \text{for a.e. } x \in \Omega. \tag{28}$$

However, in this case we can restrict ourselves to N-cyclically sub-antisymmetric functions of two variables and establish the following extension of the Theorem of Krauss.

Theorem 5 *If u is N-cyclically monotone on Ω, then there exists a concave-convex function of two variables F that is N-cyclically sub-antisymmetric, such that*

$$(-u(x), u(x) \in \partial F(x, x) \text{ for all } x \in \Omega, \tag{29}$$

where ∂H is the sub-differential of H as a concave-convex function [11]. Moreover,

$$u(x) = \nabla_2 F(x, x) \text{ for a.e. } x \in \Omega. \tag{30}$$

Proof: Let $f(x,y) = \langle u(x), x-y\rangle$ and let $f^1(x,y)$ be its convexification in x for fixed y, that is

$$f^1(x,y) = \inf\left\{\sum_{k=1}^{n}\lambda_k f(x_k,y) : \lambda_k \geq 0, \sum_{k=1}^{n}\lambda_k = 1, \sum_{k=1}^{n}\lambda_k x_k = x\right\}. \tag{31}$$

Since $f(x,x)=0$, f is linear in y, and $\sum_{i=1}^{N} f(x_i,x_{i+1}) \geq 0$ for any cyclic family $x_1,...,x_N, x_{N+1}=x_1$ in Ω, it is easy to show that $f \geq f^1$ on Ω, f^1 is convex in the first variable and concave with respect to the second, $f^1(x,x)=0$ for each $x\in\Omega$, and that f^1 is N-cyclically supersymmetric in the sense that for any cyclic family $x_1,...,x_N,x_{N+1}=x_1$ in Ω, we have $\sum_{i=1}^{N} f^1(x_i,x_{i+1}) \geq 0$.

Consider now $F(x,y) = -f^1(x,y)$ and note that $x \to F(x,y)$ is concave, $y\to F(x,y)$ is convex, $F(x,y) \geq -f(x,y) = \langle u(x), y-x\rangle$ and F is N-cyclically sub-antisymmetric. By the antisymmetry, we have

$$\langle u(x_1), x_2 - x_1\rangle \leq F(x_1,x_2) \leq \langle u(x_2), x_2-x_1\rangle, \tag{32}$$

which yields that $(-u(x),u(x)) \in \partial F(x,x)$ for all $x\in\Omega$.

Since F is anti-symmetric and concave-convex, the possibly multivalued map $x \to \partial_2 F(x,x)$ is monotone on Ω, and therefore single-valued and differentiable almost everywhere [10]. This completes the proof.

Remark 6 *Note that we cannot expect to have a function F such that $\sum\limits_{i=1}^{N} F(x_i,x_{i+1}) = 0$ for all cyclic families $x_1,...,x_N,x_{N+1}=x_1$ in Ω. Actually, we believe that the only function satisfying such an N-anti-symmetry for $N\geq 3$ must be of the form $F(x,y) = f(x)-f(y)$. This is the reason why one needs to consider functions of N-variables in order to get N-antisymmetry. In other words, the function defined by*

$$H(x_1,x_2,...,x_N) := \frac{(N-1)F(x_1,x_2) - \sum_{i=2}^{N-1} F(x_i,x_{i+1})}{N}, \tag{33}$$

is N-antisymmetric in the sense of (6) and $H(x_1,x_2...,x_N) \geq F(x_1,x_2)$ for all $(x_1,x_2...,x_N)$ in Ω^N.

3 Variational characterization of monotone vector fields

In order to simplify the exposition, we shall always assume in the sequel that $d\mu$ is Lebesgue measure dx normalized to be a probability on Ω. We shall also assume that Ω is convex and that its boundary has measure zero.

Theorem 7 *Let $u_1,...,u_{N-1} : \Omega \to \mathbb{R}^d$ be bounded measurable vector fields. The following properties are then equivalent:*

1. *The $(N-1)$-tuple $(u_1,...,u_{N-1})$ is jointly N-monotone a.e., that is there exists a measure zero set Ω_0 such that $(u_1,...,u_{N-1})$ is jointly N-monotone on $\Omega\setminus\Omega_0$.*
2. *The infimum of the following Monge-Kantorovich problem*

$$\inf\left\{\int_{\Omega^N}\sum_{\ell=1}^{N-1}\langle u(x_1), x_1 - x_{\ell+1}\rangle d\pi(x_1,x_2,...,x_N));\, \pi\in\mathcal{P}^{\mu}_{\rm sym}(\Omega^N)\right\} \tag{34}$$

   is equal to zero, and is therefore attained by the push-forward of μ by the map $x\to(x,x,...,x)$.
3. *$(u_1,...,u_{N-1})$ is in the polar of $\mathcal{S}_N(\Omega,\mu)$ in the following sense,*

$$\inf\left\{\int_\Omega\sum_{\ell=1}^{N-1}\langle u_\ell(x), x - S^\ell x\rangle\, d\mu; S\in\mathcal{S}_N(\Omega,\mu)\right\} = 0. \tag{35}$$

4. *The following holds:*

$$\inf\left\{\int_\Omega\sum_{\ell=1}^{N-1}|u_\ell(x) - S^\ell x|^2 d\mu; S\in\mathcal{S}_N(\Omega,\mu)\right\} = \sum_{\ell=1}^{N-1}\int_\Omega |u_\ell(x)-x|^2 d\mu. \tag{36}$$

5. *There exists a N-sub-antisymmetric Hamiltonian H which is concave in the first variable, convex in the last $(N-1)$ variables such that*

$$(u_1(x), ..., u_{N-1}(x)) = \nabla_{2,...,N} H(x, x, ..., x) \quad \text{for a.e. } x \in \Omega. \tag{37}$$

Moreover, H is N-symmetric in the sense of (16).

6. *The following duality holds:*

$$\inf\{\int_\Omega L_H(x, u_1(x), ..., u_{N-1}(x))d\mu;\ H \in \mathcal{H}_N(\Omega)\} = \sup\{\int_\Omega \sum_{\ell=1}^{N-1} \langle u_\ell(x), S^\ell x\rangle\, d\mu; S \in \mathcal{S}_N(\Omega, \mu)\}$$

and the latter is attained at the identity map.

We start with the following lemma, which identifies those probabilities in $\mathcal{P}^\mu_{\rm sym}(\Omega^N)$ that are carried by graphs of functions from Ω to Ω^N.

Lemma 8 *Let $S : \Omega \to \Omega$ be a μ-measurable map, then the following properties are equivalent:*

1. *The image of μ by the map $x \to (x, Sx, ..., S^{N-1}x)$ belongs to $\mathcal{P}^\mu_{\rm sym}(\Omega^N)$.*
2. *S is μ-measure preserving and $S^N(x) = x$ μ-a.e.*
3. *For any bounded Borel measurable N-antisymmetric H on Ω^N, we have $\int_\Omega H(x, Sx, ..., S^{N-1}x)\, d\mu = 0$.*

Proof. It is clear that 1) implies 3) since $\int_{\Omega^N} H(\mathbf{x})\, d\pi(\mathbf{x}) = 0$ for any N-antisymmetric Hamiltonian H and any $\pi \in \mathcal{P}^\mu_{\rm sym}(\Omega^N)$.

That 2) implies 1) is also straightforward since if π is the push-forward of μ by a map of the form $x \to (x, Sx, ..., S^{N-1}x)$, where S is a μ-measure preserving S with $S^N x = x$ μ a.e. on Ω, then for all $h \in L^1(\Omega^N, d\pi)$, we have

$$\begin{aligned}
\int_{\Omega^N} h(x_1, ..., x_N) d\pi &= \int_{\Omega^N} h(x, Sx, ..., S^{N-1}x)\, d\mu(x) = \int_{\Omega^N} h(Sx, S^2x, ..., S^{N-1}x, S^N x)\, d\mu(x)\\
&= \int_{\Omega^N} h(Sx, S^2x, ..., S^{N-1}x, x)\, d\mu(x) = \int_{\Omega^N} h(\sigma(x_1, ..., x_N)) d\pi.
\end{aligned}$$

We now prove that 2) and 3) are equivalent. Assuming first that S is μ-measure preserving such that $S^N = I$ μ a.e., then for every Borel bounded N-antisymmetric H, we have

$$\begin{aligned}
\int_\Omega H(x, Sx, S^2x, ..., S^{N-1}x) d\mu &= \int_\Omega H(Sx, S^2x, ..., S^{N-1}x, x) d\mu\\
&= ... = \int_\Omega H(S^{N-1}x, x, Sx, ..., S^{N-2}x) d\mu.
\end{aligned}$$

Since H is N-antisymmetric, we can see that

$$H(x, Sx, ..., S^{N-1}x) + H(Sx, S^2x, ..., S^{N-1}x, x) + ...H(S^{N-1}x, x, Sx, .., S^{N-2}x) = 0.$$

It follows that $N\int_\Omega H(x, Sx, S^2x, .., S^{N-1}x) d\mu = 0$.

For the reverse implication, assume $\int_\Omega H(x, Sx, S^2x, ..., S^{N-1}x) d\mu = 0$ for every N-antisymmetric Hamiltonian H. By testing this identity with the Hamiltonians

$$H(x_1, x_2, ..., x_N) = f(x_1) - f(x_i),$$

where f is any continuous function on Ω, one gets that S is μ-measure preserving. Now take the Hamiltonian

$$H(x_1, x_2, ..., x_N) = |x_1 - Sx_N| - |Sx_1 - x_2| - |x_2 - Sx_1| + |Sx_2 - x_3|.$$

Note that $H \in \mathcal{H}_N(\Omega)$ since it is of the form $H(x_1, ..., x_N) = f(x_1, x_2, x_N) - f(x_2, x_3, x_1)$. Now test the above identity with such an H to obtain

$$0 = \int_\Omega H(x, Sx, S^2x, ..., S^{N-1}x)d\mu = \int_\Omega |x - SS^{N-1}x|\, d\mu.$$

It follows that $S^N = I$ μ a.e. on ω, and we are done. □

Proof of Theorem 7: To show that (1) implies (2), it suffices to notice that if π is a σ-invariant probability measure on Ω^N such that $\text{proj}_1\pi = \mu$, then

$$\begin{aligned}
\int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_1 - x_{\ell+1}\rangle d\pi\,(x_1, ..., x_N) &= \frac{1}{N}\sum_{i=1}^{N}\int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_\ell\,(x_i)\,, x_i - x_{i+\ell}\rangle d\pi\,(x_1, ..., x_N) \\
&= \frac{1}{N}\int_{\Omega^N} \left(\sum_{i=1}^{N}\sum_{\ell=1}^{N-1} \langle u_\ell\,(x_i)\,, x_i - x_{i+\ell}\rangle\right) d\pi\,(x_1, ..., x_N) \\
&\geq 0,
\end{aligned}$$

since $(u_1, ..., u_{N-1})$ is jointly N-monotone. On the other hand, if π is the σ-invariant measure obtained by taking the image of $\mu := dx$ by $x \to (x, ..., x)$, then

$$\int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_1 - x_{\ell+1}\rangle d\pi\,(x_1, ..., x_N) = 0.$$

To show that (2) implies (3), let S be a μ-measure preserving transformation on Ω such that $S^N = I$ μ a.e. on Ω. Then the image π_S of μ by the map

$$x \to \left(x, Sx, S^2x, ..., S^{N-1}x\right)$$

is σ-invariant, hence

$$\int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_1 - x_{\ell+1}\rangle d\pi_S\,(x_1, ..., x_N) = \int_\Omega \sum_{\ell=1}^{N-1} \langle u_\ell(x), x - S^\ell x\rangle\, d\mu \geq 0.$$

By taking $S = I$, we get that the infimum is necessarily zero.

The equivalence of (3) and (4) follows immediately from developing the square.

We now show that (3) implies (1). For that take N points $x_1, x_2, ..., x_N$ in Ω, and let $R > 0$ be such that $B\,(x_i, R) \subset \Omega$. Consider the transformation

$$S_R\,(x) = \begin{cases} x - x_1 + x_2 \text{ for } x \in B\,(x_1, R) \\ x - x_2 + x_3 \text{ for } x \in B\,(x_2, R) \\ ... \\ x - x_N + x_1 \text{ for } x \in B\,(x_N, R) \\ x \text{ otherwise} \end{cases}$$

It is easy to see that S_R is a measure preserving transformation and that $S_R^N = Id$. We then have

$$0 \leq \int_\Omega \sum_{\ell=1}^{N-1} \langle u_\ell(x), x - S_R^\ell x\rangle\, d\mu \leq \sum_{i=1}^{N} \int_{B(x_i,R)} \sum_{\ell=1}^{N-1} \langle u_\ell\,(x)\,, x_i - x_{\ell+i}\rangle\, d\mu.$$

Letting $R \to 0$, we get from Lebesgue's density theorem, that

$$\frac{1}{|B\,(x_i, R)|}\int_{B(x_i,R)} \langle u_\ell\,(x)\,, x_i - x_{\ell+i}\rangle\, d\mu \to \langle u_\ell\,(x_i)\,, x_i - x_{\ell+i}\rangle\,,$$

from which follows that $(u_1, ..., u_{N-1})$ are jointly N-monotone a.e. on Ω.

The fact that (1) is equivalent to (5) follows immediately from Theorem 2.
To prove that 5) implies 6) note that for all $p_i \in \mathbb{R}^d, x \in \Omega, y_i \in \Omega, i = 1, ..., N-1$,

$$L_H(x, p_1, ..., p_{N-1}) + H(x, y_1, ..., y_{N-1}) \geq \sum_{i=1}^{N-1} \langle p_i, y_i \rangle,$$

which yields that for any $S \in \mathcal{S}_N(\Omega, \mu)$,

$$\int_\Omega [L_H(x, u_1(x), ..., u_{N-1}(x))\, d\mu + H(x, Sx, ..., S^{N-1}x)]\, d\mu \geq \int_\Omega \sum_{\ell=1}^{N-1} \langle u_\ell(x), S^\ell x \rangle\, d\mu.$$

If $H \in \mathcal{H}_N(\Omega)$ and $S \in \mathcal{S}_N(\Omega, \mu)$, we then have $\int_\Omega H(x, Sx, ..., S^{N-1}x) d\mu = 0$, and therefore

$$\int_\Omega L_H(x, u_1(x), ..., u_{N-1}(x))\, d\mu \geq \int_\Omega \sum_{\ell=1}^{N-1} \langle u_\ell(x), S^\ell x \rangle\, d\mu.$$

If now H is the N-sub-antisymmetric Hamiltonian obtained by 5), which is concave in the first variable, convex in the last $(N-1)$ variables, then

$$L_H(x, u_1(x), ..., u_{N-1}(x)) + H(x, x, ..., x) = \textstyle\sum_{\ell=1}^{N-1} \langle u_\ell(x), x \rangle \quad \text{for all } x \in \Omega \setminus \Omega_0,$$

and therefore $\int_\Omega L_H(x, u_1(x), ..., u_{N-1}(x))\, d\mu = \sum_{\ell=1}^{N-1} \int_\Omega \langle u_\ell(x), x \rangle\, d\mu$.

Consider now

$$\bar{H}(\mathbf{x}) = \frac{(N-1)H(\mathbf{x}) - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x}))}{N}.$$

As before, we have that $\bar{H} \in \mathcal{H}_N(\Omega)$ and $\bar{H} \geq H$. Since $L_{\bar{H}} \leq L_H$, we have that $\int_\Omega L_{\bar{H}}(x, u_1(x), ..., u_{N-1}(x))\, d\mu = \sum_{\ell=1}^{N-1} \int_\Omega \langle u_\ell(x), x \rangle\, d\mu$ and (6) is proved.

Finally, note that (6) readily implies (3), which means that $(u_1, ..., u_{N-1})$ is then jointly N-monotone. □

We now consider again the case of a single N-cyclically monotone vector field.

Corollary 9 *Let $u : \Omega \to \mathbb{R}^d$ be a bounded measurable vector field. The following properties are then equivalent:*

1. *u is N-cyclically monotone a.e., that is there exists a measure zero set Ω_0 such that u is N-cyclically monotone on $\Omega \setminus \Omega_0$.*
2. *The infimum of the following Monge-Kantorovich problem*

$$\inf\{\textstyle\int_{\Omega^N} \langle u(x_1), x_1 - x_2 \rangle d\pi(\mathbf{x});\ \pi \in \mathcal{P}^\mu_{\text{sym}}(\Omega^N)\} \tag{38}$$

   is equal to zero, and is therefore attained by the push-forward of μ by the map $x \to (x, x, ..., x)$.

3. *The vector field u is in the polar of $\mathcal{S}_N(\Omega, \mu)$, that is*

$$\inf\{\int_\Omega \langle u(x), x - Sx \rangle\, d\mu; S \in \mathcal{S}_N(\Omega, \mu)\} = 0. \tag{39}$$

4. *The projection of u on $\mathcal{S}_N(\Omega, \mu)$ is the identity map, that is*

$$\inf\{\int_\Omega |u(x) - Sx|^2 d\mu; S \in \mathcal{S}_N(\Omega, \mu)\} = \int_\Omega |u(x) - x|^2 d\mu. \tag{40}$$

5. *There exists a N-cyclically sub-antisymmetric function H of two variables, which is concave in the first variable, convex in the second variable such that*

$$u(x) = \nabla_2 H(x,x) \quad \text{for a.e. } x \in \Omega. \tag{41}$$

6. *The following duality holds:*

$$\inf\{\int_\Omega L_H(x, u(x), 0, ..., 0) d\mu;\ H \in \mathcal{H}_N(\Omega)\} = \sup\{\int_\Omega \langle u(x), Sx\rangle\, d\mu; S \in \mathcal{S}_N(\Omega,\mu)\}$$

and the latter is attained at the identity map.

Proof: This is an immediate application of Theorem 7 applied to the $(N-1)$-tuplet vector fields $(u, 0, ..., 0)$, which is clearly jointly N-monotone on $\Omega \setminus \Omega_0$, whenever u is N-monotone on $\Omega \setminus \Omega_0$.

Remark 10 Note that the sets of μ-measure preserving N-involutions $(\mathcal{S}_N(\Omega,\mu))_N$ do not form a nested family, that is $\mathcal{S}_N(\Omega,\mu)$ is not necessarily included in $\mathcal{S}_M(\Omega,\mu)$, whenever $N \leq M$, unless of course M is a multiple of N. On the other hand, the above theorem shows that their polar sets, i.e.,

$$\mathcal{S}_N(\Omega,\mu)^0 = \{u \in L^2(\Omega,\mathbb{R}^d);\ \int_\Omega \langle u(x), x - Sx\rangle\, d\mu \geq 0 \text{ for all } S \in \mathcal{S}_N(\Omega,\mu)\},$$

which coincide with the N-cyclically monotone maps, satisfy

$$\mathcal{S}_{N+1}(\Omega,\mu)^0 \subset \mathcal{S}_N(\Omega,\mu)^0,$$

for every $N \geq 1$. This can also be seen directly. Indeed, it is clear that a 2-involution is a 4-involution but not necessarily a 3-involution. On the other hand, assume that u is 3-cyclically monotone operator, then for any transformation $S: \Omega \to \Omega$, we have

$$\int_\Omega \langle u(x), x - Sx\rangle d\mu + \int_\Omega \langle u(Sx), Sx - S^2x\rangle d\mu + \int_\Omega \langle u(S^2x), S^2x - x\rangle d\mu \geq 0.$$

If now S is measure preserving, we have

$$\int_\Omega \langle u(x), x - Sx\rangle d\mu + \int_\Omega \langle u(x), x - Sx\rangle d\mu + \int_\Omega \langle u(S^2x), S^2x - x\rangle d\mu \geq 0,$$

and if $S^2 = I$, then $\int_\Omega \langle u(x), x - Sx\rangle d\mu \geq 0$, which means that $u \in \mathcal{S}_2(\Omega,\mu)^0$. Similarly, one can show that any $(N+1)$-cyclically monotone operator belongs to $\mathcal{S}_N(\Omega,\mu)^0$. In other words, $\mathcal{S}_{N+1}(\Omega,\mu)^0 \subset \mathcal{S}_N(\Omega,\mu)^0$ for all $N \geq 2$. Note that $\mathcal{S}_1(\Omega,\mu)^0 = \{I\}^0 = L^2(\Omega,\mathbb{R}^d)$, while

$$\mathcal{S}(\Omega,\mu)^0 = \cap_N \mathcal{S}_N(\Omega,\mu)^0 = \{u \in L^2(\Omega,\mathbb{R}^d), u = \nabla\phi \text{ for some convex function } \phi \text{ in } W^{1,2}(\mathbb{R}^d)\},$$

in view of classical results of Rockafellar [11] and Brenier [1].

Remark 11 In a forthcoming paper [6], the above result is extended to give a similar decomposition for any family of bounded measurable vector fields $u_1, u_2, ..., u_{N-1}$ on Ω. It is shown there that there exists a measure preserving N-involution S on Ω and an N-antisymmetric Hamiltonian H on Ω^N such that for $i = 1, ..., N-1$, we have

$$u_i(x) = \nabla_{i+1} H(x, Sx, S^2x, ...S^{N-1}x) \quad \text{for a.e. } x \in \Omega.$$

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