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Variational representations for N -cyclically monotone vector fields

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Abstract

Given a convex bounded domain Ω in \mathbb{R}^d and an integer $N \geq 2$, we associate to any *jointly N -monotone* $(N - 1)$ -tuple $(u_1, u_2, \dots, u_{N-1})$ of vector fields from Ω into \mathbb{R}^d , a Hamiltonian H on $\mathbb{R}^d \times \mathbb{R}^d \dots \times \mathbb{R}^d$, that is concave in the first variable, jointly convex in the last $(N - 1)$ variables such that for almost all $x \in \Omega$,

$$(u_1(x), u_2(x), \dots, u_{N-1}(x)) = \nabla_{2, \dots, N} H(x, x, \dots, x).$$

Moreover, H is N -sub-antisymmetric, meaning that $\sum_{i=0}^{N-1} H(\sigma^i(\mathbf{x})) \leq 0$ for all $\mathbf{x} = (x_1, \dots, x_N) \in \Omega^N$, σ

being the cyclic permutation on \mathbb{R}^d defined by $\sigma(x_1, x_2, \dots, x_N) = (x_2, x_3, \dots, x_N, x_1)$. Furthermore, H is N -antisymmetric in a sense to be defined below. This can be seen as an extension of a theorem of E. Krauss, which associates to any monotone operator, a concave-convex antisymmetric saddle function. We also give various variational characterizations of vector fields that are almost everywhere N -monotone, showing that they are dual to the class of measure preserving N -involutions on Ω .

1 Introduction

Given a domain Ω in \mathbb{R}^d , recall that a single-valued map u from Ω to \mathbb{R}^d is said to be *N -cyclically monotone* if for every cycle $x_1, \dots, x_N, x_{N+1} = x_1$ of points in Ω , one has

$$\sum_{i=1}^N \langle u(x_i), x_i - x_{i+1} \rangle \geq 0. \quad (1)$$

A classical theorem of Rockafellar [10] states that a map u from Ω to \mathbb{R}^d is *N -cyclically monotone for every $N \geq 2$* if and only if

$$u(x) \in \partial\phi(x) \text{ for all } x \in \Omega, \quad (2)$$

where $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a convex function. On the other hand, a result of E. Krauss [9] yields that u is a monotone map, i.e., a 2-cyclically monotone map, if and only if

$$u(x) \in \partial_2 H(x, x) \text{ for all } x \in \Omega, \quad (3)$$

where H is a concave-convex antisymmetric Hamiltonian on $\mathbb{R}^d \times \mathbb{R}^d$, and $\partial_2 H$ is the subdifferential of H as a convex function in the second variable.

In this paper, we extend the result of Krauss to the class of N -cyclically monotone vector fields, where $N \geq 3$. We shall give a representation for a family of $(N - 1)$ vector fields, which may or may not be individually N -cyclically monotone. Here is the needed concept.

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Definition 1 Let u_1, \dots, u_{N-1} be bounded vector fields from a domain $\Omega \subset \mathbb{R}^d$ into \mathbb{R}^d . We shall say that the $(N-1)$ -tuple $(u_1, u_2, \dots, u_{N-1})$ is jointly N -monotone, if for every cycle $x_1, \dots, x_{N+\ell}$ of points in Ω such that $x_{N+i} = x_i$ for $1 \leq i \leq \ell$, one has

$$\sum_{i=1}^N \sum_{\ell=1}^{N-1} \langle u_\ell(x_i), x_i - x_{\ell+i} \rangle \geq 0. \quad (4)$$

Examples of jointly N -monotone families of vector fields:

- It is clear that $(u, 0, 0, \dots, 0)$ is jointly N -monotone if and only if u is N -monotone.
- More generally, if each u_ℓ is N -monotone, then the family $(u_1, u_2, \dots, u_{N-1})$ is jointly N -monotone. Actually, one only needs that for $1 \leq \ell \leq N-1$, the vector field u_ℓ be (N, ℓ) -monotone, in the following sense: for every cycle $x_1, \dots, x_{N+\ell}$ of points in Ω such that $x_{N+i} = x_i$ for $1 \leq i \leq \ell$, we have

$$\sum_{i=1}^N \langle u_\ell(x_i), x_i - x_{\ell+i} \rangle \geq 0. \quad (5)$$

This notion is sometimes weaker than N -monotonicity since if ℓ divides N , then it suffices for u to be $\frac{N}{\ell}$ -monotone in order to be an (N, ℓ) -monotone vector field. For example, if u_1 and u_3 are 4-monotone operators and u_2 is 2-monotone, then the triplet (u_1, u_2, u_3) is jointly 4-monotone.

- Another example is when (u_1, u_2, u_3) are vector fields such that u_2 is 2-monotone and

$$\langle u_1(x) - u_3(y), x - y \rangle \geq 0 \text{ for every } x, y \in \mathbb{R}^d.$$

In this case, the triplet (u_1, u_2, u_3) is jointly 4-monotone. In particular, if u_1 and u_2 are both 2-monotone, then the triplet (u_1, u_2, u_1) is jointly 4-monotone.

- More generally, it is easy to show that (u, u, \dots, u) is jointly N -monotone if and only if u is 2-cyclically monotone.

In the sequel, we shall denote by σ the cyclic permutation on $\mathbb{R}^d \times \dots \times \mathbb{R}^d$, defined by

$$\sigma(x_1, x_2, \dots, x_{N-1}, x_N) = (x_2, x_3, \dots, x_N, x_1),$$

and consider the family of continuous N -antisymmetric Hamiltonians on Ω^N , that is

$$\mathcal{H}_N(\Omega) = \{H \in C(\Omega^N); \sum_{i=0}^{N-1} H(\sigma^i(x_1, \dots, x_N)) = 0\} \quad (6)$$

We shall say that H is N -sub-antisymmetric on Ω if

$$\sum_{i=0}^{N-1} H(\sigma^i(x_1, \dots, x_N)) \leq 0 \text{ on } \Omega^N \quad \text{and} \quad H(x, x, \dots, x) = 0 \text{ on the diagonal.} \quad (7)$$

We shall also say that a function F of two variables is N -cyclically sub-antisymmetric on Ω , if

$$F(x, x) = 0 \text{ and } \sum_{i=1}^N F(x_i, x_{i+1}) \leq 0 \text{ for all cyclic families } x_1, \dots, x_N, x_{N+1} = x_1 \text{ in } \Omega. \quad (8)$$

Note that if a function $H(x_1, \dots, x_N)$ is N -sub-antisymmetric and if it only depends on the first two variables, then the function $F(x_1, x_2) := H(x_1, x_2, \dots, x_N)$ is N -cyclically sub-antisymmetric.

We associate to any function H on Ω^N , the following functional on $\Omega \times (\mathbb{R}^d)^{N-1}$,

$$L_H(x, p_1, \dots, p_{N-1}) = \sup \left\{ \sum_{i=1}^{N-1} \langle p_i, y_i \rangle - H(x, y_1, \dots, y_{N-1}); y_i \in \Omega \right\}. \quad (9)$$

Note that if Ω is convex and if H is convex in the last $(N-1)$ variables, then L_H is nothing but the Legendre transform of \tilde{H} with respect to the last $(N-1)$ variables, where \tilde{H} is the extension of H over $(\mathbb{R}^d)^N$, defined as: $\tilde{H} = H$ on Ω^N and $\tilde{H} = +\infty$ outside of Ω^N . Since $H(x, \dots, x) = 0$ for any $H \in \mathcal{H}_N(\Omega)$, then for any such H , we have for $x \in \Omega$ and $p_1, \dots, p_{N-1} \in \mathbb{R}^d$,

$$L_H(x, p_1, \dots, p_{N-1}) \geq \sum_{i=1}^{N-1} \langle x, p_i \rangle. \quad (10)$$

To formulate variational principles for such vector fields, we shall consider the class of σ -invariant probability measures on Ω^N , which are those $\pi \in \mathcal{P}(\Omega^N)$ such that for all $h \in L^1(\Omega^N, d\pi)$, we have

$$\int_{\Omega^N} h(x_1, \dots, x_N) d\pi = \int_{\Omega^N} h(\sigma(x_1, \dots, x_N)) d\pi. \quad (11)$$

We denote

$$\mathcal{P}_{\text{sym}}(\Omega^N) = \{\pi \in \mathcal{P}(\Omega^N); \pi \text{ } \sigma\text{-invariant probability on } \Omega^N\}. \quad (12)$$

For a given probability measure μ on Ω , we also consider the class

$$\mathcal{P}_{\text{sym}}^\mu(\Omega^N) = \{\pi \in \mathcal{P}_{\text{sym}}(\Omega^N); \text{proj}_1 \pi = \mu\}, \quad (13)$$

i.e., the set of all $\pi \in \mathcal{P}_{\text{sym}}(\Omega^N)$ with a given first marginal μ , meaning that

$$\int_{\Omega^N} f(x_1) d\pi(x_1, \dots, x_N) = \int_{\Omega} f(x_1) d\mu(x_1) \text{ for every } f \in L^1(\Omega, \mu). \quad (14)$$

Consider now the set $\mathcal{S}(\Omega, \mu)$ of μ -measure preserving transformations on Ω , which can be identified with a closed subset of the sphere of $L^2(\Omega, \mathbb{R}^d)$. We shall also consider the subset of $\mathcal{S}(\Omega, \mu)$ consisting of N -involutions, that is

$$\mathcal{S}_N(\Omega, \mu) = \{S \in \mathcal{S}(\Omega, \mu); S^N = I \text{ } \mu \text{ a.e.}\}.$$

2 Monotone vector fields and N -antisymmetric Hamiltonians

In this section, we establish the following extension of a theorem of Krauss.

Theorem 2 *Let $N \geq 2$ be an integer, and consider u_1, \dots, u_{N-1} to be bounded vector fields from a convex domain $\Omega \subset \mathbb{R}^d$ into \mathbb{R}^d .*

1. *If the $(N-1)$ -tuple (u_1, \dots, u_{N-1}) is jointly N -monotone, then there exists an N -sub-antisymmetric Hamiltonian H that is concave in the first variable, convex in the other $(N-1)$ variables such that*

$$(u_1(x), \dots, u_{N-1}(x)) = \nabla_{2, \dots, N} H(x, x, \dots, x) \quad \text{for a.e. } x \in \Omega. \quad (15)$$

Moreover, H is N -antisymmetric in the following sense

$$H(x_1, x_2, \dots, x_N) + H_{2, \dots, N}(x_1, x_2, \dots, x_N) = 0, \quad (16)$$

where $H_{2, \dots, N}$ is the concavification of the function $K(\mathbf{x}) = \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x}))$ with respect to the last $(N-1)$ variables.

Furthermore, there exists a continuous N -antisymmetric Hamiltonian \bar{H} on Ω^N , such that

$$L_{\bar{H}}(x, u_1(x), u_2(x), \dots, u_{N-1}(x)) = \sum_{i=1}^{N-1} \langle u_i(x), x \rangle \text{ for all } x \in \Omega. \quad (17)$$

2. *Conversely, if (u_1, \dots, u_{N-1}) satisfy (15) for some N -sub-antisymmetric Hamiltonian H that is concave in the first variable, convex in the other variables, then the $(N-1)$ -tuple (u_1, \dots, u_{N-1}) is jointly N -monotone.*

Remark 3 Note that in the case $N = 2$, $K(\mathbf{x}) = H(x_2, x_1)$ is concave with respect to x_2 , hence $H_2(x_1, x_2) = H(x_2, x_1)$, and (16) becomes

$$H(x_1, x_2) + H(x_2, x_1) = 0,$$

thus H is antisymmetric, recovering well-known results [9], [4], [7], [8].

We start with the following lemma.

Lemma 4 Assume the $(N - 1)$ -tuple of bounded vector fields (u_1, \dots, u_{N-1}) on Ω is jointly N -monotone. Let $f(x_1, \dots, x_N) := \sum_{l=1}^{N-1} \langle u_l(x_1), x_1 - x_{l+1} \rangle$ and consider the function $\tilde{f}(x_1, \dots, x_N)$ to be the convexification of f with respect to the first variable, that is

$$\tilde{f}(x_1, x_2, \dots, x_N) = \inf \left\{ \sum_{k=1}^n \lambda_k f(x_1^k, x_2, \dots, x_N) : n \in \mathbb{N}, \lambda_k \geq 0, \sum_{k=1}^n \lambda_k = 1, \sum_{k=1}^n \lambda_k x_1^k = x_1 \right\}. \quad (18)$$

Then, \tilde{f} satisfies the following properties:

1. $f \geq \tilde{f}$ on Ω^N ;
2. \tilde{f} is convex in the first variable and concave with respect to the other variables;
3. $\tilde{f}(x, x, \dots, x) = 0$ for each $x \in \Omega$,
4. \tilde{f} satisfies

$$\sum_{i=0}^{N-1} \tilde{f}(\sigma^i(x_1, \dots, x_N)) \geq 0 \text{ on } \Omega^N. \quad (19)$$

Proof: Since the $(N - 1)$ -tuple (u_1, \dots, u_{N-1}) is jointly N -monotone, it is easy to see that the function

$$f(x_1, \dots, x_N) := \sum_{l=1}^{N-1} \langle u_l(x_1), x_1 - x_{l+1} \rangle$$

is linear in the last $(N - 1)$ variables, that $f(x, x, \dots, x) = 0$, and that

$$\sum_{i=0}^{N-1} f(\sigma^i(x_1, \dots, x_N)) \geq 0 \text{ on } \Omega^N. \quad (20)$$

It is also clear that $f \geq \tilde{f}$, that \tilde{f} is convex with respect to the first variable x_1 , and that it is concave with respect to the other variables x_2, \dots, x_N , since f itself is concave (actually linear) with respect to x_2, \dots, x_N . We now show that \tilde{f} satisfies (19).

For that, we fix x_1, x_2, \dots, x_N in Ω and consider $(x_1^k)_{k=1}^n$ in Ω , and $(\lambda_k)_k$ in \mathbb{R} such that $\lambda_k \geq 0$ such that $\sum_{k=1}^n \lambda_k = 1$ and $\sum_{k=1}^n \lambda_k x_1^k = x_1$. For each k , we have

$$f(x_1^k, x_2, \dots, x_N) + f(x_2, \dots, x_N, x_1^k) + \dots + f(x_N, x_1^k, x_2, \dots, x_{N-1}) \geq 0.$$

Multiplying by λ_k , summing over k , and using that f is linear in the last $(N - 1)$ -variables, we have

$$\sum_{k=1}^n \lambda_k f(x_1^k, x_2, \dots, x_N) + f(x_2, \dots, x_N, x_1) + \dots + f(x_N, x_1, x_2, \dots, x_{N-1}) \geq 0.$$

By taking the infimum, we obtain

$$\tilde{f}(x_1, x_2, \dots, x_N) + \sum_{i=1}^{N-1} f(\sigma^i(x_1, x_2, \dots, x_N)) \geq 0.$$

Let now $n \in \mathbb{N}$, $\lambda_k \geq 0$, $x_N^k \in \Omega$ be such that $\sum_{k=1}^n \lambda_k = 1$ and $\sum_{k=1}^n \lambda_k x_2^k = x_2$. We have for every $1 \leq k \leq n$,

$$\tilde{f}(x_1, x_2^k, x_3, \dots, x_N) + f(x_2^k, x_3, \dots, x_1) + \dots + f(x_N, x_1, x_2^k, x_3, \dots, x_{N-1}) \geq 0.$$

Multiplying by λ_k , summing over k and using that \tilde{f} is convex in the first variable and f is linear in the last $(N - 1)$ -variables, we obtain

$$\begin{aligned} & \tilde{f}(x_1, x_2, x_3, \dots, x_N) + \sum_{k=1}^n \lambda_k f(x_2^k, x_3, \dots, x_1) + \dots + f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \\ & \geq \sum_{k=1}^n \lambda_k \tilde{f}(x_1, x_2^k, x_3, \dots, x_N) + \sum_{k=1}^n \lambda_k f(x_2^k, x_3, \dots, x_1) + \dots + \sum_{k=1}^n \lambda_k f(x_N, x_1, x_2^k, x_3, \dots, x_{N-1}) \geq 0. \end{aligned}$$

By taking the infimum over all possible such choices, we get

$$\tilde{f}(x_1, x_2, x_3, \dots, x_N) + \tilde{f}(x_2, x_3, \dots, x_1) + \dots + f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \geq 0.$$

By repeating this procedure with x_3, \dots, x_{N-1} , we get

$$\sum_{i=0}^{N-2} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N)) + f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \geq 0.$$

Finally, since

$$f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \geq - \sum_{i=0}^{N-2} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N)).$$

and since \tilde{f} is concave in the last $(N - 1)$ variables, we have for fixed x_1, x_2, \dots, x_{N-1} , that the function

$$x_N \rightarrow - \sum_{i=0}^{N-2} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N))$$

is a convex minorant of $x_N \rightarrow f(x_N, x_1, x_2, x_3, \dots, x_{N-1})$. It follows that

$$f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \geq \tilde{f}(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \geq - \sum_{i=0}^{N-2} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N)),$$

which finally implies that $\sum_{i=0}^{N-1} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N)) \geq 0$.

This clearly implies that $\tilde{f}(x, x, \dots, x) \geq 0$ for any $x \in \Omega$. On the other hand, since $\tilde{f}(x, x, \dots, x) \leq f(x, x, \dots, x) = 0$, we get that $\tilde{f}(x, x, \dots, x) = 0$ for all $x \in \Omega$. \square

Proof of Theorem 2: Assume the $(N - 1)$ -tuple of vector fields (u_1, \dots, u_{N-1}) is jointly N -monotone on Ω , and consider the function $f(x_1, \dots, x_n) := \sum_{l=1}^{N-1} \langle u_l(x_1), x_1 - x_{l+1} \rangle$ as well as its convexification with respect to the first variable $\tilde{f}(x_1, \dots, x_n)$.

By Lemma 4, the function $\psi(x_1, \dots, x_n) := -\tilde{f}(x_1, \dots, x_n)$ satisfies the following properties

- (i) $x_1 \rightarrow \psi(x_1, \dots, x_n)$ is concave;
- (ii) $(x_2, x_3, \dots, x_n) \rightarrow \psi(x_1, \dots, x_n)$ is convex;
- (iii) $\psi(x_1, \dots, x_n) \geq -f(x_1, \dots, x_n) = \sum_{l=1}^{N-1} \langle u_l(x_1), x_{l+1} - x_1 \rangle$;
- (iv) ψ is N -sub-antisymmetric.

Consider now the family \mathcal{H} of functions $H : \Omega^N \rightarrow \mathbb{R}$ such that

1. $H(x_1, x_2, \dots, x_N) \geq \sum_{l=1}^{N-1} \langle u_l(x_1), x_{l+1} - x_1 \rangle$ for every N -tuple (x_1, \dots, x_N) in Ω^N ;
2. H is concave in the first variable;
3. H is jointly convex in the last $(N - 1)$ variables;
4. H is N -sub-antisymmetric.

Note that $\mathcal{H} \neq \emptyset$ since ψ belongs to \mathcal{H} . Moreover, by N -subsymmetry, any $H \in \mathcal{H}$ satisfies for all $\mathbf{x} = (x_1, \dots, x_N) \in \Omega^N$,

$$H(\mathbf{x}) \leq - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x})) \leq - \sum_{i=1}^{N-1} \psi(\sigma^i(\mathbf{x})). \quad (21)$$

This also yields that

$$\sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_{\ell+1} - x_1 \rangle \leq H(\mathbf{x}) \leq - \sum_{i=2}^N \sum_{\ell=1}^{N-1} \langle u_\ell(x_i), x_i - x_{i+\ell} \rangle, \quad (22)$$

where we denote $x_{i+N} := x_i$ for $i = 1, \dots, \ell$. This yields that $H(x, x, \dots, x) = 0$ for every $H \in \mathcal{H}$ and any $x \in \Omega$.

On the other hand, it is easy to see that every directed family $(H_i)_i$ in \mathcal{H} has a supremum $H_\infty \in \mathcal{H}$, meaning that \mathcal{H} is a Zorn family, and therefore it has a maximal element H .

Consider now the function

$$\bar{H}(\mathbf{x}) = \frac{(N-1)H(\mathbf{x}) - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x}))}{N},$$

and note that

(i) \bar{H} is N -antisymmetric, since

$$\bar{H}(\mathbf{x}) = \frac{1}{N} \sum_{i=1}^{N-1} [H(\mathbf{x}) - H(\sigma^i(\mathbf{x}))],$$

and each $K_i(\mathbf{x}) := H(\mathbf{x}) - H(\sigma^i(\mathbf{x}))$ is N -antisymmetric.

(ii) $\bar{H} \geq H$ on Ω^N , since

$$N[\bar{H}(\mathbf{x}) - H(\mathbf{x})] = - \sum_{i=0}^{N-1} H(\sigma^i(\mathbf{x})) \geq 0,$$

because H itself is N -sub-antisymmetric.

The maximality of H would have implied that $H = \bar{H}$ is N -antisymmetric if only \bar{H} was jointly convex in the last $(N-1)$ -variables, but since this is not necessarily the case, we consider for $\mathbf{x} = (x_1, x_2, \dots, x_N)$, the function

$$K(x_1, x_2, \dots, x_N) = K(\mathbf{x}) := - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x})),$$

which is already concave in the first variable x_1 . Its convexification in the last $(N-1)$ -variables, that is

$$K^{2, \dots, N}(\mathbf{x}) = \inf \left\{ \sum_{i=1}^n \lambda_i K(x_1, x_2^i, \dots, x_N^i); \lambda_i \geq 0, \sum_{i=1}^n \lambda_i (x_2^i, \dots, x_N^i, 1) = (x_2, \dots, x_N, 1) \right\},$$

is still concave in the first variable, but is now convex in the last $(N-1)$ variables. Moreover,

$$H \leq K^{2, \dots, N} \leq K = - \sum_{i=1}^{N-1} H \circ \sigma^i. \quad (23)$$

Indeed, $K^{2, \dots, N} \leq K$ from the definition of $K^{2, \dots, N}$, while $H \leq K^{2, \dots, N}$ because $H \leq K$ and H is already convex in the last $(N-1)$ -variables. It follows that

$$H \leq \frac{(N-1)H + K^{2, \dots, N}}{N} \leq \frac{(N-1)H + K}{N} = \frac{(N-1)H - \sum_{i=1}^{N-1} H \circ \sigma^i}{N} = \bar{H}.$$

The function $H' = \frac{(N-1)H + K^{2 \cdots N}}{N}$ belongs to the family \mathcal{H} and therefore $H = H'$ by the maximality of H .

This finally yields that \bar{H} is N -sub-antisymmetric, that $H(x, x, x) = 0$ for all $x \in \Omega$ and that

$$H(\mathbf{x}) + H_{2, \dots, N}(\mathbf{x}) = 0 \text{ for every } \mathbf{x} \in \Omega^N,$$

where $H_{2, \dots, N} = -K^{2 \cdots N}$, which for a fixed x_1 , is nothing but the concavification of $(x_2, \dots, x_N) \rightarrow \sum_{i=1}^{N-1} H(\sigma^i(x_1, x_2, \dots, x_N))$.

Note now that since for any x_1, \dots, x_N in Ω ,

$$H(x_1, x_2, \dots, x_N) \geq \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_{\ell+1} - x_1 \rangle, \quad (24)$$

and

$$H(x_1, x_1, \dots, x_1) = 0, \quad (25)$$

we have

$$H(x_1, x_2, \dots, x_N) - H(x_1, \dots, x_1) \geq \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_{\ell+1} - x_1 \rangle. \quad (26)$$

Since H is convex in the last $(N-1)$ variables, this means that for all $x \in \Omega$, we have

$$(u_1(x), u_2(x), \dots, u_{N-1}(x)) \in \partial_{2, \dots, N} H(x, x, \dots, x). \quad (27)$$

as claimed in (15). Note that this also yield that

$$L_H(x, u_1(x), \dots, u_{N-1}(x)) + H(x, x, \dots, x) = \sum_{\ell=1}^{N-1} \langle u_\ell(x), x \rangle \text{ for all } x \in \Omega.$$

In other words, $L_H(x, u_1(x), \dots, u_{N-1}(x)) = \sum_{\ell=1}^{N-1} \int_\Omega \langle u_\ell(x), x \rangle$ for all $x \in \Omega$. As above, consider

$$\bar{H}(\mathbf{x}) = \frac{(N-1)H(\mathbf{x}) - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x}))}{N}.$$

We have that $\bar{H} \in \mathcal{H}_N(\Omega)$ and $\bar{H} \geq H$, and therefore $L_{\bar{H}} \leq L_H$. On the other hand, we have for all $x \in \Omega$,

$$L_{\bar{H}}(x, u_1(x), \dots, u_{N-1}(x)) = L_{\bar{H}}(x, u_1(x), \dots, u_{N-1}(x)) + \bar{H}(x, x, \dots, x) \geq \sum_{\ell=1}^{N-1} \langle u_\ell(x), x \rangle.$$

To prove (17), we use the appendix in [6] to deduce that for $i = 2, \dots, N$, the gradients $\nabla_i H(x, x, \dots, x)$ actually exist for a.e. x in Ω .

The converse is straightforward since if (27) holds, then (26) does, and since we also have (25), then the property that (u_1, \dots, u_{N-1}) is jointly N -monotone follows from (24) and the sub-antisymmetry of H . \square

In the case of a single N -monotone vector field, we can obviously apply the above theorem to the $(N-1)$ -tuple $(u, 0, \dots, 0)$ which is then N -monotone to find a N -sub-antisymmetric Hamiltonian H , which is concave in the first variable, convex in the last $(N-1)$ variables such that

$$(-u(x), u(x), 0, \dots, 0) = \nabla H(x, x, \dots, x) \text{ for a.e. } x \in \Omega. \quad (28)$$

However, in this case we can restrict ourselves to N -cyclically sub-antisymmetric functions of two variables and establish the following extension of the Theorem of Krauss.

Theorem 5 *If u is N -cyclically monotone on Ω , then there exists a concave-convex function of two variables F that is N -cyclically sub-antisymmetric, such that*

$$(-u(x), u(x)) \in \partial F(x, x) \text{ for all } x \in \Omega, \quad (29)$$

where ∂H is the sub-differential of H as a concave-convex function [11]. Moreover,

$$u(x) = \nabla_2 F(x, x) \text{ for a.e. } x \in \Omega. \quad (30)$$

Proof: Let $f(x, y) = \langle u(x), x - y \rangle$ and let $f^1(x, y)$ be its convexification in x for fixed y , that is

$$f^1(x, y) = \inf \left\{ \sum_{k=1}^n \lambda_k f(x_k, y) : \lambda_k \geq 0, \sum_{k=1}^n \lambda_k = 1, \sum_{k=1}^n \lambda_k x_k = x \right\}. \quad (31)$$

Since $f(x, x) = 0$, f is linear in y , and $\sum_{i=1}^N f(x_i, x_{i+1}) \geq 0$ for any cyclic family $x_1, \dots, x_N, x_{N+1} = x_1$ in Ω , it is easy to show that $f \geq f^1$ on Ω , f^1 is convex in the first variable and concave with respect to the second, $f^1(x, x) = 0$ for each $x \in \Omega$, and that f^1 is N -cyclically supersymmetric in the sense that for any cyclic family $x_1, \dots, x_N, x_{N+1} = x_1$ in Ω , we have $\sum_{i=1}^N f^1(x_i, x_{i+1}) \geq 0$.

Consider now $F(x, y) = -f^1(x, y)$ and note that $x \rightarrow F(x, y)$ is concave, $y \rightarrow F(x, y)$ is convex, $F(x, y) \geq -f(x, y) = \langle u(x), y - x \rangle$ and F is N -cyclically sub-antisymmetric. By the antisymmetry, we have

$$\langle u(x_1), x_2 - x_1 \rangle \leq F(x_1, x_2) \leq \langle u(x_2), x_2 - x_1 \rangle, \quad (32)$$

which yields that $(-u(x), u(x)) \in \partial F(x, x)$ for all $x \in \Omega$.

Since F is anti-symmetric and concave-convex, the possibly multivalued map $x \rightarrow \partial_2 F(x, x)$ is monotone on Ω , and therefore single-valued and differentiable almost everywhere [10]. This completes the proof.

Remark 6 Note that we cannot expect to have a function F such that $\sum_{i=1}^N F(x_i, x_{i+1}) = 0$ for all cyclic families $x_1, \dots, x_N, x_{N+1} = x_1$ in Ω . Actually, we believe that the only function satisfying such an N -antisymmetry for $N \geq 3$ must be of the form $F(x, y) = f(x) - f(y)$. This is the reason why one needs to consider functions of N -variables in order to get N -antisymmetry. In other words, the function defined by

$$H(x_1, x_2, \dots, x_N) := \frac{(N-1)F(x_1, x_2) - \sum_{i=2}^{N-1} F(x_i, x_{i+1})}{N}, \quad (33)$$

is N -antisymmetric in the sense of (6) and $H(x_1, x_2, \dots, x_N) \geq F(x_1, x_2)$ for all (x_1, x_2, \dots, x_N) in Ω^N .

3 Variational characterization of monotone vector fields

In order to simplify the exposition, we shall always assume in the sequel that $d\mu$ is Lebesgue measure dx normalized to be a probability on Ω . We shall also assume that Ω is convex and that its boundary has measure zero.

Theorem 7 Let $u_1, \dots, u_{N-1} : \Omega \rightarrow \mathbb{R}^d$ be bounded measurable vector fields. The following properties are then equivalent:

1. The $(N-1)$ -tuple (u_1, \dots, u_{N-1}) is jointly N -monotone a.e., that is there exists a measure zero set Ω_0 such that (u_1, \dots, u_{N-1}) is jointly N -monotone on $\Omega \setminus \Omega_0$.
2. The infimum of the following Monge-Kantorovich problem

$$\inf \left\{ \int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_1 - x_{\ell+1} \rangle d\pi(x_1, x_2, \dots, x_N); \pi \in \mathcal{P}_{\text{sym}}^\mu(\Omega^N) \right\} \quad (34)$$

is equal to zero, and is therefore attained by the push-forward of μ by the map $x \rightarrow (x, x, \dots, x)$.

3. (u_1, \dots, u_{N-1}) is in the polar of $\mathcal{S}_N(\Omega, \mu)$ in the following sense,

$$\inf \left\{ \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_\ell(x), x - S^\ell x \rangle d\mu; S \in \mathcal{S}_N(\Omega, \mu) \right\} = 0. \quad (35)$$

4. The following holds:

$$\inf \left\{ \int_{\Omega} \sum_{\ell=1}^{N-1} |u_\ell(x) - S^\ell x|^2 d\mu; S \in \mathcal{S}_N(\Omega, \mu) \right\} = \sum_{\ell=1}^{N-1} \int_{\Omega} |u_\ell(x) - x|^2 d\mu. \quad (36)$$

5. There exists a N -sub-antisymmetric Hamiltonian H which is concave in the first variable, convex in the last $(N - 1)$ variables such that

$$(u_1(x), \dots, u_{N-1}(x)) = \nabla_{2, \dots, N} H(x, x, \dots, x) \quad \text{for a.e. } x \in \Omega. \quad (37)$$

Moreover, H is N -symmetric in the sense of (16).

6. The following duality holds:

$$\inf \left\{ \int_{\Omega} L_H(x, u_1(x), \dots, u_{N-1}(x)) d\mu; H \in \mathcal{H}_N(\Omega) \right\} = \sup \left\{ \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), S^{\ell} x \rangle d\mu; S \in \mathcal{S}_N(\Omega, \mu) \right\}$$

and the latter is attained at the identity map.

We start with the following lemma, which identifies those probabilities in $\mathcal{P}_{\text{sym}}^{\mu}(\Omega^N)$ that are carried by graphs of functions from Ω to Ω^N .

Lemma 8 *Let $S : \Omega \rightarrow \Omega$ be a μ -measurable map, then the following properties are equivalent:*

1. The image of μ by the map $x \rightarrow (x, Sx, \dots, S^{N-1}x)$ belongs to $\mathcal{P}_{\text{sym}}^{\mu}(\Omega^N)$.
2. S is μ -measure preserving and $S^N(x) = x$ μ -a.e.
3. For any bounded Borel measurable N -antisymmetric H on Ω^N , we have $\int_{\Omega} H(x, Sx, \dots, S^{N-1}x) d\mu = 0$.

Proof. It is clear that 1) implies 3) since $\int_{\Omega^N} H(\mathbf{x}) d\pi(\mathbf{x}) = 0$ for any N -antisymmetric Hamiltonian H and any $\pi \in \mathcal{P}_{\text{sym}}^{\mu}(\Omega^N)$.

That 2) implies 1) is also straightforward since if π is the push-forward of μ by a map of the form $x \rightarrow (x, Sx, \dots, S^{N-1}x)$, where S is a μ -measure preserving S with $S^N x = x$ μ a.e. on Ω , then for all $h \in L^1(\Omega^N, d\pi)$, we have

$$\begin{aligned} \int_{\Omega^N} h(x_1, \dots, x_N) d\pi &= \int_{\Omega^N} h(x, Sx, \dots, S^{N-1}x) d\mu(x) = \int_{\Omega^N} h(Sx, S^2x, \dots, S^{N-1}x, S^N x) d\mu(x) \\ &= \int_{\Omega^N} h(Sx, S^2x, \dots, S^{N-1}x, x) d\mu(x) = \int_{\Omega^N} h(\sigma(x_1, \dots, x_N)) d\pi. \end{aligned}$$

We now prove that 2) and 3) are equivalent. Assuming first that S is μ -measure preserving such that $S^N = I$ μ a.e., then for every Borel bounded N -antisymmetric H , we have

$$\begin{aligned} \int_{\Omega} H(x, Sx, S^2x, \dots, S^{N-1}x) d\mu &= \int_{\Omega} H(Sx, S^2x, \dots, S^{N-1}x, x) d\mu \\ &= \dots = \int_{\Omega} H(S^{N-1}x, x, Sx, \dots, S^{N-2}x) d\mu. \end{aligned}$$

Since H is N -antisymmetric, we can see that

$$H(x, Sx, \dots, S^{N-1}x) + H(Sx, S^2x, \dots, S^{N-1}x, x) + \dots H(S^{N-1}x, x, Sx, \dots, S^{N-2}x) = 0.$$

It follows that $N \int_{\Omega} H(x, Sx, S^2x, \dots, S^{N-1}x) d\mu = 0$.

For the reverse implication, assume $\int_{\Omega} H(x, Sx, S^2x, \dots, S^{N-1}x) d\mu = 0$ for every N -antisymmetric Hamiltonian H . By testing this identity with the Hamiltonians

$$H(x_1, x_2, \dots, x_N) = f(x_1) - f(x_i),$$

where f is any continuous function on Ω , one gets that S is μ -measure preserving. Now take the Hamiltonian

$$H(x_1, x_2, \dots, x_N) = |x_1 - Sx_N| - |Sx_1 - x_2| - |x_2 - Sx_1| + |Sx_2 - x_3|.$$

Note that $H \in \mathcal{H}_N(\Omega)$ since it is of the form $H(x_1, \dots, x_N) = f(x_1, x_2, x_N) - f(x_2, x_3, x_1)$. Now test the above identity with such an H to obtain

$$0 = \int_{\Omega} H(x, Sx, S^2x, \dots, S^{N-1}x) d\mu = \int_{\Omega} |x - SS^{N-1}x| d\mu.$$

It follows that $S^N = I$ μ a.e. on ω , and we are done. \square

Proof of Theorem 7: To show that (1) implies (2), it suffices to notice that if π is a σ -invariant probability measure on Ω^N such that $\text{proj}_1 \pi = \mu$, then

$$\begin{aligned} \int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x_1), x_1 - x_{\ell+1} \rangle d\pi(x_1, \dots, x_N) &= \frac{1}{N} \sum_{i=1}^N \int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x_i), x_i - x_{i+\ell} \rangle d\pi(x_1, \dots, x_N) \\ &= \frac{1}{N} \int_{\Omega^N} \left(\sum_{i=1}^N \sum_{\ell=1}^{N-1} \langle u_{\ell}(x_i), x_i - x_{i+\ell} \rangle \right) d\pi(x_1, \dots, x_N) \\ &\geq 0, \end{aligned}$$

since (u_1, \dots, u_{N-1}) is jointly N -monotone. On the other hand, if π is the σ -invariant measure obtained by taking the image of $\mu := dx$ by $x \rightarrow (x, \dots, x)$, then

$$\int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x_1), x_1 - x_{\ell+1} \rangle d\pi(x_1, \dots, x_N) = 0.$$

To show that (2) implies (3), let S be a μ -measure preserving transformation on Ω such that $S^N = I$ μ a.e. on Ω . Then the image π_S of μ by the map

$$x \rightarrow (x, Sx, S^2x, \dots, S^{N-1}x)$$

is σ -invariant, hence

$$\int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x_1), x_1 - x_{\ell+1} \rangle d\pi_S(x_1, \dots, x_N) = \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), x - S^{\ell}x \rangle d\mu \geq 0.$$

By taking $S = I$, we get that the infimum is necessarily zero.

The equivalence of (3) and (4) follows immediately from developing the square.

We now show that (3) implies (1). For that take N points x_1, x_2, \dots, x_N in Ω , and let $R > 0$ be such that $B(x_i, R) \subset \Omega$. Consider the transformation

$$S_R(x) = \begin{cases} x - x_1 + x_2 & \text{for } x \in B(x_1, R) \\ x - x_2 + x_3 & \text{for } x \in B(x_2, R) \\ \dots & \\ x - x_N + x_1 & \text{for } x \in B(x_N, R) \\ x & \text{otherwise} \end{cases}$$

It is easy to see that S_R is a measure preserving transformation and that $S_R^N = Id$. We then have

$$0 \leq \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), x - S_R^{\ell}x \rangle d\mu \leq \sum_{i=1}^N \int_{B(x_i, R)} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), x_i - x_{\ell+i} \rangle d\mu.$$

Letting $R \rightarrow 0$, we get from Lebesgue's density theorem, that

$$\frac{1}{|B(x_i, R)|} \int_{B(x_i, R)} \langle u_{\ell}(x), x_i - x_{\ell+i} \rangle d\mu \rightarrow \langle u_{\ell}(x_i), x_i - x_{\ell+i} \rangle,$$

from which follows that (u_1, \dots, u_{N-1}) are jointly N -monotone a.e. on Ω .

The fact that (1) is equivalent to (5) follows immediately from Theorem 2. To prove that 5) implies 6) note that for all $p_i \in \mathbb{R}^d, x \in \Omega, y_i \in \Omega, i = 1, \dots, N-1$,

$$L_H(x, p_1, \dots, p_{N-1}) + H(x, y_1, \dots, y_{N-1}) \geq \sum_{i=1}^{N-1} \langle p_i, y_i \rangle,$$

which yields that for any $S \in \mathcal{S}_N(\Omega, \mu)$,

$$\int_{\Omega} [L_H(x, u_1(x), \dots, u_{N-1}(x)) d\mu + H(x, Sx, \dots, S^{N-1}x)] d\mu \geq \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), S^{\ell}x \rangle d\mu.$$

If $H \in \mathcal{H}_N(\Omega)$ and $S \in \mathcal{S}_N(\Omega, \mu)$, we then have $\int_{\Omega} H(x, Sx, \dots, S^{N-1}x) d\mu = 0$, and therefore

$$\int_{\Omega} L_H(x, u_1(x), \dots, u_{N-1}(x)) d\mu \geq \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), S^{\ell}x \rangle d\mu.$$

If now H is the N -sub-antisymmetric Hamiltonian obtained by 5), which is concave in the first variable, convex in the last $(N-1)$ variables, then

$$L_H(x, u_1(x), \dots, u_{N-1}(x)) + H(x, x, \dots, x) = \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), x \rangle \quad \text{for all } x \in \Omega \setminus \Omega_0,$$

and therefore $\int_{\Omega} L_H(x, u_1(x), \dots, u_{N-1}(x)) d\mu = \sum_{\ell=1}^{N-1} \int_{\Omega} \langle u_{\ell}(x), x \rangle d\mu$.

Consider now

$$\bar{H}(\mathbf{x}) = \frac{(N-1)H(\mathbf{x}) - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x}))}{N}.$$

As before, we have that $\bar{H} \in \mathcal{H}_N(\Omega)$ and $\bar{H} \geq H$. Since $L_{\bar{H}} \leq L_H$, we have that $\int_{\Omega} L_{\bar{H}}(x, u_1(x), \dots, u_{N-1}(x)) d\mu = \sum_{\ell=1}^{N-1} \int_{\Omega} \langle u_{\ell}(x), x \rangle d\mu$ and (6) is proved.

Finally, note that (6) readily implies (3), which means that (u_1, \dots, u_{N-1}) is then jointly N -monotone. \square

We now consider again the case of a single N -cyclically monotone vector field.

Corollary 9 *Let $u : \Omega \rightarrow \mathbb{R}^d$ be a bounded measurable vector field. The following properties are then equivalent:*

1. u is N -cyclically monotone a.e., that is there exists a measure zero set Ω_0 such that u is N -cyclically monotone on $\Omega \setminus \Omega_0$.
2. The infimum of the following Monge-Kantorovich problem

$$\inf \left\{ \int_{\Omega^N} \langle u(x_1), x_1 - x_2 \rangle d\pi(\mathbf{x}); \pi \in \mathcal{P}_{\text{sym}}^{\mu}(\Omega^N) \right\} \quad (38)$$

is equal to zero, and is therefore attained by the push-forward of μ by the map $x \rightarrow (x, x, \dots, x)$.

3. The vector field u is in the polar of $\mathcal{S}_N(\Omega, \mu)$, that is

$$\inf \left\{ \int_{\Omega} \langle u(x), x - Sx \rangle d\mu; S \in \mathcal{S}_N(\Omega, \mu) \right\} = 0. \quad (39)$$

4. The projection of u on $\mathcal{S}_N(\Omega, \mu)$ is the identity map, that is

$$\inf \left\{ \int_{\Omega} |u(x) - Sx|^2 d\mu; S \in \mathcal{S}_N(\Omega, \mu) \right\} = \int_{\Omega} |u(x) - x|^2 d\mu. \quad (40)$$

5. There exists a N -cyclically sub-antisymmetric function H of two variables, which is concave in the first variable, convex in the second variable such that

$$u(x) = \nabla_2 H(x, x) \quad \text{for a.e. } x \in \Omega. \quad (41)$$

6. The following duality holds:

$$\inf \left\{ \int_{\Omega} L_H(x, u(x), 0, \dots, 0) d\mu; H \in \mathcal{H}_N(\Omega) \right\} = \sup \left\{ \int_{\Omega} \langle u(x), Sx \rangle d\mu; S \in \mathcal{S}_N(\Omega, \mu) \right\}$$

and the latter is attained at the identity map.

Proof: This is an immediate application of Theorem 7 applied to the $(N-1)$ -tuple vector fields $(u, 0, \dots, 0)$, which is clearly jointly N -monotone on $\Omega \setminus \Omega_0$, whenever u is N -monotone on $\Omega \setminus \Omega_0$.

Remark 10 Note that the sets of μ -measure preserving N -involutions $(\mathcal{S}_N(\Omega, \mu))_N$ do not form a nested family, that is $\mathcal{S}_N(\Omega, \mu)$ is not necessarily included in $\mathcal{S}_M(\Omega, \mu)$, whenever $N \leq M$, unless of course M is a multiple of N . On the other hand, the above theorem shows that their polar sets, i.e.,

$$\mathcal{S}_N(\Omega, \mu)^0 = \{u \in L^2(\Omega, \mathbb{R}^d); \int_{\Omega} \langle u(x), x - Sx \rangle d\mu \geq 0 \text{ for all } S \in \mathcal{S}_N(\Omega, \mu)\},$$

which coincide with the N -cyclically monotone maps, satisfy

$$\mathcal{S}_{N+1}(\Omega, \mu)^0 \subset \mathcal{S}_N(\Omega, \mu)^0,$$

for every $N \geq 1$. This can also be seen directly. Indeed, it is clear that a 2-involution is a 4-involution but not necessarily a 3-involution. On the other hand, assume that u is 3-cyclically monotone operator, then for any transformation $S : \Omega \rightarrow \Omega$, we have

$$\int_{\Omega} \langle u(x), x - Sx \rangle d\mu + \int_{\Omega} \langle u(Sx), Sx - S^2x \rangle d\mu + \int_{\Omega} \langle u(S^2x), S^2x - x \rangle d\mu \geq 0.$$

If now S is measure preserving, we have

$$\int_{\Omega} \langle u(x), x - Sx \rangle d\mu + \int_{\Omega} \langle u(x), x - Sx \rangle d\mu + \int_{\Omega} \langle u(S^2x), S^2x - x \rangle d\mu \geq 0,$$

and if $S^2 = I$, then $\int_{\Omega} \langle u(x), x - Sx \rangle d\mu \geq 0$, which means that $u \in \mathcal{S}_2(\Omega, \mu)^0$. Similarly, one can show that any $(N+1)$ -cyclically monotone operator belongs to $\mathcal{S}_N(\Omega, \mu)^0$. In other words, $\mathcal{S}_{N+1}(\Omega, \mu)^0 \subset \mathcal{S}_N(\Omega, \mu)^0$ for all $N \geq 2$. Note that $\mathcal{S}_1(\Omega, \mu)^0 = \{I\}^0 = L^2(\Omega, \mathbb{R}^d)$, while

$$\mathcal{S}(\Omega, \mu)^0 = \cap_N \mathcal{S}_N(\Omega, \mu)^0 = \{u \in L^2(\Omega, \mathbb{R}^d), u = \nabla \phi \text{ for some convex function } \phi \text{ in } W^{1,2}(\mathbb{R}^d)\},$$

in view of classical results of Rockafellar [11] and Brenier [1].

Remark 11 In a forthcoming paper [6], the above result is extended to give a similar decomposition for any family of bounded measurable vector fields u_1, u_2, \dots, u_{N-1} on Ω . It is shown there that there exists a measure preserving N -involution S on Ω and an N -antisymmetric Hamiltonian H on Ω^N such that for $i = 1, \dots, N-1$, we have

$$u_i(x) = \nabla_{i+1} H(x, Sx, S^2x, \dots, S^{N-1}x) \quad \text{for a.e. } x \in \Omega.$$

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