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François Le Grand, Xavier Ragot. Managing Inequality over Business Cycles: Optimal Policies with Heterogeneous Agents and Aggregate Shocks. 2020. hal-03476095

HAL Id: hal-03476095

https://hal-sciencespo.archives-ouvertes.fr/hal-03476095

Preprint submitted on 12 Dec 2021

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Discussion Paper

MANAGING INEQUALITY OVER BUSINESS CYCLES: OPTIMAL POLICIES WITH HETEROGENEOUS AGENTS AND AGGREGATE SHOCKS

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SCIENCES PO ECONOMICS DISCUSSION PAPER

No. 2020-10

Managing Inequality over Business Cycles: Optimal Policies with Heterogeneous Agents and Aggregate Shocks*

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June 12, 2020

Abstract

We present a truncation theory of idiosyncratic histories for heterogeneous agent models. This method allows us to derive optimal Ramsey policies in heterogeneous agent models with aggregate shocks, in general frameworks. We use this method to characterize the optimal level of unemployment insurance over the business cycle in a production economy, with occasionally binding credit constraints.

Keywords: Incomplete markets, optimal policies, heterogeneous agent models.
JEL codes: E21, E44, D91, D31.

*A previous version of the paper was circulated as "Optimal fiscal policy with heterogeneous agents and aggregate shocks". We thank Kurt Mitman for having shared his code with us and Wouter Den Haan and Albert Marcet for their useful suggestions. We also thank Edouard Challe, Varadarajan Chari, Jonathan Heathcote, Christos Koulovatianos, Dirk Krueger, Felix Kubler, Etienne Lehmann, Alaïs Martin-Baillon, Ben Moll, Fabrizio Perri, Elisabeth Pröhl, Morten Ravn, Michael Reiter, Stephanie Schmitt-Grohe, Vincent Sterck, Martin Uribe, Gianluca Violante, Pierre-Olivier Weill, Sevin Yeltekin, and the seminar participants at the Federal Reserve Bank of Minneapolis, University of Zurich, University of Luxembourg, University of Cambridge, New York University, National Bank of Belgium, and European University Institute, at the "Macroeconomics of aggregate fluctuations and stabilization policy" conference, the T2M 2019, EEA 2018, SED 2019, and NBER SI 2017 MPF conference at the RED Conference in Barcelona. We thank Sumudu Kankamange and Thomas Weitzenblum for their excellent research assistance. The research leading to these results received funding from the European Community's Seventh Framework Programme (FP7/2007-2013) under grant agreement Integrated Macro-Financial Modeling for Robust Policy Design (MACFINROBODS, grant no. 612796).

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1 Introduction

Incomplete insurance market economies provide a useful framework for examining many relevant aspects of inequalities and individual risks. In these models, infinitely-lived agents face incomplete insurance markets and borrowing limits that prevent them from perfectly hedging their idiosyncratic risk, in line with the Bewley-Huggett-Aiyagari literature (Bewley, 1983; Imrohoroğlu, 1989; Huggett, 1993; Aiyagari, 1994; Krusell and Smith, 1998). These frameworks are now widely used, since they fill a gap between micro- and macroeconomics, and enable the inclusion of aggregate shocks and a number of additional frictions on both the goods and labor markets. However, little is known about optimal policies in these environments due to the difficulties generated by the large and time-varying heterogeneity across agents. This is unfortunate, since a vast literature suggests that the interaction between wealth heterogeneity and capital accumulation has first-order implications for the design of optimal policies. An important example is the optimal design of time-varying unemployment benefits in an economy with fluctuating unemployment risk, which has not yet been studied in the general case due to the difficulties generated by the variations in precautionary savings over the business cycle.

This paper presents a truncation theory that can be used to derive optimal policies in incomplete insurance market economies with aggregate shocks. In standard incomplete insurance market economies, agents differ according to the full history of their idiosyncratic risk realizations. Huggett (1993) and Aiyagari (1994), using the results of Hopenhayn and Prescott (1992), have shown that economies without aggregate risk have a recursive structure when the distribution of wealth is introduced as a state variable. Unfortunately, the distribution of wealth has an infinite number of possible values, which is at the root of many difficulties. Our main idea is to go back to the sequential formulation of incomplete-market models to construct a consistent finite state-space representation. More precisely, we build an environment where agents' heterogeneity depends only on a finite but arbitrarily large number, denoted by N, of consecutive past realizations of idiosyncratic risk. Agents having the same idiosyncratic risk history for the previous N periods choose the same consumption and wealth levels. As such, our model features a simple structure with a finite number of heterogeneous agents.

In standard incomplete-market economies, there is a distribution of agents with the same truncated idiosyncratic history of any length N, due to their idiosyncratic history $N + 1$ periods ago (and before). Hence, the outcome of our truncated model, which assigns a unique wealth level to a given truncated history, is a simplification, which reduces the amount of heterogeneity. However, we show that this missing heterogeneity (within each truncated history) can be captured in our truncated model by some additional preference parameters, that we call preference shifters. These preference shifters can easily be computed using the steady-state distribution of the underlying full-fledged Bewley model and are useful for the quantitative accuracy of the model.

Our findings regarding our truncation method are twofold. First, we prove convergence results. The allocations of the truncated model, such as the distribution of wealth, converge to those of the Bewley model when the truncation length becomes increasingly long. We also show that the convergence result is maintained in the presence of aggregate shocks that are not too large – as long as the solution is computed with perturbation method. Second, in terms of numerical performance, the truncation method delivers dynamics similar to those implied by other numerical techniques, in particular those of Reiter (2009) and Boppart et al. (2018). Several accuracy tests furthermore confirm the quantitative relevance of the method.[1]

Second, the advantage of this truncation theory is that the tools developed in dynamic contracts, sometimes referred to as the Lagrangian approach and developed by Marcet and Marimon (2019), can be used to solve the Ramsey problem with an arbitrary set of instruments for the planner. We show how the Lagrangian approach must be adapted to deal with our model and occasionally-binding credit constraints, where some regularity conditions may not hold. It is then possible to derive optimal policies with heterogeneous agents and aggregate shocks. More precisely, we use our truncation theory to characterize optimal unemployment benefits over the business cycle in an economy where agents face both productivity risk and time-varying employment risk, as in Krueger et al. (2018). The economy is hit by aggregate shocks that affect technology and labor market transitions. Agents choose their labor supply when working, consume, save, and face incomplete markets for idiosyncratic risk and credit constraints. In this economy, a planner chooses the level of unemployment benefits in each period, which must be fully financed by a distorting labor tax. Although the economic trade-off is the standard trade-off between insurance and efficiency, this problem is very hard to solve in a general equilibrium setting. The level of unemployment benefits directly affects agents' welfare as well as their saving decisions and the dynamics of interest rates and wages. We find that the replacement rate is countercyclical, increasing the transfer to unemployed agents in recessions. This policy reduces the volatility of the total income of unemployed agents, what is welfare improving.

Literature review. Our paper contributes to the recent literature on optimal policies in heterogeneous agent models. Aiyagari (1995) presents an initial paper studying the Ramsey allocation in a general setup, with a characterization of the optimal capital tax. Other papers, such as Aiyagari and McGrattan (1998) or Krueger and Ludwig (2016), derive optimal policies by maximizing the aggregate steady-state welfare rather than by determining the optimal Ramsey policy, which does not account for the welfare cost of transitions. Açikgöz (2015), further developed in Açikgöz et al. (2018), uses an explicit Lagrangian approach to derive the planner's first-order conditions at the steady state and relies on a numerical procedure to approximate the value of Lagrange multipliers. Dyrda and Pedroni (2016) and Chang et al. (2018) compute optimal policies without considering the planner's first-order conditions, and instead directly maximize the intertemporal welfare over all possible paths for the planner's instruments. This

[1] The truncation method is also straightforward to implement as it draws on existing tools, such as Dynare.

method is computationally very intensive, which limits the number of possible applications. Nuño and Moll (2018) consider a continuous-time framework in which they use the techniques of Ahn et al. (2017) to simplify the derivation of the planner's first-order conditions. All these papers solve for optimal policy without aggregate shocks. Our truncation method allows us to solve for optimal policies with no restrictions, both at the steady state and with aggregate shocks. To the best of our knowledge, the only paper deriving optimal Ramsey policy in a general environment with incomplete insurance markets and aggregate shocks is Bhandari et al. (2020). Their method can account for large aggregate shocks but relies on a "primal approach" in which credit constraints can be always binding or never binding, but not occasionally binding. Compared to their model, our solution strategy works well with occasionally binding credit constraints, which may be the relevant case in some environments.[2]

Our paper also contributes to the literature on solution methods for incomplete insurance market economies with aggregate shocks. Our truncation method is related to other projection and perturbation methods (Rios-Rull, 1999, Reiter, 2009, and Young, 2010), which have been shown to be accurate approximations compared to global solution techniques (see Boppart et al., 2018 or Auclert et al., 2019). The main difference is that our solution is based on idiosyncratic histories and not on the space of wealth, which turns out to be crucial for solving Ramsey programs. In particular, our method keeps track of the wealth distribution, which is necessary for deriving optimal policies, which depend on the distributive effects of the Ramsey instruments. Finally, our truncation of idiosyncratic histories is related to, but different from, the truncation of aggregate histories (see for instance Chien et al., 2011, 2012). Our truncation is used to derive a limited-heterogeneity representation of the full-fledged model, which we simulate using perturbation methods. To the best of our knowledge, this paper is the first to develop a truncation in the space of idiosyncratic histories.

Finally, regarding the application, our paper contributes to the literature on optimal unemployment benefits. This literature is huge and a large part of it employs the sufficient-statistics approach (see the surveys of Chetty, 2009, Chetty and Finkelstein, 2013, and Kolsrud et al., 2018 for recent developments), based on partial-equilibrium analysis. A handful of papers introduce general equilibrium effects, such as Mitman and Rabinovich (2015), Landais et al. (2018a,b), or Ábrahám et al. (2019), but they focus on labor market externalities and not on saving distortions. To the best of our knowledge, the only paper analyzing optimal unemployment insurance in general equilibrium with saving choices is Krusell et al. (2010). To simplify the quantitative exercise, the authors perform a welfare analysis by comparing steady states with different levels of unemployment benefits. We adopt a different approach, deriving the time-varying solution of

[2]Another strategy used in the literature is to focus on a simplified economy, in which the wealth distribution has only one or two mass points. This solution generates a tractable equilibrium (see McKay and Reis, 2016, Bilbiie and Ragot, 2017, Ravn and Sterk, 2017, and Challe, 2018, among others). These models provide important economic insights but they cannot identify certain properties related to the time-varying wealth distribution. Their quantitative relevance is thus hard to assess.

a general Ramsey problem in an economy with aggregate shocks.

The rest of the paper is organized as follows. In Section 2 we present the environment. In Section 3 we present the general truncation theory in the space of idiosyncratic histories. In Section 4 we construct the approximated model, and in Section 5 we derive optimal Ramsey policies and discuss the economic trade-off for optimal unemployment benefits over the business cycle. Section 6 sets out our quantitative analysis.

2 The economy

Time is discrete and indexed by $t = 0, 1, 2, \ldots$ The economy is populated by a continuum of agents of measure 1, distributed on an interval $\mathcal{I}$ according to a measure $\ell(\cdot)$. We follow Green (1994) and assume that the law of large numbers holds.

2.1 Preferences

In each period, agents derive utility from private consumption c and disutility from labor l. The period utility function, denoted by $U(c, l)$, is assumed to be of the Greenwood-Hercowitz-Huffman (GHH) type, exhibiting no wealth effect for the labor supply, as in Heathcote (2005), for instance:

$$U(c, l) = u\left(c - \chi^{-1}\frac{l^{1+1/\varphi}}{1 + 1/\varphi}\right), \tag{1}$$

where $\varphi > 0$ is the Frisch elasticity of labor supply, $\chi > 0$ scales labor disutility, and $u : \mathbb{R}_+ \to \mathbb{R}$ is twice continuously derivable, increasing, and concave, with $u'(0) = \infty$. Our results do not rely on the GHH functional form and we could consider a more general utility function U. The algebra is simplified, however, especially in the Ramsey program, because of the absence of a wealth effect for the labor supply. In Appendix D.3, we show how to use our truncation method with a more general utility function.

Agents have standard additive intertemporal preferences, with a constant discount factor $0 < \beta < 1$. They therefore rank consumption and labor streams, denoted respectively by $(c_t)_{t\geq 0}$ and $(l_t)_{t\geq 0}$, using the intertemporal utility criterion $\sum_{t=0}^{\infty} \beta^t U(c_t, l_t)$.

2.2 Risks

We consider a general setup where agents face an aggregate risk, a time-varying unemployment risk, and a productivity risk, as modeled by Krueger et al. (2018). As will be clear in the quantitative analysis below, this general setup allows us to match realistic labor market wealth distribution and dynamics.[3]

[3] Compared to Krueger et al. (2018), we introduce endogenous labor supply, such that labor taxes are distorting. In addition, we simplify the economy and remove the age dimension. We follow Krueger et al. (2018) and denote all transitions by Π. They will only be distinguished by their subscripts.

Aggregate risk. The aggregate risk affects both aggregate productivity and unemployment risk. At a given date t, the aggregate state is denoted by z_t and takes values in the (possibly continuous) state space $\mathcal{Z} \subset \mathbb{R}^+$. We assume that the aggregate risk is a Markov process. The history of aggregate shocks up to time t is denoted by $z^t = \{z_0, \ldots, z_t\} \in \mathcal{Z}^{t+1}$. For the sake of clarity, we will denote the realization of any random variable $X_t : \mathcal{Z}^{t+1} \to \mathbb{R}$ in state z^t by X_t, instead of $X_t(z^t)$, when there is no ambiguity.

Employment risk. At the beginning of each period, each agent $i \in \mathcal{I}$ faces an uninsurable idiosyncratic employment risk, denoted by e^i_t at date t. The employment status e^i_t can take two values, e and u, corresponding to employment and unemployment, respectively. We denote the set of possible employment statuses by $\mathcal{E} = \{e, u\}$. An employed agent with $e^i_t = e$ can freely choose her labor supply l^i_t. An unemployed agent with $e^i_t = u$ cannot work and will receive an unemployment benefit financed by a distorting tax on labor and will suffer from a fixed disutility reflecting a domestic effort. These aspects are further described below.

The employment status $(e^i_t)_{t\geq 0}$ follows a discrete Markov process with transition matrix $\Pi(z^t) \in [0,1]^{2\times 2}$ – that will simply denoted as Π_t –, which is assumed to depend on the history of aggregate shocks up to date t. The job-separation rate between periods $t-1$ and t is denoted by $\Pi_{t,eu} = 1 - \Pi_{t,ee}$, while $\Pi_{t,ue} = 1 - \Pi_{t,uu}$ is the job-finding rate between $t-1$ and t. We denote the implied population shares of unemployed and employed agents by $S_{t,u}$ and $S_{t,e}$, respectively, with $S_{t,u} + S_{t,e} = 1$.

Productivity risk. Agents' individual productivity, denoted by y^i_t, is stochastic and takes values in a finite set $\mathcal{Y} \subset \mathbb{R}_+$. Large values in $\mathcal{Y}$ correspond to high productivities. The before-tax wage earned by an employed agent i is the product of the aggregate wage w_t (dependent on aggregate shock), the labor effort l^i_t, and individual productivity y^i_t. The total before-tax wage is therefore $y^i_t w_t l^i_t$. An unemployed agent will also carry an idiosyncratic productivity level that will affect her unemployment benefits and her disutility level, denoted by ζ_y (for productivity $y \in \mathcal{Y}$), associated with domestic production.

The productivity status follows a first-order Markov process where the transition probability from state $y^i_{t-1} = y$ to $y^i_t = y'$ is constant and denoted by $\Pi_{yy'}$. In particular, it is independent of the agent's employment status. We denote by S_y the share of agents endowed with individual productivity level y. This share is constant over time because of the assumptions regarding transition probabilities $(\Pi_{yy'})$.

The individual state of any agent i is characterized by her employment status and her productivity level. We denote by $s^i_t = (e^i_t, y^i_t)$ the date-t individual status of any agent, whose possible values lie in the set $\mathcal{S} = \mathcal{E} \times \mathcal{Y}$. Finally, we denote by $s^{i,t} = \{s^i_0, \ldots, s^i_t\}$ a history until period t. We can then use the transition probabilities for employment and productivity to derive the measure $\mu_t : \mathcal{S}^{t+1} \to [0,1]$ – $\mu_t(s^t)$ being the measure of agents with history s^t in period t.

2.3 Production

The good is produced by one profit-maximizing representative firm. This firm is endowed with production technology that transforms, at date t, labor L_t (in efficient units) and capital K_{t-1} into Y_t output units of the single good. The production function F is a Cobb-Douglas function with parameter $\alpha \in (0,1)$ featuring constant returns-to-scale. Capital must be installed one period before production and the total productivity factor Z_t is stochastic. Constant capital depreciation is denoted by $\delta > 0$, and net output Y_t is formally defined as follows:

$$Y_t = F(Z_t, K_{t-1}, L_t) = Z_t K_{t-1}^{\alpha} L_t^{1-\alpha} - \delta K_{t-1}, \tag{2}$$

where the total productivity factor is the exponential of the aggregate shock z_t: $Z_t = \exp(z_t)$.

The two factor prices at date t are the aggregate before-tax wage rate w_t and the capital return r_t. The profit maximization of the producing firm implies the following factor prices:

$$w_t = F_L(Z_t, K_{t-1}, L_t) \text{ and } r_t = F_K(Z_t, K_{t-1}, L_t). \tag{3}$$

2.4 Unemployment insurance

A benevolent government manages an unemployment insurance (UI) scheme, in which labor taxes are raised to finance unemployment benefits. As labor supply is endogenous, labor tax is distorting. The government thus faces the standard trade-off between efficiency and insurance.

At any date t, unemployed agents receive an unemployment benefit that is equal to a constant fraction of the wage the agent would earn if she were employed (with the same productivity level). The replacement rate is denoted by ϕ_t and the unemployment benefit of an agent i endowed with productivity y_t equals $\phi_t w_t y_t \bar{l}_t\,(y_t)$, where $\bar{l}_t\,(y_t)$ is the average labor supply of employed agents with productivity y_t. We follow Krueger et al. (2018) for this specification, which usefully reduces the state space. From the agents' perspective, the replacement rate is an exogenous process that depends on the aggregate state $\phi_t = \phi_t(z^t)$.

Unemployment benefits are financed solely by the labor tax, which is only paid by employed agents. Taxes amount to a constant share τ_t of employed agents' wages with this proportion being identical for all employed agents. The contribution τ_t is set such that the UI scheme budget is balanced at any date t, no social debt being allowed:

$$\phi_t w_t \int_{i \in \mathcal{U}_t} y_t^i \bar{l}_t^i(y_t^i)\ell(di) = \tau_t w_t \int_{i \in \mathcal{I} \setminus \mathcal{U}_t} y_t^i l_t^i \ell(di), \tag{4}$$

where $\mathcal{U}_t \subset \mathcal{I}$ is the set of unemployed agents at t and $\mathcal{I} \setminus \mathcal{U}_t$ is the set of employed agents.

2.5 Agents' program and resource constraints

2.5.1 Sequential formulation

We consider an agent $i \in \mathcal{I}$. She can save in an asset that pays the gross interest rate $1 + r_t$. She is prevented from borrowing too much and her savings must remain above an exogenous threshold, $-\bar{a} \leq 0$. At date 0, the agent chooses the consumption $(c_t^i)_{t\geq 0}$, labor supply $(l_t^i)_{t\geq 0}$, and saving plans $(a_t^i)_{t\geq 0}$ that maximize her intertemporal utility, subject to a budget constraint and the previous borrowing limit. Formally, for a given initial wealth a_{-1}^i, her program is:[4]

$$\max_{\{c_t^i, l_t^i, a_t^i\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\Big(c_t^i - \chi^{-1} \frac{l_t^{i,1+1/\varphi} 1_{e_t^i=e} + \zeta_{y_t^i}^{1+1/\varphi} 1_{e_t^i=u}}{1+1/\varphi}\Big), \tag{5}$$

$$c_t^i + a_t^i = (1 + r_t)a_{t-1}^i + ((1 - \tau_t)l_t^i 1_{e_t^i=e} + \phi_t \bar{l}_t(y_t^i) 1_{e_t^i=u}) y_t^i w_t, \tag{6}$$

$$a_t^i \geq -\bar{a},\ c_t^i > 0,\ l_t^i > 0. \tag{7}$$

Objective (5) accounts for the disutility of unemployed agents associated with domestic production. The budget constraint (6) is standard and the expression $((1 - \tau_t)l_t^i 1_{e_t^i=e} + \phi_t \bar{l}_t(y_t^i) 1_{e_t^i=u}) y_t^i w_t$ is a compact formulation for the net wage (i.e., after taxes and unemployment benefits).

We denote by $\beta^t \nu_t^i$ the Lagrange multiplier on the credit constraint of agent i. The Lagrange multiplier is obviously null when the agent is not credit constrained. Taking advantage of the GHH utility function, the first-order conditions of an employed agent's program (5)–(7) are:

$$u'(c_t^i - \chi^{-1} \frac{\hat{l}_t^{i,1+1/\varphi}}{1+1/\varphi}) = \beta \mathbb{E}_t \Big[(1 + r_{t+1}) u'(c_{t+1}^i - \chi^{-1} \frac{\hat{l}_{t+1}^{i,1+1/\varphi}}{1+1/\varphi})\Big] + \nu_t^i, \tag{8}$$

$$(l_t^i)^{1/\varphi} = \chi(1 - \tau_t) w_t y_t^i, \tag{9}$$

where for all $t \geq 0$ and $i \in \mathcal{I}$, we introduce the notation $\hat{l}_t^i$:

$$\hat{l}_t^i \equiv l_t^i 1_{e_t^i=e} + \zeta_{y_t^i} 1_{e_t^i=u}. \tag{10}$$

The GHH utility function implies that the labor supply in equation (9) only depends on current productivity and the after-tax real wage, which implies: $\bar{l}_t(y_t^i) = l_t^i$. Unemployed agents have the same Euler equation (8). They supply no labor, but they earn unemployment benefits and suffer from disutility (terms in ζ_y) related to home production.

We now turn to economy-wide constraints. Financial- and labor-market clearing implies the following relationships for the supply of capital K_t and labor L_t (in efficient units):

$$\int_i a_t^i \ell(di) = K_t \text{ and } L_t = \int_{i \in \mathcal{I} \setminus \mathcal{U}_t} y_t^i l_t^i \ell(di). \tag{11}$$

[4] In the remainder of the paper, 1_A will denote an indicator function equal to 1 if A is true and 0 otherwise. For any $t \geq 0$, $\mathbb{E}_t$ will denote an expectation operator, conditional on the information available at date t.

The clearing of the goods market implies that total consumption and the new capital stock equals total supply, itself the sum of output net of depreciation and past capital:

$$\int_i c_t^i \ell(di) + K_t = Y_t + K_{t-1}. \tag{12}$$

Using labor market transition probabilities, we deduce that the law of motion for the employed and unemployed agent populations, denoted respectively by $S_{t,e}$ and $S_{t,u}$, is:

$$S_{t,u} = 1 - S_{t,e} = \Pi_{t,eu} S_{t-1,e} + \Pi_{t,uu} S_{t-1,u}. \tag{13}$$

The constant share of agents S_y with productivity y verifies: $S_y = \sum_{y \in \mathcal{Y}} S_{y'} \Pi_{y'y}$.

Using individual labor Euler conditions (9), the UI budget constraint (4) can be written as: $\phi_t \int_{i \in \mathcal{U}_t} \left(y_t^i\right)^{1+\varphi} \ell(di) = \tau_t \int_{i \in \mathcal{I} \setminus \mathcal{U}_t} \left(y_t^i\right)^{1+\varphi} \ell(di)$. We observe that the budget balance only depends on the current idiosyncratic state. With equation (13), this can therefore be simplified into $\phi_t \sum_{y \in \mathcal{Y}} S_{t,u} S_y y^{1+\varphi} = \tau_t \sum_{y \in \mathcal{Y}} S_{t,e} S_y y^{1+\varphi}$, or:

$$\phi_t S_{t,u} = \tau_t S_{t,e}. \tag{14}$$

We can now formulate our equilibrium definition.

Definition 1 (Sequential equilibrium) *A sequential competitive equilibrium is a collection of individual allocations* $(c_t^i, l_t^i, a_t^i)_{t \geq 0, i \in \mathcal{I}}$*, of aggregate quantities* $(K_t, L_t, Y_t)_{t \geq 0}$*, of price processes* $(w_t, r_t)_{t \geq 0}$*, and of UI policy* $(\tau_t, \phi_t)_{t \geq 0}$ *such that, for an initial wealth distribution* $\left(a_{-1}^i\right)_{i \in \mathcal{I}}$*, and for initial values of capital stock* $K_{-1} = \int_i a_{-1}^i \ell(di)$*, and of the aggregate shock* z_{-1}*, we have:*

1. *given prices, individual strategies* $\left(c_t^i, l_t^i, a_t^i\right)_{t \geq 0, i \in \mathcal{I}}$ *solve the agent's optimization program in equations (5)–(7);*

2. *financial, labor, and goods markets clear: for any* $t \geq 0$*, equations (11) and (12) hold;*

3. *the UI budget is balanced: equation (4) holds for all* $t \geq 0$*;*

4. *factor prices* $(w_t, r_t)_{t \geq 0}$ *are consistent with the firm's program (3).*

The goal of this paper is to determine the replacement rate process that generates the sequential equilibrium maximizing aggregate welfare using a utilitarian welfare criterion. This is a difficult question, as the replacement rate affects the saving decisions of all agents, the capital stock, and the price dynamics. We propose a solution that involves three steps. First, we build a model in which agents' decisions depend only on a truncated history of idiosyncratic shocks (Section 3). Second, we use this truncation theory to construct a consistent approximated model, which has a finite state-space representation (Section 4). Finally, we rely on this approximated model to solve the Ramsey problem (Section 5).

3 The truncation theory

In general, the previous economy features growing heterogeneity in wealth levels over time, because agents with different idiosyncratic histories will choose different savings. This heterogeneity can be represented by a time-varying distribution of wealth levels with a continuum of possible values, which raises considerable theoretical and computational challenges. We now present an environment in which the savings of each agent depend on the idiosyncratic risk realizations for a given number of consecutive past periods, rather than on the whole history. As an endogenous outcome, the heterogeneity among the population is summarized by a finite (but possibly large) number of agent types, and each agent's behavior is determined by her truncated idiosyncratic history instead of her whole history. We denote the truncation length by $N \geq 0$.

To simplify the exposition, we split our presentation into two distinct steps. First, in Section 3.1, we develop an island metaphor (see Lucas, 1975, 1990, or Heathcote et al., 2017 for a more recent reference) to present our truncation construction. In Section 3.2, we then derive a decentralization scheme based on a fiscal system of lump-sum transfers.

Agents are assumed to face utility shocks, which will be called preference shifters, that depend on their idiosyncratic histories over the last N periods. They will prove to be simple to calibrate and useful for the accuracy of the model and its ability to generate consistent dynamics, even when the length of the truncation remains short. Their role is to account for the heterogeneity within truncated histories. See Section 4.1 for further details.

3.1 The island metaphor

Island description. There are S^N different islands, where S is the cardinal of the set $\mathcal{S} = \mathcal{E} \times \mathcal{Y}$ of idiosyncratic risk realizations – including unemployment and productivity risks. Agents with the same idiosyncratic history for the last N periods are located on the same island. Any island is represented by a vector $s^N = (s^N_{-N+1}, \ldots, s^N_0) \in \mathcal{S}^N$ summarizing the last N-period idiosyncratic history of all island inhabitants. At the beginning of each period, agents face a new idiosyncratic shock. Agents with history $\hat{s}^N \in \mathcal{S}^N$ in the previous period are endowed with the new history s^N in the current period. The history s^N will be said to be a continuation of $\hat{s}^N$, and will be denoted by $s^N \succeq \hat{s}^N$. Because histories are of finite length, the continuation s^N consists in appending a new state s^N_0 at the end of the vector $\hat{s}^N$ and dropping the first element of $\hat{s}^N$, $\hat{s}^N_{-N+1}$.

The probability of transitioning from island $\hat{s}^N$ at $t-1$ to island s^N at date t is denoted by $\Pi_{t,\hat{s}^N,s^N}$. The transition probabilities verify $\sum_{s^N \in \mathcal{S}^N} \Pi_{t,\hat{s}^N,s^N} = 1$ and can be computed from the transition probabilities for the unemployment and productivity processes:

$$\Pi_{t,\hat{s}^N,s^N} = 1_{s^N \succeq \hat{s}^N} \Pi_{t,\hat{e}^N_0,e^N_0} \Pi_{\hat{y}^N_0 y^N_0}, \tag{15}$$

where $s^N_0 = (e^N_0, y^N_0)$ and $\hat{s}^N_0 = (\hat{e}^N_0, \hat{y}^N_0)$ are the current idiosyncratic states for s^N and $\hat{s}^N$.

The specification $N = 0$ (one island) corresponds to the full insurance case, and thus to the standard representative-agent model. Symmetrically, the case $N = \infty$ corresponds to a standard incomplete-market economy with aggregate shocks, as in Krusell and Smith (1998).

The island-planner. The island-planner maximizes the welfare of agents over all islands, attributing an identical weight to all agents and behaving as a price-taker.[5] The island-planner can freely transfer resources among agents on the same island, but cannot do so across islands. All agents belonging to the same island are treated identically and therefore receive the same allocation, as is consistent with welfare maximization. For agents on island s^N, the island-planner will choose the per capita consumption level c_{t,s^N}, the labor supply l_{t,s^N}, and the savings a_{t,s^N}.

Island sizes. As $\Pi_{t,\hat{s}^N,s^N}$ is the individual probability of transitioning from island $\hat{s}^N$ in period $t-1$ to island s^N in period t, the island sizes $(S_{t,s^N})_{s^N \in \mathcal{S}^N}$ verify the following recursion:

$$S_{t,s^N} = \sum_{\hat{s}^N \in \mathcal{S}^N} S_{t-1,\hat{s}^N} \Pi_{t,\hat{s}^N,s^N}, \tag{16}$$

where the initial size of islands $(S_{-1,s^N})_{s^N \in \mathcal{S}^N}$, with $\sum_{s^N \in \mathcal{S}^N} S_{-1,s^N} = 1$, is given.

Wealth pooling and heterogeneity reduction. At the beginning of each period t, agents learn about their current idiosyncratic shock and have to move from island $\hat{s}^N$ to island s^N. Agents take their wealth – equal to the per capita saving $a_{t-1,\hat{s}^N}$ – with them when they move. On island s^N, the wealth of all agents coming from island $\hat{s}^N$ (equal to $S_{t-1,\hat{s}^N} \Pi_{t,\hat{s}^N,s^N} a_{t-1,\hat{s}^N}$) – and for all islands $\hat{s}^N$– is pooled together and then equally divided among the S_{t,s^N} agents of island s^N. Therefore, at the *beginning of period* t, each agent of s^N holds wealth $\tilde{a}_{t,s^N}$ equal to:

$$\tilde{a}_{t,s^N} = \sum_{\hat{s}^N \in \mathcal{S}^N} \frac{S_{t-1,\hat{s}^N}}{S_{t,s^N}} \Pi_{t,\hat{s}^N,s^N} a_{t-1,\hat{s}^N}. \tag{17}$$

We denote by $(a_{-1,s^N})_{s^N \in \mathcal{S}^N}$ the initial wealth endowment.

As explained above, agents face island-specific preference shifters, denoted by ξ_{s^N}, that multiply their utility function. The island-planner's program can be expressed as:

$$\max_{(c_{t,s^N}, l_{t,s^N}, a_{t,s^N}, \tilde{a}_{t,s^N})_{t \geq 0, s^N \in \mathcal{S}^N}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{s^N \in \mathcal{S}^N} S_{t,s^N} \xi_{s^N} U(c_{t,s^N}, \hat{l}_{t,s^N}), \tag{18}$$

$$a_{t,s^N} + c_{t,s^N} = ((1-\tau_t) l_{t,s^N} 1_{e_0^N = e} + \phi_t l_{t,s^N,e} 1_{e_0^N = u}) y_0^N w_t + (1 + r_t)\tilde{a}_{t,s^N}, \text{ for } s^N \in \mathcal{S}^N, \tag{19}$$

$$c_{t,s^N}, \, l_{t,s^N} \geq 0, a_{t,s^N} \geq -\bar{a}, \text{ for } s^N \in \mathcal{S}^N, \tag{20}$$

[5] As the island-planner does not internalize the effect of its choices on prices, the allocation is not constrained-efficient, and the distortions identified by Dávila et al. (2012) are present in the equilibrium allocation. Section 5 introduces a Ramsey planner that will choose an UI policy to reduce these distortions.

and subject to the law of motion (16) for $(S_{t,s^N})_{t\geq 0}^{s^N\in\mathcal{S}^N}$, and to the definition (17) of $(\tilde{a}_{t,s^N})_{t\geq 0}^{s^N\in\mathcal{S}^N}$. Note that initial island sizes $(S_{-1,s^N})_{s^N\in\mathcal{S}^N}$ and initial wealth $(a_{-1,s^N})_{s^N\in\mathcal{S}^N}$ are given.[6]

As the objective function is increasing and concave, constraints are linear (i.e., the admissible set is convex), and the existence of the equilibrium can be proved using standard techniques (see Stokey et al., 1989, Chap. 15 and 16). We therefore omit this proof in the interest of conciseness.

Let $\beta^t\nu_{t,s^N}$ be the Lagrange multiplier of the credit constraint. The first-order conditions are:

$$\xi_{s^N}U_c(c_{t,s^N},\hat{l}_{t,s^N}) = \beta\mathbb{E}_t\left[\sum_{\tilde{s}^N\succeq s^N}\Pi_{t+1,s^N,\tilde{s}^N}\xi_{\tilde{s}^N}U_c(c_{t+1,\tilde{s}^N},\hat{l}_{t+1,\tilde{s}^N})(1+r_{t+1})\right]+\nu_{t,s^N}, \tag{21}$$

$$l_{t,s^N}^{1/\varphi} = \chi(1-\tau_t)w_t y_0^N 1_{e_0^N=e}, \tag{22}$$

$$\nu_{t,s^N}(a_{t,s^N}+\bar{a}) = 0 \text{ and } \nu_{t,s^N}\geq 0. \tag{23}$$

The first-order conditions (21)–(23) have the same form as equations (8) and (9) derived in the standard incomplete insurance-market economy of Section 2, except for the $(\xi_{s^N})_{s^N}$. We provide further details in Section 4 regarding the ξs and how they help replicate the allocation of the Bewley model of Section 2.

Market clearing and equilibrium. On any island s^N, the clearing for labor and capital markets implies the following equalities:

$$L_t = \sum_{s^N\in\mathcal{S}^N} y_0^N S_{t,s^N}l_{t,s^N}, \text{ and } K_t = \sum_{s^N\in\mathcal{S}^N} S_{t,s^N}a_{t,s^N}. \tag{24}$$

We can now state our sequential equilibrium definition, which is standard.

Definition 2 (Sequential equilibrium) *A sequential truncated competitive equilibrium is a collection of individual allocations $(c_{t,s^N},l_{t,s^N},a_{t,s^N},\tilde{a}_{t,s^N})_{t\geq 0}^{s^N\in\mathcal{S}^N}$, of island population sizes $(S_{t,s^N})_{t\geq 0}^{s^N\in\mathcal{S}^N}$, of aggregate quantities $(L_t,K_t,Y_t)_{t\geq 0}$, of price processes $(w_t,r_t)_{t\geq 0}$, and of UI policy $(\tau_t,\phi_t)_{t\geq 0}$, such that, for an initial distribution of island population and wealth $(S_{-1,s^N},a_{-1,s^N})_{s^N\in\mathcal{S}^N}$, and for initial values of capital stock K_{-1} and of the initial aggregate shock z_{-1}, we have:*

1. given prices, individual strategies $(c_{t,s^N},l_{t,s^N},a_{t,s^N},\tilde{a}_{t,s^N})_{t\geq 0}^{s^N\in\mathcal{S}^N}$ solve the agents' optimization program in equations (18)–(20);

2. island sizes and beginning-of-period individual wealth $(S_{t,s^N},\tilde{a}_{t,s^N})_{t\geq 0}^{s^N\in\mathcal{S}^N}$ are consistent with the laws of motion (16) and (17);

3. capital and labor markets clear at all dates: for any $t\geq 0$, equations (24) hold;

4. the UI budget is balanced at all dates: equation (14) holds for all $t\geq 0$;

5. factor prices $(w_t,r_t)_{t\geq 0}$ are consistent with the firm's program (3).

[6] In equation (19), $l_{t,s^N,e}$ is the labor supply of an employed agent with productivity y_0^N, which determines the UI benefits of unemployed agents of history s^N. Furthermore, as in (10), $\hat{l}_{t,s^N} = l_{t,s^N}1_{e_0^N=e}+\zeta_{y_0^N}1_{e_0^N=u}$.

The sequential equilibrium has a simple structure defined at each date by $5S^N + 5$ variables and $5S^N + 5$ equations for a given UI policy $(\tau_t, \phi_t)_{t\geq 0}$. The equilibrium features a finite number of different allocations, characterized by the N-period history of agents.

3.2 A decentralization mechanism

We now prove that the finite-state equilibrium of Definition 2 can be decentralized through fiscal transfers, which are shown to measure the degree of idiosyncratic risk sharing achieved by asset pooling in the island economy. The economy is now similar to that in Section 2 – in particular, agents are expected-utility maximizers – except for two differences. First, agents are endowed with preference shifters. Second, at each date, each agent receives a lump-sum transfer Γ_{N+1}, which is contingent on her individual history s^{N+1} over the previous $N+1$ periods. This fiscal system will be key in mimicking the pooling operation of Section 3.1. Using standard recursive notation, the agents' program can be written as:[7]

$$V(a, s^{N+1}, X) = \max_{a',c,l} \; \xi_{s^N} U(c, \hat{l}) + \beta \mathbb{E}\left[\sum_{(s^{N+1})'} \Pi'_{s^{N+1},(s^{N+1})'} V(a', (s^{N+1})', X')\right], \tag{25}$$

$$a' + c = ((1-\tau) l 1_{e_0^N = e} + \phi l_{y,e} 1_{e_0^N = u}) w(X) + (1 + r(X)) a + \Gamma_{N+1}(s^{N+1}, X), \tag{26}$$

$$\hat{l} = l 1_{e_0^N = e} + \zeta_{y_0^N} 1_{e_0^N = u}, \tag{27}$$

$$c,\, l \geq 0,\, a' \geq -\bar{a}, \tag{28}$$

where $l_{y,e}$ denotes the labor supply of an employed agent with productivity y, and where the state vector X encompasses all variables necessary to forecast prices, including aggregate shocks. Compared to the economies studied by Huggett (1993) and Aiyagari (1994), the individual history s^{N+1} is a state variable, as it determines the transfer $\Gamma_{N+1}(s^{N+1}, X)$.

We now state our first result, which explains that we can find a particular set of transfers – denoted by $(\Gamma^*_{N+1}(s^{N+1}, X))_{s^{N+1}\in\mathcal{S}^{N+1}}$ – such that the decentralized economy allocations match those of the island economy.

Proposition 1 (Finite state space) *A set of balanced transfers exists, that are denoted by $(\Gamma^*_{N+1}(s^{N+1}, X))_{s^{N+1}\in\mathcal{S}^{N+1}}$, such that any optimal allocation of the island program (18)–(20) is also a solution to the decentralized program (25)–(28).*

The previous proposition states that the island program presented in Section 3.1 can be decentralized by the balanced lump-sum transfers $(\Gamma^*_{N+1}(s^{N+1}, X))_{s^{N+1}\in\mathcal{S}^{N+1}}$ (shortened to Γ^*_{N+1} henceforth). This transfer is formally provided in equation (48) of Appendix A.1.

The transfers Γ^*_{N+1} mimic the wealth pooling of the island economy, formalized in equation (17), when agents transfer from one island to another. It consists of two steps: (i) putting together

[7] As standard, we denote the current savings choice by a'; a is thus the beginning-of-period wealth.

the beginning-of-period wealth of all agents with the same idiosyncratic history for the last N periods, independently of their idiosyncratic status $N+1$ periods ago, and (ii) redistributing consistently the same amount to agents with the same idiosyncratic history for N periods, such that there are only S^N possible wealth levels. Finally, the transfers (Γ^*_{N+1}) that operate a strict redistribution among agents sum up to zero, by construction.

4 Constructing an approximated model

We have presented two apparently different representations of the economy. The first (Section 2) is the standard full-fledged economy – also referred to as the Bewley economy in the absence of aggregate shocks – where atomistic agents are expected-utility maximizers. The second is a "truncated" economy that can be viewed as either an island metaphor with a particular risk-sharing arrangement (Section 3.1), or as a decentralized economy with a well-designed tax-system (Section 3.2).

This section aims to show how the preference shifters ξs can be constructed, to make these two representations consistent with each other. The main objective of this construction is that, in the absence of aggregate shocks, equilibrium allocations in the full-fledged and truncated economies are consistent with each other. More precisely, the savings of agents with history s^N in the truncated economy will be exactly equal to the average, in the Bewley model, of individual savings for agents whose idiosyncratic shock realizations in the last N periods are s^N. The same also holds for consumption levels and labor supplies. A first consequence of this property is that aggregate quantities (capital, total labor supply, aggregate consumption) and prices are identical in both the truncated and Bewley economies. A second consequence is that, for large N, the allocations of the truncated economy will converge toward those of the full-fledged economy.

We proceed in two steps. First, in Section 4.1, we focus on the steady-state economy to specify the construction of the ξs and state our first convergence result. Second, in Section 4.2, we explain how to use the truncated model in the presence of aggregate shocks as an approximation of the full-fledged model. As a preliminary step, we formally define the truncated aggregation of an allocation, or more concisely the *truncation* of an allocation. Consider a generic variable, denoted by $X_t(s^t, z^t)$, which depends on the history s^t of idiosyncratic risk and z^t of aggregate risk. For a N-period history s^N, the truncation of X, denoted by X_{t,s^N} is formally defined as:

$$X_{t,s^N} = \frac{\sum_{s^t \in \mathcal{S}^t | (s^t_{t-N+1}, \ldots, s^t_t) = s^N} X_t(s^t, z^t)\mu_t(s^t)}{S_{t,s^N}}, \tag{29}$$

where we recall that $\mu_t(s^t)$ is the measure of agents with history s^t, and S_{t,s^N} is the population of agents with history s^N, at period t. In other words, the truncation X_{t,s^N} is the average value of the variable X among the population of agents experiencing history s^N over the last N periods.

4.1 Steady-state economy

The equations characterizing prices and allocations in the truncated economy are: the Euler equations for consumption and labor, (21) and (22), the collection of budget constraints (19), and the dynamics of history sizes (16). At the steady state, dropping the subscript t, these equations become, respectively, for all $s^N \in \mathcal{S}^N$:

$$\xi_{s^N} U_c(c_{s^N}, \hat{l}_{s^N}) = \nu_{s^N} + \beta(1+r) \sum_{\tilde{s}^N \succeq s^N} \Pi_{s^N \tilde{s}^N} \xi_{\tilde{s}^N} U_c(c_{\tilde{s}^N}, \hat{l}_{\tilde{s}^N}), \tag{30}$$

$$l_{s^N}^{1/\varphi} = \chi(1-\tau) w y_0^N, \tag{31}$$

$$a_{s^N} + c_{s^N} = ((1-\tau_t) l_{s^N} 1_{e_0^N = e} + \phi l_{s^N, e} 1_{e_0^N = u}) y_0^N w + (1+r) \tilde{a}_{s^N}, \tag{32}$$

$$S_{s^N} = \sum_{\hat{s}^N \in \mathcal{S}^N} \Pi_{\hat{s}^N s^N} S_{\hat{s}^N}, \tag{33}$$

where the quantity $\hat{l}_{s^N} = l_{s^N} 1_{e_0^N = e} + \zeta_{y_0^N} 1_{e_0^N = u}$ is defined in equation (10). The steady-state equilibrium is further characterized by the following aggregate equations, which are unchanged: market clearing (24), UI budget balance (14), and factor prices (3).

The next proposition will state that the ξs can be chosen such that the steady-state allocations of the truncated model – determined by equations (30)–(33) – are equal to the "truncation" of the steady-state allocations of the full-fledged model.

Proposition 2 (Constructing the ξs) *The preference shifters $(\xi_{s^N})_{s^N \in \mathcal{S}^N}$ can be computed at the steady state, such that the truncation – following equation (29) – of the steady-state equilibrium allocations of the full-fledged model (Definition 1) is an equilibrium allocation in the truncated model (Definition 2).*

The proof is provided in Appendix B.2.1 and the logic is as follows. In the steady state, we can characterize the stationary wealth distribution of the full-fledged model. From this stationary distribution, we can integrate policy rules and transition probabilities to determine all individual variables, which can then be combined to compute the truncated quantities using equation (29). Finally, we use the Euler equations (30) of the truncated economy to compute the preference shifters $(\xi_{s^N})_{s^N \in \mathcal{S}^N}$. We can also identify the set of credit-constrained histories. Technically, computing the ξs simply requires inverting a matrix. A detailed account of this construction is provided in Appendix B.2.1.

Note that preference shifters are defined up to a positive rescaling: if $(\xi_{s^N})_{s^N}$ are admissible (in the sense of Proposition 2) then $(\lambda_\xi \xi_{s^N})_{s^N}$ are also admissible for any $\lambda_\xi > 0$. We will henceforth assume uniqueness by further imposing a normalization constraint: $\sum_{s^N \in \mathcal{S}^N} S_{s^N} \xi_{s^N} = 1$.

A noteworthy consequence of constructing the ξs according to Proposition 2 is that a truncated economy with such ξs will feature prices and aggregate quantities (consumption, labor supply, and capital) that are, in the steady state, exactly identical to those of the Bewley model. This statement is formalized in the following corollary.

Corollary 1 (Price and aggregate quantities) *Consider a truncated economy, where the coefficients $(\xi_{s^N})_{s^N \in \mathcal{S}^N}$ have been constructed following Proposition 2, based on the underlying Bewley equilibrium. Prices and aggregate quantities (aggregate consumption, total labor supply, and capital) in the truncated economy are then identical to those of the Bewley economy.*

Understanding the ξs. The preference shifters ξs connect the competitive equilibrium of the full-fledged Bewley model to the truncated one. More precisely, the truncation involves assigning the same allocation to agents with the same truncated history. Consequently, truncated allocations are generally not consistent with history-level Euler equations. The reason for this is as follows: since marginal utilities are not linear, the truncation of Euler equations does not give the Euler equations for truncated allocations, because there is a non-degenerate distribution of agents with the same truncated history in the full-fledged Bewley model. The role of the ξs is precisely to reconcile Euler equations, with consumption levels of truncated histories, such that truncated allocations can be seen as the result of a competitive equilibrium (provided that agents are endowed with preference shifters).

The correction coefficients ξs could be set to 1, when there is no heterogeneity within truncated histories in the Bewley model.[8] The role of the ξs is thus to account for the within-history heterogeneity. We show in the numerical investigation of Section 6 that the ξs efficiently capture this within-history heterogeneity in a parsimonious way, even with a short truncation, as $N = 2$.

Convergence properties. Two steps remain: (i) showing that for large N, the truncated allocations converge to the allocation of the full-fledged model, and that the ξs converge to 1, and (ii) determining how to use the truncated model in the presence of aggregate shocks. The second step will be explained in Section 4.2, while the first is formalized in the next proposition.

Proposition 3 (Convergence of allocations) *For any truncation length $N > 0$, we denote by $(c_{t,s^N}, a_{t,s^N}, l_{t,s^N})_{s^N}$ the allocations – consumption, savings, and labor supply – and by $(\xi_{s^N})_{s^N}$ the preference shifters associated with the truncated economy in the absence of aggregate shocks. We have the following convergence result for allocations:*

$$(c_{t,s^N}, a_{t,s^N}, l_{t,s^N})_{s^N} \longrightarrow_N (c_t(s^\infty), a_t(s^\infty), l_t(s^\infty))_{s^\infty \in \mathcal{S}^\infty}, \text{ almost surely,}$$

[8]This argument explains why, thanks to the GHH utility function, there is no such truncation wedge for the labor supply Euler equation because the latter is linear in current productivity (see equation (31)). However, it would be present with a more general utility function and correcting coefficients would be needed for both Euler equations. Though slightly more involved, it is noteworthy that this would not impair the construction of the truncated economy. See Appendix D.3 for further details.

where $(c_t(s^\infty), a_t(s^\infty), l_t(s^\infty))_{s^\infty \in \mathcal{S}^\infty}$ *are the steady-state allocations of the full-fledged model. Similarly, for the preference shifters, we have:* $\xi_{s^N} \longrightarrow_N 1$, *almost surely.*

This proposition states that the truncated allocations converge (almost surely) to the steady-state allocations of the full-fledged model (or the Bewley allocations in short) when the history length becomes increasingly long. It also states that the preference shifters, constructed according to Proposition 2, converge to 1. The proof can be found in Appendix A.2.

The intuition for this result originates in the pooling operation at the heart of the truncated economy, through either the asset pooling of equation (17) or the taxation scheme Γ_N^* of Proposition 1. As we explained in the construction of the ξs, the histories of the truncated economy feature some within-heterogeneity (due to the pooling) that is partly captured by the ξs. As N increases, the agents' shared history becomes longer and the first period with a potentially different idiosyncratic status becomes more distant. This means that as N increases, the agents assigned to a given history become more "similar" and the within-history heterogeneity becomes smaller. In other words, the pooling concerns agents that become increasingly similar to each other when N becomes larger. This implies that when N increases, each truncated history converges to an infinitely long history and the choices become those of the atomistic agent endowed with this limit history. Furthermore, since the within-history heterogeneity vanishes, the ξs have an increasingly small role to play and converge to 1 for large N.

Since the result of Corollary 1 holds for any truncation length N, this is also the case at the limit. So, a consequence of Proposition 3 is that, at the steady state, the truncated economy converges to the Bewley economy.

4.2 The dynamics of the truncated model

We now formally present the structure of the approximated model. Our main assumptions regarding the truncated model are stated below.

Assumption A *We make the following two assumptions.*

1. *The* preference shifters $(\xi_{s^N})_{s^N \in \mathcal{S}^N}$ *remain constant and equal to their steady-state values.*
2. *The set of credit-constrained histories, denoted by* $\mathcal{C} \subset \mathcal{S}^N$, *is time-invariant.*

Assumption A enables us to use our truncated representation to simulate the economy in the presence of aggregate shocks. The first item of Assumption A means that the model features within-history heterogeneity, but that this heterogeneity is not time-varying, as the preference shifters, determined at the steady state, remain constant in the presence of aggregate shocks. The second item assumes that if a history $s^N \in \mathcal{S}^N$ is credit constrained at the steady state, it also remains credit constrained in the dynamic version of the model. Symmetrically, unconstrained histories at the steady state remain unconstrained in the dynamic version. Note that with aggregate shocks, the number of credit-constrained households can be time-varying, since the size of islands can be time-varying because of time-varying transition probabilities.[9] The relevance of these assumptions is a quantitative issue. Our quantitative analysis of Section 6 shows that, for a standard calibration, these assumptions provide accurate dynamics. We can now formally present our model in the presence of aggregate shocks.

Definition 3 (Model with aggregate shocks) *In the presence of aggregate shocks, the truncated model is defined by the following set of equations:*

$$\forall s^N \in \mathcal{S}^N \setminus \mathcal{C}, \xi_{s^N} U_c(c_{t,s^N}, \hat{l}_{t,s^N}) = \beta \mathbb{E}_t (1 + r_{t+1}) \sum_{\tilde{s}^N \in \mathcal{S}^N} \Pi_{t+1,s^N,\tilde{s}^N} \xi_{\tilde{s}^N} U_c(c_{t+1,\tilde{s}^N}, \hat{l}_{t+1,\tilde{s}^N}), \quad (34)$$

$$\forall s^N \in \mathcal{C}, \; a_{t,s^N} = -\bar{a}, \quad (35)$$

$$\forall s^N \in \mathcal{S}^N, \; l_{t,s^N}^{\frac{1}{\varphi}} = \chi(1-\tau_t) w_t y_0^N, \quad (36)$$

$$\forall s^N \in \mathcal{S}^N, \; c_{t,s^N} + a_{t,s^N} \leq (1+r_t) \sum_{\tilde{s}^N \in \mathcal{S}^N} \Pi_{t+1,s^N,\tilde{s}^N} \frac{S_{t,\tilde{s}^N}}{S_{t,s^N}} a_{t-1,s^N} \quad (37)$$
$$+ \left((1-\tau_t) 1_{e_0^N = e} l_{t,s^N} + \phi_t 1_{e_0^N = u} l_{t,s^N,e} \right) y_0^N w_t,$$

together with equations (3) for factor prices, (24) for market clearing conditions, and (14) for UI budget balance.

Using Definition 3, we can easily simulate the model using standard perturbation methods. Although our goal is to solve for optimal policies with aggregate shocks, an additional advantage of the truncation theory is that it is on a par with current simulation techniques, including those of Reiter (2009) and Boppart et al. (2018). We discuss quantitative comparisons in Section 6.

We conclude this section with a convergence result in the presence of aggregate shocks.

Proposition 4 (Allocation convergence in the presence of aggregate shocks) *For any truncation length $N > 0$, we denote by $(c_{t,s^N}, a_{t,s^N}, l_{t,s^N})_{s^N}$ the allocations – consumption, savings, and labor supply – and by $(\xi_{s^N})_{s^N}$ the preference shifters (associated with the truncated economy in the absence of aggregate shocks), and computed using a first-order perturbation method.*

For large N, $(c_{t,s^N}, a_{t,s^N}, l_{t,s^N})_{s^N}$ converge to the solution of a first-order perturbation of the full-fledged model (if it exists).

This convergence result partially extends the convergence result of Proposition 3 to the presence of aggregate shocks. The limitation is that the solution must be computed using a first-order perturbation method, implying that aggregate shocks cannot be too large. The intuition of the proof is rather straightforward. The perturbation method involves constructing

[9] It is noteworthy that this second point is a restriction imposed by the perturbation method and is not specific to our construction. In particular, it could be relaxed by using history-specific penalty functions to model (possibly large) aggregate shocks. We leave this development for future work.

the economy as a first-order polynomial of the model aggregate shocks. The coefficients of this polynomial are functions of the model truncated variables (i.e., they include the ξs), which converge toward their Bewley counterparts. The polynomial therefore converges toward the polynomial associated with the full-fledged model. Obviously, this method assumes that the full-fledged model with aggregate shocks can be properly solved via the perturbation method.

5 Ramsey program

The previous construction provides a solid foundation for solving Ramsey policies with aggregate shocks. Computing such Ramsey policies in the general case is a very difficult task. It is necessary to introduce additional state variables, such as Lagrange multipliers, for the relevant individual constraints. The Ramsey problem thus involves a joint distribution of two individual state variables – in our case, beginning-of-period wealth and Lagrange multipliers. Characterizing this joint distribution is particularly difficult and, to the best of our knowledge, there is no general method for such a characterization even in the absence of aggregate shocks. In our approach, the state space has a finite support, allowing us to resolve the Ramsey program using the tools of Marcet and Marimon (2019) adjusted to our heterogeneous-agent model. An additional benefit is that it is possible to derive analytical expressions for first-order conditions of the Ramsey program. This eases the interpretation of results and the comparison with other approaches.

5.1 Formulation of the Ramsey program

The Ramsey problem involves determining the unemployment insurance policy (which consists here of the replacement rate ϕ_t and the labor tax rate τ_t) that corresponds to the "best" competitive equilibrium, according to a utilitarian welfare criterion. Aggregate welfare is measured by the sum $\sum_{t=0}^{\infty} \beta^t \sum_{s^N \in \mathcal{S}^N} S_{t,s^N} \xi_{s^N} U(c_{t,s^N}, l_{t,s^N})$, where ξ_{s^N} are the preference shifters of Proposition 2. The Ramsey problem can be written as – with $\hat{l}_{t,s^N}$ defined in (10):

$$\max_{\left((a_{t,s^N}, c_{t,s^N}, l_{t,s^N}, \hat{l}_{t,s^N})_{s \in \mathcal{S}^N}, \phi_t, \tau_t, r_t, w_t\right)_{t \geq 0}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \sum_{s^N \in \mathcal{S}^N} S_{t,s^N} \xi_{s^N} U(c_{t,s^N}, \hat{l}_{t,s^N}) \right], \tag{38}$$

subject to: (i) the budget constraints (37), (ii) the labor Euler equations (36), (iii) the consumption Euler equations (34), (iv) the UI scheme budget balance (14), (v) the market clearing constraints (24), and finally (vi) the factor prices (3).

A reformulation of the Ramsey problem. We simplify the formulation of the Ramsey problem exposed in equation (38), using the factorization of the Lagrangian employed by Marcet and Marimon (2019). However, two difficulties arise when applying this approach in our environment. The first difficulty is the consistency of the method with our truncation methodology. In fact, we show below in Lemma 2 that truncation adds no complexity to the formulation of the planner's objective. The second difficulty is the application of Marcet and Marimon (2019) to models with occasionally binding credit constraints, for which the Slater (1950) condition may not be fulfilled. Loosely speaking, this condition requires the existence of an interior solution of the primal problem. To show that the first-order conditions of the Lagrangian approach are valid, we derive an additional theoretical result in Proposition 6 of Appendix D.2, where we prove that the first-order conditions of our Ramsey problem can be understood as the limit of those of a Ramsey problem featuring penalty functions (whose concavity become infinitely high), for which the Slater (1950) condition applies.[10] We here directly provide the first-order conditions of our Ramsey problem to simplify the exposition.

We denote by $\beta^t S_{t,s^N}\lambda_{t,s^N}$ the Lagrange multiplier of the Euler equation (34) for history s^N at date t. These Lagrange multipliers are key to understanding the planner's program. If agents' private incentives to save in history s^N at date t are socially optimal, then their Euler equation is not a constraint and the Lagrange multiplier is 0: $\lambda_{t,s^N} = 0$. Depending on how the planner perceives the saving incentive distortions (i.e., whether agents save too much or too little from the planner's perspective), these coefficients can be either positive or negative. The sign of these multipliers helps us to understand the saving incentive distortions as seen by the planner. We provide an example in Appendix D.1 to clarify this aspect. Let us define for $s^N \in \mathcal{S}^N$:

$$\Lambda_{t,s^N} \equiv \frac{\sum_{\tilde{s}^N \in \mathcal{S}^N} S_{t-1,\tilde{s}^N}\Pi_{t,\tilde{s}^N s^N}\lambda_{t-1,\tilde{s}^N}}{S_{t,s^N}}, \tag{39}$$

which, for agents in history s^N, can be interpreted as the average of their previous period Lagrange multipliers for the Euler equation. Finally, note that the multiplier λ_{t,s^N} is null when the credit constraint is binding. The product $\lambda_{t,s^N}\nu_{t,s^N}$ (for any t and s^N) is thus always null. The following lemma summarizes our simplified Ramsey problem.

Lemma 1 (Simplified Ramsey problem) *The Ramsey problem (38) can be simplified into:*

$$\max_{\left((a_{t,s^N},c_{t,s^N},l_{t,s^N},\hat{l}_{t,s^N})_{s^N\in\mathcal{S}^N},\phi_t,\tau_t,r_t,w_t\right)_{t\geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty}\beta^t \sum_{s^N\in\mathcal{S}^N} S_{t,s^N}\Big(\xi_{s^N}U(c_{t,s^N},\hat{l}_{t,s^N}) \tag{40}$$

$$- \Big(\lambda_{t,s^N} - (1+r_t)\Lambda_{t,s^N}\Big)\xi_{s^N}U_c(c_{t,s^N},\hat{l}_{t,s^N})\Big)$$

$$s.t.\ a_{t,s^N} \geq -\overline{a} \text{ and } \lambda_{t,s^N} = 0 \text{ if } a_{t,s^N} = -\overline{a}, \tag{41}$$

and subject to equations (37), (24), (14), and (3).

Lemma 1 shows that the factorization of the Lagrangian can be performed with truncated histories, as in Marcet and Marimon (2019). This considerably simplifies the derivation of first-order conditions of the Ramsey program. The proof is provided in Appendix A.5.1.

[10]We thank Albert Marcet for this suggestion.

5.2 Ramsey conditions and economic interpretation

Using proper substitution, the program (40)–(41) can be written as a maximization problem with only two sets of choice variables: the labor tax τ_t and saving choices $(a_{t,s^N})_{t\geq 0}^{s^N\in\mathcal{S}^N}$. The current section derives the planner's first-order conditions and discusses the economic trade-offs that determine the time-varying replacement rate.

To ease the economic interpretation of the first-order conditions, we define for $s^N \in \mathcal{S}^N$:[11]

$$\Psi_{t,s^N} = \xi_{s^N} U_c(c_{t,s^N}, \hat{l}_{t,s^N}) - \Big(\lambda_{t,s^N} - (1+r_t)\Lambda_{t,s^N}\Big)\, \xi_{s^N} U_{cc}(c_{t,s^N}, \hat{l}_{t,s^N}), \tag{42}$$

which will be called the *marginal social valuation of liquidity* for agents in history s^N, because it is the marginal gain for the planner of transferring resources to history s^N at date t. If an agent receives one additional unit of goods today, this additional unit will have a value proportional to U_c. This value only accounts for private valuation, but should also reflect, for the planner, the effect on the saving incentives, i.e., on the Euler equations. This additional unit therefore affects the agent's saving incentive from period $t-1$ to period t and from period t to period $t+1$. This effect is captured by the second term, which is proportional to U_{cc}.

The saving decision in history s^N at date t affects the individual welfare of all agents due to general equilibrium effects on capital and prices. The first-order condition of the Ramsey program related to saving choices summarizes all of these effects. It can be written as follows for unconstrained histories $s^N \in \mathcal{S}^N \setminus \mathcal{C}$:

$$\begin{aligned}
\Psi_{t,s^N} = {} & \underbrace{\beta \sum_{\tilde{s}^N\in\mathcal{S}^N} \mathbb{E}_t\left[(1+r_{t+1})\Pi_{t+1,s^N\tilde{s}^N}\Psi_{t+1,\tilde{s}^N}\right]}_{\text{liquidity smoothing}} + \beta\frac{\alpha\,(1-\alpha)}{1+\alpha\varphi}\frac{1}{L_{t+1}}\left(\frac{K_t}{L_{t+1}}\right)^{\alpha-1} \qquad (43)\\
& \times \Bigg\{\underbrace{\mathbb{E}_t\Bigg[(1-\tau_{t+1})\sum_{\tilde{s}^N\in\mathcal{S}^N}\Psi_{t+1,\tilde{s}^N}S_{t+1,\tilde{s}^N}l_{t+1,\tilde{s}^N}\tilde{y}_0^N 1_{\tilde{e}_0^N=e}\Bigg]}_{\text{wage effect on labor supply for employed}} \\
& + (1+\varphi)\underbrace{\mathbb{E}_t\Bigg[\phi_{t+1}\sum_{\tilde{s}^N\in\mathcal{S}^N}\Psi_{t+1,\tilde{s}^N}S_{t+1,\tilde{s}^N}l_{t+1,\tilde{s}^N,e}\tilde{y}_0^N 1_{\tilde{e}_0^N=u}\Bigg]}_{\text{wage effect on unemployment benefits for unemployed}} \\
& - \left(\frac{K_t}{L_{t+1}}\right)^{-1}\sum_{\tilde{s}^N\in\mathcal{S}^N}\mathbb{E}_t\underbrace{\Big[S_{t+1,\tilde{s}^N}\Big(\Lambda_{t+1,\tilde{s}^N}\xi_{\tilde{s}^N}U_c(c_{t+1,s^N},\hat{l}_{t+1,s^N}) + \Psi_{t+1,\tilde{s}^N}\tilde{a}_{t+1,\tilde{s}^N}\Big)\Big]}_{\text{interest rate effect on smoothing and wealth}}\Bigg\},
\end{aligned}$$

where $\tilde{a}_{t,s^N} = \sum_{\tilde{s}^N\in\mathcal{S}^N}\Pi_{t,\tilde{s}^N s^N}\frac{S_{t,\tilde{s}^N}}{S_{t,s^N}}a_{t-1,\tilde{s}^N}$ is the beginning-of-period wealth for history s^N at t.

[11] Unlike Chien et al. (2011) or Marcet and Marimon (2019), we do not use cumulative Lagrange multipliers to analyze the dynamics. Instead, we use the period multipliers to derive the planner's first-order conditions. These conditions are easier to interpret and the simulation of the model relies on a smaller number of variables. See these two references for a discussion of the existence of these multipliers in such economies.

Equation (43) features the first-order condition on the liquidity allocation (i.e., saving choices) for unconstrained histories. Although it appears complicated, the equation has a straightforward interpretation. Four effects are at play. The first is a direct effect that measures the expected future value of liquidity tomorrow. In other words, this component states that liquidity value should be smoothed over time. This first part is very similar to a standard Euler equation. We refer to this first term as "liquidity smoothing". The three other components alter the pure smoothing effect and reflect the fact that the planner also takes into account the consequences of liquidity allocation on prices. More precisely, the second and third components correspond to the marginal effect of additional saving on the wage rate. This affects employed agents (second component) and unemployed (third component) agents, because UI benefits are proportional to the labor income of employed agents with the same productivity. Finally, the fourth and last component reflects the distortions of the interest rate on saving incentives.

The second first-order condition, relating to the labor tax, can be written as follows:

$$\underbrace{\frac{S_{t,e}}{S_{t,u}}\left(\frac{1}{\varphi}+1-\frac{1-\alpha}{1-\tau_t}\right)\sum_{\tilde{s}^N\in\mathcal{S}^N}\Psi_{t,\tilde{s}^N}S_{t,\tilde{s}^N}\frac{l_{t,\tilde{s}^N,e}}{L_t}\tilde{y}_0^N 1_{\tilde{e}_0^N=u}}_{\text{gain of unemployment benefits for unemployed}}= \tag{44}$$

$$\underbrace{\frac{\alpha}{(1-\tau_t)K_{t-1}}\sum_{\tilde{s}^N\in\mathcal{S}^N}S_{t,\tilde{s}^N}\left(\Lambda_{t,\tilde{s}^N}\xi_{\tilde{s}^N}U_{c,t,s^N}+\Psi_{t,\tilde{s}^N}\tilde{a}_{t,\tilde{s}^N}\right)}_{\text{effect on prices, smoothing, and redistribution}}+\underbrace{\frac{1}{\varphi}\sum_{\tilde{s}^N\in\mathcal{S}^N}S_{t,\tilde{s}^N}\Psi_{t,\tilde{s}^N}\frac{l_{t,\tilde{s}^N}}{L_t}\tilde{y}_0^N 1_{\tilde{e}_0^N=e}}_{\text{cost of the tax for employed}}.$$

Equation (44) determines the optimal labor tax rate by setting the marginal costs of a higher tax rate equal to the marginal benefits. On the right-hand side of (44), marginal costs comprise two effects. The first reflects the tax distortion on the interest rate and thus on saving incentives. The second accounts for the impact of the labor tax on employed agents, taking into account the negative net effect on the labor supply (inversely proportional to the Frisch elasticity φ). On the left-hand side of equation (44), the marginal benefit comprises the marginal gain of tax (and UI benefit) for unemployed agents. Note that equation (44) embeds, in a compact form, the general equilibrium effect on wages, which are captured by both the Frisch elasticity of the labor supply, φ, and the concavity of the production function, α.

5.3 Convergence result

Although we consider the Ramsey problem of an approximated model, the solution of the approximated Ramsey model converges to the solution of the actual Ramsey program, if this solution exists. This result is the Ramsey equilibria parallel of Proposition 3, which held for competitive equilibria. The following proposition formalizes this statement.

Proposition 5 (Convergence of the approximate Ramsey program) *We assume that savings choices are bounded from above by $a_{\max} > 0$. For any truncation length N, we denote the* *solutions of the Ramsey program in the absence of aggregate shocks (equations (30)–(33) and first-order equations (43) and (44)) by:* $(c_{s^N,t}, a_{t,s^N}, l_{t,s^N})_{s^N,t}$ *for allocations,* $(\lambda_{s^N,t})_{s^N,t}$ *for Lagrange multipliers,* $(K_{N,t}, L_{N,t})$ *for aggregate quantities,* $(r_{N,t}, w_{N,t})$ *for factor prices, and* $(\tau_{N,t}, \phi_{N,t})$ *for the UI policy.*

These quantities converge almost surely to the solution of an exact Ramsey program (if it exists), and more precisely to the allocations $(c_t(s^\infty), a_t(s^\infty), l_t(s^\infty))_{s^\infty \in \mathcal{S}^\infty}$*, the Lagrange multipliers* $(\lambda_t(s^\infty))_{s^\infty \in \mathcal{S}^\infty}$*, the aggregate quantities* (K_t, L_t)*, the factor prices* (r_t, w_t)*, and the UI policy* (τ_t, ϕ_t)*, respectively, such that:*

- *allocations* $(c_t(s^\infty), a_t(s^\infty), l_t(s^\infty))_{s^\infty \in \mathcal{S}^\infty}$ *are those of a Bewley model;*
- *factor prices* (r_t, w_t) *are consistent with aggregate quantities* (K_t, L_t) *and verify (3);*
- *Lagrange multipliers and the UI policy are consistent with the FOC of an exact Ramsey program (without aggregation).*

The proof of this result can be found in Appendix A.4. We need a technical assumption imposing an upper bound on saving choices $a_{\max}$ (which can be arbitrarily large) which is required to apply Berge theorem. Proposition 5 states that in the absence of aggregate shocks, the solution of the approximate Ramsey program converges to the solution of an exact Ramsey program (if it exists) when the history length becomes increasingly long. The truncation can therefore asymptotically approximate not only competitive equilibria but also Ramsey equilibria.

As was the case for the competitive equilibrium in Corollary 1, this convergence result can be extended to aggregate shocks in the context of a perturbation method. We state this result formally in the following corollary.

Corollary 2 (Convergence of the Ramsey program with aggregate shocks) *The convergence result of Proposition 5 can be extended to the presence of aggregate shocks when the Ramsey program is solved with the perturbation method (assuming that a steady-state Ramsey equilibrium exists).*

A key aspect of this corollary is that the steady-state equilibrium must exist for the Ramsey program. The "perturbation" is in fact realized in the neighborhood of the steady state.

5.4 An algorithm to calibrate and simulate the Ramsey allocation using the approximated model

We now provide the main steps of the algorithm used to approximate the Ramsey allocation at the steady state.

1. Solve the "true" Bewley model (i.e., without aggregate shocks) for a given UI policy.

2. Construct the truncated model, and compute the values of the Lagrange multipliers, using equation (43).

3. Iterate on the UI policy until the optimality condition (44) is satisfied.

This strategy has three advantages. First, the derivation of preference shifters for the approximated model is consistent with the true Bewley model at each step. Second, since the Bewley model is required to exist at each step, the perturbation method cannot be used around non-existing steady-state equilibria.[12] Another advantage of our method is that the steady-state value of Lagrange multipliers can be simply expressed in closed-form using matrix calculus, as the state space has a finite dimension. This provides a new and very efficient algorithm with which to compute the steady-state solution of the Ramsey program, as shown in Appendix B.2.

Once the steady state of the model has been computed, the dynamics of the model – provided in Appendix B.1 – can be solved by perturbation techniques using software such as Dynare (Adjemian et al., 2011). This implies that the (truncated) wealth distribution and the distribution of Lagrange multipliers are used as state variables. As shown in the numerical analysis of Section 6, the planner's instruments depend on these two distributions.

6 Numerical analysis

We calibrate the model and perform two exercises. First, we simulate the model with aggregate shocks and a constant replacement rate to check the accuracy of the truncated model. Second, we compute the optimal dynamics of the replacement rate and discuss the mechanisms and the accuracy of the solution.

6.1 The calibration

6.1.1 Preferences

The period is a quarter. The discount factor is $\beta = 0.99$. The period utility function is $\log(c - \chi^{-1}\frac{l^{1+1/\varphi}}{1+1/\varphi})$. The Frisch elasticity of labor supply is set to $\varphi = 0.5$, which is the value recommended by Chetty et al. (2011) for the intensive margin in heterogeneous agent models. The scaling parameter is set to $\chi = 0.04$, which implies normalizing the aggregate labor supply, defined in (11), to 1/3.

Unemployed workers cannot choose their labor supply. Their utility function is $\log(c - \chi^{-1}\frac{\zeta_y^{1+1/\varphi}}{1+1/\varphi})$, also with $\chi = 0.04$ and $\varphi = 0.5$. We recall that ζ_y is the exogenous labor supply for home production for a worker with productivity y. For agents to be worse-off when unemployed than employed, ζ_y is set to the steady-state labor supply of a worker with productivity y.

[12] In the general case, it is possible that the set of equilibria is larger in the approximated model than in the true Bewley model. This procedure ensures that we only select equilibria that exist for the original Bewley model.

6.1.2 Technology and TFP shock

The production function is Cobb-Douglas: $Y = ZK^{\alpha}L^{1-\alpha}$. The capital share is set to $\alpha = 36\%$ and the depreciation rate is $\delta = 2.5\%$, as in Krueger et al. (2018) among others. The TFP process is a standard AR(1) process, with $Z_t = \exp(z_t)$ and:

$$z_t = \rho_z z_{t-1} + \varepsilon_t^z, \tag{45}$$

where $\varepsilon_t^z \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_z^2)$. We use the standard values $\rho_z = 0.95$ and $\sigma_z = 0.31\%$ to obtain a deviation of the TFP shock z_t equal to 1% at a quarterly frequency (see Den Haan, 2010 for instance).

6.1.3 Idiosyncratic risk

To calibrate the uninsurable labor risk, we follow the strategy of Krueger et al. (2018) and introduce both employment and productivity risks. We adapt their calibration to our economy, which features a GHH utility function and endogenous labor supply.

Unemployment risk. For the unemployment risk, we follow Shimer (2003) and assume that the job-separation rate is constant over the business cycle, while the job-finding rate is time-varying and procyclical. We set $\Pi_{eu}^{SS} = 4.87\%$ for the average job-separation rate and $\Pi_{ue}^{SS} = 78.6\%$ for the average job-finding rate. The standard deviation of the job-finding rate is set to 6%, based on US estimates (see Abeille-Becker and Clerc, 2013 or Challe and Ragot, 2016). As the standard deviation of z_t is 1%, we assume that the job-finding rate is defined as follows:

$$\Pi_{t,ue} = \Pi_{ue}^{SS} + \sigma_{ue} z_t, \text{ with } \sigma_{ue} = 6.$$

Idiosyncratic productivity risk. Idiosyncratic productivity risk is a key ingredient for the model to generate a realistic earning and wealth distribution. We calibrate a productivity process:

$$\log y_t = \rho_y \log y_{t-1} + \varepsilon_t^y,$$

with $\varepsilon_t^y \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_y^2)$. As we are considering a model with endogenous labor supply, there is a difference between the earning process and the productivity process. With the GHH utility function, the labor supply is $l_y = (w(1-\tau)\chi y)^{\varphi}$. The log of labor income yl_y is thus related to $(1+\varphi)\log y$ and accordingly depends on the value chosen for the Frisch elasticity. We calibrate the process y such that the persistence and variance of the labor income yl_y match the estimated values of Krueger et al. (2018). They estimate a process with a persistent and transitory shock on productivity. Following Boppart et al. (2018), we use persistent shocks and consider transitory shocks as measurement errors. Using a Frisch elasticity of 0.5, we compute a quarterly persistence of $\rho_y = 0.9923$ and a standard deviation of $\sigma_y = 6.60\%$, which generate, for the log of earnings, an annual persistence of 0.9695 and a variance of $\frac{3.84\%}{1-0.9695^2}$. [13] The Rouwenhorst (1995) procedure is used to discretize the productivity process into 7 idiosyncratic states with a constant transition matrix. As agents can be either employed or unemployed, each agent can be in one of the $14 = 7 \times 2$ idiosyncratic states. Table 1 provides a summary of the model parameters.

Parameter	Description	Value
β	Discount factor	0.99
α	Capital share	0.36
δ	Depreciation rate	0.025
Π_{ue}^{SS}	Average job-finding rate	78.6%
$\bar{a}$	Credit limit	0
Π_{eu}^{SS}	Average job-separation rate	4.87%
S_u^{SS}	Steady-state unemployment rate	5.83%
ρ_z	Autocorrelation TFP	0.95
σ_z	Standard deviation TFP shock	0.31%
σ_{ue}	Cov. job find. rate with TFP	6
ρ_y	Autocorrelation idio. income	0.992
σ_y	Standard dev. idio. income	6.60%
χ	Scaling param. labor supply	0.04
φ	Frisch elasticity labor supply	0.5

Table 1: Parameter values in the baseline calibration. See text for descriptions and targets.

6.2 Steady-state equilibrium distribution

We simulate a Bewley model for a constant and exogenous replacement rate ϕ. The computational details are provided in Appendix C.1 and accuracy tests are discussed in Section 6.5 below.

In Table 2, we report the wealth distribution generated by the model and compare it to the empirical distribution. We compute a number of standard statistics – listed in the first column – including the quartiles, the Gini coefficient, and the 90-95 and 95-100 intercentiles.

The empirical wealth distribution, reported in the second and third columns of Table 2, is computed using two sources, the PSID for the year 2006 and the SCF for the year 2007. The fourth and fifth columns report the wealth distribution generated by our model with two different values for the exogenous replacement rate ϕ, set either to 50% (column 4) or to 42% (column 5). The former value of $\phi = 50\%$ corresponds to the standard value used in the literature for the

[13] We follow the procedure of Footnote 19 in Krueger et al. (2018). The quarterly persistence ρ_y is such that $\rho_y^4 = 0.9695$ equals the annual persistence. The variance of the log of labor income yl_y (at the quarterly frequency) is the same as the variance at the annual frequency, so: $(1+\varphi)\frac{\sigma_y^2}{1-\rho_y^2} = \frac{0.0384}{1-0.9695^2}$.

	Data		Models	
Wealth statistics	PSID, 06	SCF, 07	$\phi = 50\%$	$\phi = 42\%$
Q1	-0.9	-0.2	0.2	0.3
Q2	0.8	1.2	1.4	1.8
Q3	4.4	4.6	6.2	6.4
Q4	13.0	11.9	19.5	18.7
Q5	82.7	82.5	71.6	68.7
90-95	13.7	11.1	16.9	16.9
95-100	36.5	36.4	32.9	32.8
Gini	0.77	0.78	0.70	0.69

Table 2: Wealth distribution in the data and in the model.

US, as in Krueger et al. (2018) among others. The latter value of $\phi = 42\%$ corresponds to the optimal steady-state replacement rate that we compute below (see Section 6.4).

Overall, the distribution of wealth generated by the model is quite similar for the two replacement rate values and is close to the data. In particular, the model does a good job in matching the wealth distribution with a high Gini of 0.70. The concentration of wealth at the top of the distribution is higher in the data than in the model. It is known that additional model features must be introduced to match the high wealth inequality in the US, such as heterogeneous discount rates, as in Krusell and Smith (1998), or entrepreneurship, as in Quadrini (1999).

Steady-state effect of a change in the replacement rate. To better understand the effect of the replacement rate, Table 3 compares some steady-state statistics for two economies, featuring the replacement rate of either 42% or 50%. The first two rows correspond to the two economies while the third row corresponds to the relative change for the variable of interest, reported in the columns. It can be observed that an increase in the replacement rate of 8% decreases the capital stock, K, and steady-state consumption, C, by a small amount. Due to the higher replacement rate, unemployed agents are better insured by the UI scheme and agents thus express a smaller demand for self-insurance. They therefore save less, which diminishes total savings and capital. Furthermore, due to the higher replacement rate, the higher labor tax deters labor supply and agents' labor earnings, which has a negative impact on savings and consumption. The higher replacement rate also decreases the consumption inequality c^u/c^e between unemployed and employed workers, as unemployed agents are better insured. The last column of Table 3 reports the population average Lagrange multiplier on the credit constraint, $mean\,(\lambda)$, which is discussed below to clarify the role of the optimal replacement rate.

	$\phi(\%)$	K	C	c^u/c^e	$mean\,(\lambda)$
Economy 1	42	13.1428	0.8382	0.9823	81.02
Economy 2	50	13.1304	0.8380	0.9847	78.89
Variation (%)	$8^{\dagger}$	−0.1	−0.02	−0.24	−1.41

Note: See the text for definitions of the variables. † indicates an absolute variation, while the variation is by default a relative variation.

Table 3: Implications of a variation in the replacement rate

6.3 Model dynamics with a fixed replacement rate

We now compute the dynamics of the model with a fixed replacement rate, set to $\phi = 42\%$, and with aggregate shocks affecting both TFP and the job-finding rate. We set the truncation length to $N = 2$, which implies that we simulate $14^2 = 196$ histories. We use these steady-state allocations to compute the ξs, which ensures that the truncated model has the same truncated wealth distribution as the steady-state Bewley model. Importantly, this low value of N is enough to replicate the dynamics of the model, thanks to the ξs. The model captures relevant heterogeneity in productivity levels and transitions, as discussed in Section 4. The good quality of approximation for $N = 2$ is indirect evidence that the within-history heterogeneity has a second-order effect on the dynamics of the model – but not on the steady state – and that the time-varying heterogeneity across histories is sufficient to capture the dynamics.

For the sake of clarity, we summarize the equations of the model in Appendix B.1. We simulate the model over 10,000 periods. The dynamics of the truncated model are compared to an alternative solution method, developed by Rios-Rull (1999), Reiter (2009), and Young (2010) among others, which we call the Reiter method for brevity. This method uses a histogram representation and a perturbation technique to solve the model. The method is known to provide accurate results, when compared to the global method of Krusell and Smith (1998), as shown in Boppart et al. (2018) or in Auclert et al. (2019).[14]

The comparison of the two methods can be found in Table 4, in the columns labeled simulations (1) and (2). Column (3) reports the outcomes of a representative agent (RA) economy. This last model features a single island and one Euler equation. All parameters are otherwise the same. Economies (4)–(6) are discussed below. Table 4 reports, for each of the three economies, the mean and the normalized standard deviation (i.e., the standard deviation divided by the mean) of the main aggregate variables: output Y, aggregate consumption C, total labor L, and the replacement rate ϕ. The table also reports the autocorrelations and correlations for consumption

[14] We also compare the results of the truncation approach to those of the Boppart et al. (2018) and Auclert et al. (2019) algorithms – hereafter BKM – for TFP shocks. The results are reported in Appendix C.2. The three methods (Reiter, BKM, and truncation) generate very similar results. Finally, the Reiter method can be used with bases other than histograms, such as in Winberry (2018) or Bayer et al. (2019).

Repl. rate		Exogenous			Optimal rule		
Methods		Trunc.	Reiter	RA	Trunc.	Reiter	RA
Simulations		(1)	(2)	(3)	(4)	(5)	(6)
Y	mean	1.17	1.17	1.08	1.17	1.17	1.08
	std/mean (%)	2.07	2.06	2.45	2.64	2.71	2.81
C	mean	0.84	0.84	0.80	0.84	0.84	0.80
	std/mean (%)	1.70	1.69	2.02	2.26	2.27	2.35
L	mean	0.30	0.30	0.30	0.30	0.30	0.30
	std/mean (%)	0.93	0.92	1.08	1.67	1.70	1.47
K	mean	13.14	13.14	11.09	13.14	13.14	11.09
	std/mean (%)	1.93	1.92	2.54	2.29	2.45	2.93
ϕ	mean(ϕ)(%)	42	42	42	42	42	42
	std(ϕ)(%)	0	0	0	23	23	26
$corr(C, C_{-1})$	(in %)	99.09	99.13	99.66	98.74	99.18	99.67
$corr(Y, Y_{-1})$		97.55	97.56	98.22	97.41	97.52	98.21
$corr(C, Y)$		96.45	96.22	94.60	97.18	95.30	95.22
$corr(Y, \Phi)$		0	0	0	−93.23	−96.65	−96.29

Table 4: Moments of the simulated models for different specifications and different resolution techniques.

and output.

Overall, the Reiter and truncated methods both generate very close statistics. The standard levels of GDP and of aggregate consumption are the same, up to an order of magnitude of 10^{-4}.[15] This is not due to the fact that heterogeneity does not matter. Indeed, the RA-economy (3) significantly differs from the economies implied by the Reiter and truncated methods. For instance, the normalized standard deviation of GDP in the Reiter and truncation economies is 2.06% and 2.07%, respectively, whereas it is 2.45% in the RA-economy. More specific accuracy checks are provided in Section 6.5.

6.4 Optimal replacement rate

The optimal steady-state replacement rate is computed using the algorithm described in Section 5.4. This algorithm yields an optimal replacement rate of $\phi = 42\%$, which we used in the simulations of Section 6.3. This optimal replacement rate is obtained for $N = 2$. We have checked

[15] To quantitatively assess the role of the ξs, we have computed the dynamics of the truncated model, but with all ξs set to one. The model dynamics are then very different from those implied by the truncated or Reiter methods. For instance, the normalized standard deviation of GDP is equal to 1.9, which differs from the 2.06 and 2.07 computed by the Reiter and truncated methods, respectively.

that we obtain the same optimal replacement rate, $\phi = 42\%$, for $N = 3$, with 2,744 histories. In this last case, the computations are very slow. For this reason, we focus on the case $N = 2$ which appears to be very accurate. Once the steady-state allocations and policy instruments of the Ramsey planner have been computed, we can deduce the dynamics of the model using perturbation techniques. More details are provided in Appendices B.1 and B.2.

First, the tradeoffs faced by the planner have already been presented in the discussion of Table 3. An increase in the replacement rate reduces inequality and capital accumulation. This can be seen in the last column of Table 3, which reports the average value of the Lagrange multiplier on the Euler equation. As discussed in Section 5.1 and developed in Appendix D.1, this average value is positive when the planner perceives that agents save too much. The mean is negative when the planner perceives that there are not enough savings. In both economies, the mean is positive ($mean\,(\lambda) > 0$), indicating that agents save too much on average. The mean decreases with the higher replacement rate, since agents save less due to a lower precautionary motive. Increasing the replacement rate is a distorting tool that disincentivizes savings as it decreases labor supply and consumption.

Second, the dynamic properties of the replacement rate, solved in the truncated economy, are reported in economy (4) of Table 4. The average replacement rate is 42% and its standard deviation is 23%. The comparison between economies (2) (with a fixed replacement rate of 42%) and (4) shows the effect of a time-varying replacement rate. It can be seen that the replacement rate is countercyclical, since $corr(Y, \phi) = -0.93 < 0$. It increases in recessions and decreases in booms. As the replacement rate is countercyclical, labor supply, aggregate consumption, and output are more procyclical in economy (4) than in economy (2).[16]

Optimal replacement rate dynamics. We now investigate whether a relatively simple rule can reproduce the dynamics of the optimal replacement rate in the truncated economy. We simulate the model for 10,000 periods and regress the replacement rate ϕ_t on several moments of aggregate variables. We find that the following rule has a very high $R^2 = 0.99999$:

$$\phi_t = (1 - a_1^\phi - a_2^\phi)\phi^{ss} + a_1^\phi \phi_{t-1} + a_2^\phi \phi_{t-2} + a_0^\varepsilon \varepsilon_t^z + a_1^\varepsilon \varepsilon_{t-1}^z + a_2^\varepsilon \varepsilon_{t-2}^z + a^K (K_{t-1} - \bar{K}) + \varepsilon_t^\phi, \quad (46)$$

with $(\phi^{ss}, a_1^\phi, a_2^\phi, a_0^\varepsilon, a_1^\varepsilon, a_2^\varepsilon, a^K) = (0.4200, -24.4772, 0.0104, 0.6048, -0.3317, 0.3669, -0.0326)$.[17]

We can now plug this estimated rule into the model and simulate it using the Reiter method. The purpose of this experiment is to check that the moments of the economy featuring a carefully

[16] In the literature, results about the cyclicality of the replacement rate are mixed. Mitman and Rabinovich (2015) find a procyclical replacement rate, whereas Landais et al. (2018a,b) find a countercyclical one. These papers study economies without capital, but with a much more detailed model of the labor market. We instead investigate the implication of an endogenous capital stock, but with a simpler labor market structure.

[17] We also estimated a simpler rule. We regress the replacement rate ϕ_t on the technology shock z and on the first, second, and third-order moments of the wealth distribution, to see whether moments of the wealth distribution can be sufficient statistics. We find a low R^2 of 0.73. It appears that a rich time structure is necessary to reproduce the dynamics of the optimal replacement rate.

estimated rule for the replacement rate and simulated using the Reiter method are close to the moments generated by the Ramsey model simulated by the truncation method. The results are provided in Economy (5) of Table 4. The moments are very close in both economies. In addition, in Appendix C.3, we provide the IRFs generated by both methods after a positive TFP shock and it can be seen that they are also close to each other, especially for consumption and labor affecting the utility. This confirms that the assumption of constant within-history heterogeneity is quantitatively reasonable. Again, this is not because heterogeneity has no role to play. Economy (6) of Table 4 in fact corresponds to an RA-economy with the optimal rule of equation (46) for the replacement rate and it can be verified that its moments are significantly different from those of Economies (4) and (5).

Alternative rules. To check the optimality of the time-varying rule of equation (46), we also simulate the model using the Reiter method and compute the related aggregate welfare while changing the coefficients in the rule. We first simulate an economy with a procyclical replacement rate, where we change the signs of a_0^ε, a_1^ε, a_2^ε, and a^K. The results are reported in Appendix C.3. We find that aggregate welfare decreases with this new rule: a number of agents now have a low consumption level and a high marginal utility in recessions. Second, we simulate the economy with a modified rule featuring the same cyclicality as the original rule, but a higher variance – such that the standard deviation of ϕ_t is now 35% instead of 23%. Welfare is again decreasing because the replacement rate falls considerably in good times and the consumption of unemployed agents consequently falls. From this experience, we can be confident of the rule's optimality – and thereby of the Ramsey program implied by the truncation method. We also better understand the role of the optimal rule in the business cycle, which attempts to stabilize the consumption of low-utility agents.

6.5 Convergence and additional accuracy tests

Convergence properties. We now report some convergence statistics when the length of the truncation increases. For any given N, we compute $(\xi_{s^N})_{s^N \in \mathcal{S}^N}$ as explained in Section 4 and then deduce the standard deviation $std(\xi)$ across histories. We also quantify the amount of risk sharing by providing the standard deviation of the "pooling" transfers Γ^*_{N+1} defined in Proposition 1, normalized by the total income $Inc_{s^{N+1}}$ of agents with history s^{N+1} ($Inc_{s^{N+1}} = w\theta_{s_0^{N+1}} l_{s^{N+1}} + \delta 1_{s_0^{N+1}=0} + (1+r)a_{s^{N+1}}$). This normalized standard deviation will be denoted by $std\,(\Gamma)$. This quantity captures the amount of within-history wealth heterogeneity, which needs to be compensated for, in order for the truncation representation to be consistent with the Bewley model. The two statistics, $std(\xi)$ and $std\,(\Gamma)$, for $N = 2$ to $N = 4$, are reported in Table 5. It can be observed that the standard deviations $std(\xi)$ and $std\,(\Gamma)$ decrease with N,

Truncation length N	Number of histories	$std(\xi)$ (%)	$std\,(\Gamma)$ (%)
2	196	34.1	10.03
3	2,744	32.2	9.99
4	38,416	32.1	9.95

Table 5: Convergence properties of the truncated method with different truncation lengths N.

but at a slow rate.[18] As the number of idiosyncratic states is high, equal to 14, the number of histories grows rapidly with N, which computationally limits the maximal truncation length. As explained above, our choice of N is not based on the minimization of the standard deviations of ξ or Γ. Indeed, we have shown in Section 6.3 that the case $N = 2$ accurately captures the relevant amount of heterogeneity and thus the dynamics of the model.

Accuracy tests. As an additional validation test for our truncation method, we perform several standard accuracy tests. We start with Euler Equation error tests (see Den Haan and Marcet, 1994, Aruoba et al. 2006, and Den Haan, 2010) on both the steady-state model and the model with aggregate shocks. They consist in computing the absolute errors (on a base-10 log scale) implied by the exact Euler equations using the simulated allocations. The results (including mean, standard deviation, and distribution of errors) are reported in Table 6. A

	Static	Dynamic	
	Bewley	Exo. ϕ	Endo. ϕ
Average	-3.91	-4.08	-3.95
Std. dev.	0.63	0.57	0.70
$[-2, \infty)$	0.00	0.00	0.00
$[-3, -2)$	1.47	3.01	5.11
$[-4, -3)$	96.57	32.14	53.93
$[-5, -4)$	1.78	59.72	33.73
$[-6, -5)$	0.06	4.59	5.94
$(-\infty, -6)$	0.10	0.53	1.29

Table 6: Euler equation errors

value of -3 for this error, which is approximately the value found in the three cases, means a \$1 mistake for \$1,000 of consumption. This is generally considered as being an acceptable error, as discussed in Faraglia et al. (2019) among others.

[18] More generally, simulating different economies, we find that these standard deviations decrease faster when the persistence of the idiosyncratic shocks is low, not necessarily when the number of idiosyncratic states is small.

For the steady-state model, the results can be found in the column of Table 6 labeled Bewley. For the benchmark economy with $\phi = 42\%$, the average error amounts to -3.91, which lies within the admissible range. The results for the model with aggregate shocks are reported in the third and fourth rows of Table 6. This accuracy test is all the more important since we use a perturbation approach, which ignores some potential non-linearities. We consider two cases for the replacement rate: an exogenous ϕ, set to 42%, (third column) and an optimal time-varying rate (fourth column). The mean absolute log error is -4.08 in the exogenous case and -3.95 in the endogenous one, which are in the admissible range.

Our second set of tests concerns the assumption that credit-constrained histories remain constant in the dynamics (second point of Assumption A). The test is run as follows. We simulate the model – with both exogenous and endogenous replacement rates – over $10,000$ periods. We then check that the saving decision of each unconstrained history remains above the credit limit in the simulations. Conversely, we also check that credit-constrained histories remain constrained in the dynamics by checking the sign of the Euler inequality. We find that savings remain positive for all unconstrained histories and that all Euler inequalities have the correct sign for constrained histories. Finally, as noted by Den Haan (2010), these accuracy tests are not sufficient to characterize the overall goodness-of-fit of a simulation method. We consider the results of Table 4 as the main evidence of the truncation method's relevance.

7 Conclusion

This paper presents a truncation representation of incomplete insurance market models with aggregate shocks. We construct a limited heterogeneity representation, which can be easily simulated with aggregate shocks, and for which optimal Ramsey policies can be derived. We apply the theory to characterize optimal time-varying unemployment benefits when the economy is hit by aggregate shocks. The model simulation uses perturbation methods, which considerably eases implementation. Such methods, however, rely on small aggregate shocks around a well-defined steady state. They are less relevant for models with large macroeconomic shocks, for which additional developments are needed, using penalty functions or global methods.

The theory could obviously be used for many other applications. The underlying model could be generalized to examine relevant frictions on the goods, labor, or financial markets, such as limited participation on financial markets or nominal frictions. In addition, the planner could use other tools to reduce distortions, such as a whole set of fiscal or monetary policy instruments. We are currently working on the design of general optimal fiscal policies in these environments. The simplicity of the implementation could contribute to a more systematic integration of redistributive effects in the design of economic policies.

Appendix

This Appendix is split into four main parts. Section A contains the proofs of the paper. Section B presents the details of our numerical method, such as algorithms and full-fledged list of equations. Section C contains a number of robustness checks and additional results of our numerical implementation. Section D regroups the supplemental theoretical results (penalty functions and the extension to non-GHH utility functions).

A Proofs

A.1 Proof of Proposition 1

We use a guess-and-verify strategy. The transfer is constructed such that all agents with the same N-period history have the same after-transfer wealth. The measure of agents with history s^N follows the same law of motion as (16) in the island economy and is equal to S_{s^N}. If agents with the same history $(\hat{s}^N, s), s \in \mathcal{S}$ have the same beginning-of-period wealth $a_{\hat{s}^N}$, the after-transfer wealth, denoted by $\hat{a}_{s^N}$, of agents with history $s^N \succeq \hat{s}^N$ is:

$$\hat{a}'_{s^N} = \sum_{\tilde{s}^N \in \mathcal{S}^N} \frac{S_{\tilde{s}^N}}{S_{s^N}} \Pi_{\tilde{s}^N, s^N} a'_{\tilde{s}^N}, \tag{47}$$

such that agents with the same history hold the same wealth. By construction, $\hat{a}_{s^N}$ follows dynamics similar to the "after-pooling" wealth $\tilde{a}_{t,s^N}$ in the island economy of equation (17). The transfer scheme denoted by $(\Gamma^*_{N+1}(s^{N+1}, X))_{s^{N+1} \in \mathcal{S}^{N+1}}$ that enables all agents with the same history to have the same wealth is:

$$\Gamma^*_{N+1}(s^{N+1}, X) = (1 + r) \left(\hat{a}_{s^N} - a_{\hat{s}^N}\right), \tag{48}$$

where we use $s^{N+1} = (\hat{s}^N, s) = (s_N, s^N)$ (in the former notation, s^{N+1} is seen as the history $\hat{s}^N \in \mathcal{S}^N$ with the successor state $s \in \mathcal{S}$, while in the latter notation, s^{N+1} is seen as the state $s_N \in \mathcal{S}$ followed by history $s^N \in \mathcal{S}^N$). The transfer Γ^*_{N+1} defined in (48) replaces the beginning-of-period wealth $(1 + r)a_{\hat{s}^N}$ with the *average* wealth $(1 + r)\hat{a}_{s^N}$, which only depends on the last N-period history. Since there is a continuum with mass $S_{\tilde{s}^N}$ of agents with history $\tilde{s}^N$, in which each individual agent is atomistic, all agents take the transfer Γ^*_{N+1} as given.

Finally, it is easy to check that the transfer scheme is balanced in each period. Using the definition (47) of $\hat{a}_{s^N}$, we obtain for $s^N = (s^N_{N-1}, \ldots, s^N_1, s^N_0) \in \mathcal{S}^N$, $S_{s^N}\hat{a}_{s^N} = \sum_{\hat{s}^N \in \mathcal{S}^N} S_{\hat{s}^N} \Pi_{\hat{s}^N, s^N} a_{\hat{s}^N} = \sum_{\hat{s} \in \mathcal{S}} S_{(\hat{s}, s^N_{N-1}, \ldots, s^N_1)} M_{s^N_1, s^N_0} a_{(\hat{s}, s^N_{N-1}, \ldots, s^N_1)}$. Therefore, we deduce that: $\sum_{\tilde{s} \in \mathcal{S}} S_{(\tilde{s}, s^N)} \Gamma^*_{N+1}(\tilde{s}, s^N) = (1 + r) \left[\sum_{\tilde{s} \in \mathcal{S}} S_{(\tilde{s}, s^N)} \left(\hat{a}_{s^N} - a_{(\tilde{s}, s^N_{N-1}, \ldots, s^N_1)}\right)\right] = 0$, where the last equality comes from the definition of $\hat{a}_{s^N}$ in equation (47).

A.2 Proof of the convergence of the "truncated" competitive equilibrium

The proof is performed is several steps: (i) we first show the convergence of allocations in the absence of aggregate shocks; (ii) we then show that the preference shifters $(\xi_{s^N})_{s^N}$ converge to 1; and (iii) we finally prove convergence in the presence of aggregate shocks in the context of a resolution via perturbation method.

A.2.1 Proof of the first part of Proposition 3

In this proof, there is no aggregate shock. The starting point of this proof is to remember that the allocations of the truncated model are computed using the aggregation of Bewley allocations. We recall equation (29) for the truncation of a variable X, which can be rewritten as follows:

$$X_{t,s^N} = \sum_{s^t \in \mathcal{S}^t \big| (s^t_{t-N+1},\ldots,s^t_t)=s^N} X_t(s^t,z^t) \frac{\mu_t(s^t)}{\sum_{(s^t_{t-N+1},\ldots,s^t_t)=s^N} \mu_t(s^t)}, \tag{49}$$

The variable X_{t,s^N} is thus the integral of X over some specific set, which is defined as the histories whose last N periods match the vector s^N. As N gets larger, the sum is defined over an increasingly small set. Loosely speaking, a generalized version of the fundamental theorem of calculus then guarantees the convergence of X_{t,s^N} toward the Bewley allocation.

To formalize this loose statement, we need to introduce some notation. Let $s^\infty = (s_0, s_1, \ldots)$ be an infinite idiosyncratic history, which can also be seen as a Markov chain where each s_t belongs to $(\mathcal{S}, \mathcal{F}_\mathcal{S})$, where $\mathcal{F}_\mathcal{S}$ are the σ-algebras generated by $\mathcal{S}$ (which is finite in our truncated case). The whole history s^∞ lies in the sequence space $\Omega = \mathcal{S} \times \mathcal{S} \times \ldots = \mathcal{S}^\infty$ endowed with the product σ-algebra $\mathcal{F}_\infty = \mathcal{F}_\mathcal{S}^\infty$ and the measure μ_∞. We recall that the set Ω is uncountably infinite and has the cardinality of the continuum. The measure μ_∞ exists and is defined as the infinite product measure that coincides with the standard Markov distribution for any finite sequence. This measure can be shown to be the limit of the product measure of transition kernels (generalizing the transition matrix in the case of a non-finite state space $\mathcal{S}$). An important feature of the measure μ_∞ is that it is consistent with the usual Markov measure for any finite sequence. The proof of the existence of an infinite product measure is in general quite involved and relies on the Kolmogorov extension theorem (see Tao, 2011, Theorem 2.4.3). In our case of a finite Markov chain, the infinite measure is uniquely determined by its initial distribution and its transition matrix (see Brémaud, 2014, Theorem 1.1).

We now consider the probability space $(\Omega, \mathcal{F}, \mu_\infty)$. We denote by $\mathcal{S}_N$ the partition of idiosyncratic histories, induced by the truncation: an idiosyncratic history is uniquely defined by its truncation over the last N periods. Two infinite histories with the same last N idiosyncratic realization belong to the same truncated history. A truncated history $s_N \in \mathcal{S}_N$ – that can be uniquely represented by a vector $s^N \in \mathcal{S}^N$ – can then be seen as a subset of Ω and the truncation of the variable X in equation (29) can be written as: $X_{t,s^N} := X_{t,s_N} = \int_{s^\infty \in s_N} X_t(s^\infty) \frac{\mu_\infty(ds^\infty)}{\int_{s^\infty \in s_N} \mu_\infty(ds^\infty)}$.

We denote by $\mathcal{F}_N$ the filtration associated with the partition $\mathcal{S}_N$. The conditional probability $\mathbb{E}\left[X_t|\mathcal{F}_N\right]$ – which is a random variable – verifies for any event $s^\infty \in \Omega$: $\mathbb{E}\left[X_t|\mathcal{F}_N\right]_{s^\infty} = X_{t,s^N}$, where s^N is the label of the unique partition element s_N containing s^∞. In other words, the restriction of the conditional expectation to h_n coincides with X_{t,s^N}. We can state two additional properties on the filtration sequence $(\mathcal{F}_N)_{N\geq 0}$.

1. The partition sequence $(\mathcal{S}_N)_{N\geq 0}$ is increasing by construction and is such that $\mathcal{S}_{N+1}$ is a refinement of $\mathcal{S}_N$, in the sense that any element of $\mathcal{S}_N$ is a union of elements of $\mathcal{S}_{N+1}$. The filtration sequence $(\mathcal{F}_N)_{N\geq 0}$ is thus increasing: $\mathcal{F}_N \subset \mathcal{F}_{N+1}$.
2. The partition sequence $(\mathcal{S}_N)_{N\geq 0}$ converges to the atoms of Ω – in other words, histories become infinitely long – which implies $(\mathcal{F}_N) \uparrow \mathcal{F}_\infty$.

To conclude the convergence proof, we apply the convergence theorem for conditional expectation (see Billingsley, 1965, Theorem 11.2), which yields: $\mathbb{E}\left[X|\mathcal{F}_N\right]_{s^\infty} \to_{N\to\infty} X(s^\infty)$, almost surely. This concludes the proof of the first part of Proposition 3.

A.2.2 Proof of the second part of Proposition 3

Let $(\mathcal{S}_N)_{N\geq 0}$ be an increasing partition sequence associated with history truncation, as described in Section A.2.1. We will define $\xi^u_{s^N} = \frac{\mathbb{E}\left[U_c(c_{s^N},\hat{l}_{s^N})|\mathcal{F}_N\right]_{s^\infty}}{U_c(\mathbb{E}\left[c_{s^N}|\mathcal{F}_n\right]_{s^\infty},\mathbb{E}\left[\hat{l}_{s^N}|\mathcal{F}_n\right]_{s^\infty})}$ and $\Pi^u_{t+1,\tilde{s}^N s^N} = \Pi_{t+1,\tilde{s}^N s^N}\frac{\sum_{s^t\in\mathcal{S}^t|(s^t_{t-N+1},\ldots,s^t_t)=\tilde{s}^N} U_{c,t+1,(s^t,s^N_0)}\frac{\mu_t(s^t)}{S_{t,\tilde{s}^N}}}{U_{c,t+1,s^N}}$ (where $U_{c,t+1,s^N} = U_c(c_{t+1,\tilde{s}^N},\hat{l}_{t+1,\tilde{s}^N}) > 0$ for all s^N). We then check that truncating the individual Euler equation (8) yields:

$$\xi^u_{s^N} U_{c,s^N} = \nu_{s^N} + \beta(1+r)\sum_{\tilde{s}^N\in\mathcal{S}^N}\Pi^u_{s^N\tilde{s}^N}\xi^u_{\tilde{s}^N}U_{c,\tilde{s}^N}, \tag{50}$$

which is similar to the Euler equation (21) for islands, except that preference shifters and transition probabilities are different (these preference shifters ξ^u and probabilities Π^u are actually constructed for this equality to be exactly true). The convergence result of Section A.2.1 and the continuity of U_c imply that $\xi^u_{t,s^N} \to \frac{U_c(c(s^\infty),\hat{l}(s^\infty))}{U_c(c(s^\infty),\hat{l}(s^\infty))} = 1$, almost surely. Similarly, we have:

$$U_{c,t+1,\tilde{s}^N}(\Pi^u_{t+1,s^N\tilde{s}^N} - \Pi_{t+1,s^N\tilde{s}^N}) = \Pi_{t+1,\tilde{s}^N s^N}\sum_{(s^t_{t-N+1},\ldots,s^t_t)=s^N}(U_{c,t+1,(s^t,s^N_0)} - U_{c,t+1,\tilde{s}^N})\frac{\mu_t(s^t)}{S_{t,\tilde{s}^N}},$$

which, using Section A.2.1 and the continuity of U_c, can be shown to converge to 0.

We consider the steady-state difference between equation (50) and the truncated (island) consumption Euler equation for a history $s^N \in \mathcal{S}^N$. After some manipulation, we obtain:

$$(\xi^u_{s^N} - \xi_{s^N})U_{c,s^N} = \beta(1+r) \sum_{\tilde{s}^N \in \mathcal{S}^N} \Pi_{s^N \tilde{s}^N}(\xi^u_{\tilde{s}^N} - \xi_{\tilde{s}^N})U_{c,\tilde{s}^N} + \beta(1+r) \sum_{\tilde{s}^N \in \mathcal{S}^N} (\Pi^u_{s^N \tilde{s}^N} - \Pi_{s^N \tilde{s}^N})\xi^u_{\tilde{s}^N} U_{c,\tilde{s}^N}. \tag{51}$$

Note that the second term has just been proved to converge to 0.

All previous equations now have to be stacked such that (51) is written using a matrix notation. Indeed, a history s^N can be seen as an N-length numeric vector $\{(y_{-N+1}, e_{-N+1}), \ldots, (y_0, e_0)\}$, where $e_k = 0$ if the agent is unemployed and $e_k = 1$ if she is employed, and $y_k = 1, \ldots, Y$ denotes her productivity level. The number of histories is $N_{tot} = S^N$ where $S = Card(\mathcal{S})$. We can identify each history by the integer $k_{s^N} = 1, \ldots, N_{tot}$:

$$k_{s^N} = \sum_{k=0}^{N-1} N_{tot}^{-N+1-k}(e_k \times Y + y_k - 1) + 1, \tag{52}$$

which corresponds to an enumeration in base S. In this enumeration, the first $N_{tot}/2$ histories are histories where agents are currently unemployed.

Let $\mathbf{U_c}$, $\boldsymbol{\xi}$ and $\boldsymbol{\xi}^u$ be the N_{tot}-vectors of end-of-period marginal utilities, and preference shifters. Define as $\mathbf{I}$ the identity matrix and $\mathbf{\Pi} = (\mathbf{\Pi}_{kk'})_{k,k'=1,\ldots,N_{tot}}$ as the transition matrix from history k to history k'. Equation 51 can be written as:

$$(\mathbf{I} - \beta(1+r)\mathbf{\Pi})((\boldsymbol{\xi}^u_k - \boldsymbol{\xi}_k)\mathbf{U}_{c,k})_k = \beta(1+r)(\sum_{k'=1}^{N_{tot}} (\mathbf{\Pi}^u_{k,k'} - \mathbf{\Pi}_{kk'})\boldsymbol{\xi}^u_{k'}\mathbf{U}_{c,k'})_k,$$

Then since $\beta(1+r)$ is independent of N and verifies $\beta(1+r) < 1$, the matrix $\mathbf{I} - \beta(1+r)\mathbf{\Pi}$ is invertible and its eigenvalues are bounded away from 0 for all N We finally observe that $((\xi^u_{s^N} - \xi_{s^N})U_{c,s^N})_{s^N}$ can be made arbitrarily small and that $\xi_{s^N} \to 1$, which concludes the proof.

A.3 Proof of Proposition 4

We now turn to the convergence of allocations computed using a first-order perturbation method. The proof relies on Villemot (2011), who provides a detailed account of the first-order perturbation technique. Using Villemot (2011)'s notation, it is shown that the first-order perturbation method involves writing the vector of endogenous variables, denoted by y_t, as a function of the subset of endogenous variables appearing with a lag (y^-_{t-1}), as well as the vector of exogenous variables u_t. More precisely, at the first-order, the perturbation is defined by the following recursion: $y_t = \overline{y} + g_y y^-_{t-1} + g_u(u_t - \overline{u})$, where quantities with an overbar denotes steady-state values. In our case, all of these quantities apart from u_t (which represents aggregate shocks) depend on the truncation N length, such that the recursion characterizing the perturbated system can be rewritten as $y^N_t = \overline{y}^N + g^N_y y^{N,-}_{t-1} + g^N_u(u_t - \overline{u})$. Nonexplosive dynamics impose the condition that the module of eigenvalues of g_y are smaller than one. We now show that the sequence $(y^N)_N$ can be seen as a Cauchy sequence in $\ell_\infty(\mathbb{N})$ – thereby proving convergence. The key point is to use the first point of the proof of Section A.2.1, stating that any truncation of length $N+1$ is a refinement of a truncation of length N. It is therefore meaningful to consider (up to this transformation) the difference between y_t^N and y_t^{N+1},which can be written as:

$$y_t^{N+1} - y_t^N = \overline{y}^{N+1} - \overline{y}^N + (g_y^{N+1} - g_y^N)y_{t-1}^{N+1,-} + g_y^N(y_{t-1}^{N+1,-} - y_{t-1}^{N,-}) + (g_u^{N+1} - g_u^N)(u_t - \overline{u}),$$

where the terms $\overline{y}^{N+1} - \overline{y}^N$, $g_y^{N+1} - g_y^N$ and $g_u^{N+1} - g_u^N$ can be made arbitrarily small thanks to the fact that they only depend on steady-state values and that we have a convergence result for steady-state allocations. Since the dynamic system is non-explosive and since the limit of the initial points is well-defined by construction ($\lim_N y_0^{N,-} = y_0^-$), we deduce that $(y_t^N)_N$ is a Cauchy sequence and thereby converges for any t when N increases.

We make a final remark about the existence of a well-defined perturbation method – in particular the conditions of Blanchard and Kahn (1980). If the perturbation is well-defined for the full-fledged model, then it will be well-defined for sufficiently large N.

A.4 Proof of Proposition 5

Only the proof for the steady state needs to be demonstrated. The proof for the perturbation method is not specific and follows the same lines as the proof in Section A.3.

The proof for the steady state runs as follows. Our optimization procedure selects the UI policy, such that the associated projected Bewley allocations solves the FOCs (43) and (44) – where the ξs are computed from the same Bewley equilibrium allocations. More precisely, the procedure is as follows (for a given N):

1. Select a UI policy, i.e. a pair (τ, ϕ) such that the UI budget is balanced and (14) holds.

2. Compute the associated Bewley equilibrium and deduce from these allocations, the steady-state truncated allocations $(a_{s^N}, c_{s^N}, l_{s^N})_{s^N \in \mathcal{S}^N}$, as well as the preference shifters $(\xi_{s^N})_{s^N \in \mathcal{S}^N}$, using the method of Section B.2.1.

3. Use the Ramsey FOC (43) at the steady-state to compute the Ramsey Lagrange multipliers $(\lambda_{s^N})_{s^N \in \mathcal{S}^N}$. See Section B.2.2 for further details.

4. If the Ramsey FOC (44) holds, then the steady-state Ramsey equilibrium corresponds to the UI policy under consideration and to the associated Bewley equilibrium. If not, iterate on the UI policy and go back to Step 1.

This enables to characterize the optimal policy (ϕ_N, τ_N) that we index in this section by N.

Points 1 to 3 allow us to write allocations and Lagrange multipliers as a function of the replacement rate ϕ (for any N). In Point 4, the truncated FOC (44) at the steady state implies that for any N, there exists a function $f_{\phi,N}$ such that the optimal replacement rate τ_N is characterized by $f_{\phi,N}(\phi_N) = 0$.

We now focus on the full-fledged Ramsey program, which can be expressed as follows:

$$\max_{((a_t^i,c_t^i,l_t^i)_{i\in\mathcal{I}},\phi_t,\tau_t,r_t,w_t)_{t\geq 0}} \mathbb{E}_0\Big[\sum_{t=0}^{\infty}\beta^t\int_i U(c_t^i,\hat{l}_t^i)\ell(di)\Big], \tag{53}$$

$$u'(c_t^i - \chi^{-1}\frac{\hat{l}_t^{i,1+1/\varphi}}{1+1/\varphi}) = \beta\mathbb{E}_t\Big[(1+r_{t+1})u'(c_{t+1}^i - \chi^{-1}\frac{\hat{l}_{t+1}^{i,1+1/\varphi}}{1+1/\varphi})\Big] + \nu_t^i, \tag{54}$$

$$(l_t^i)^{1/\varphi} = \chi(1-\tau_t)w_t y_t^i, \tag{55}$$

$$c_t^i + a_t^i = (1+r_t)a_{t-1}^i + ((1-\tau_t)l_t^i 1_{e_t^i=e} + \phi_t\bar{l}_t\left(y_t^i\right)1_{e_t^i=u})y_t^i w_t, \tag{56}$$

$$\int_i a_t^i\ell(di) = K_t \text{ and } L_t = \int_{i\in\mathcal{I}\setminus\mathcal{U}_t} y_t^i l_t^i \ell\left(di\right), \tag{57}$$

$$c_t^i > 0,\ a_t^i > -\bar{a}, \tag{58}$$

and subject to equations (14), and (3). As in the truncated case, we define the social valuation of liquidity for agent $i\in\mathcal{I}$, denoted by ψ_t^i, as:

$$\psi_t^i = U_c(c_t^i,\hat{l}_t^i) - (\lambda_t^i - (1+r_t)\lambda_{t-1}^i)U_{cc}(c_t^i,\hat{l}_t^i),$$

where λ_t^i is the Lagrange multiplier on the Euler equation in the Ramsey program. This is the equivalent of Ψ_{t,s^N} for an individual agent. The FOC on the asset choice a_t^i can be written as:

$$\begin{aligned}\psi_t^i &= \beta\mathbb{E}_t\left[(1+r_{t+1})\psi_{t+1}^i\right] + \beta\frac{\alpha\left(1-\alpha\right)}{1+\alpha\varphi}\frac{1}{L_{t+1}}\left(\frac{K_t}{L_{t+1}}\right)^{\alpha-1} \\ &\times\Big\{\mathbb{E}_t\Big[\left(1-\tau_{t+1}\right)\int_i\psi_{t+1}^i l_{t+1}^i y_{t+1}^i 1_{e_{t+1}^i=e}\ell(di)\Big] + \left(1+\varphi\right)\mathbb{E}_t\Big[\phi_{t+1}\int_i\psi_{t+1}^i l_{t+1}^i y_{t+1}^i 1_{e_{t+1}^i=u}\ell(di)\Big] \\ &-\left(\frac{K_t}{L_{t+1}}\right)^{-1}\mathbb{E}_t\left[\int_i\left(\lambda_t^i U_c(c_{t+1}^i,\hat{l}_{t+1}^i) + \psi_{t+1}^i a_t^i\right)\ell(di)\right]\Big\}.\end{aligned} \tag{59}$$

or at the steady-state:

$$\begin{aligned}\psi^i &= \beta\mathbb{E}_t\left[(1+r)\psi^i\right] + \beta\frac{\alpha\left(1-\alpha\right)}{1+\alpha\varphi}\frac{1}{L}\left(\frac{K}{L}\right)^{\alpha-1} \\ &\times\Big\{\left(1-\tau\right)\int_i\psi^i l^i y^i 1_{e_0^i=e}\ell(di)\Big] + \left(1+\varphi\right)\phi\int_i\psi^i l^i y^i 1_{e_0^i=u}\ell(di)\Big] \\ &-\left(\frac{K}{L}\right)^{-1}\int_i\left(\lambda_t^i U_c(c^i,\hat{l}^i) + \psi^i a^i\right)\ell(di)\Big\}.\end{aligned} \tag{60}$$

We already know that for large N, truncated allocations $(a_{s^N}, c_{s^N}, l_{s^N})_{s^N\in\mathcal{S}^N}$ converge to (exact) Bewley allocations and that preference shifters $(\xi_{s^N})_{s^N\in\mathcal{S}^N}$ converge to 1. Therefore, (60) is the limit of the truncated FOC (43) at the steady state, which implies that $(\lambda_{t,s^N})_{s^N\in\mathcal{S}^N}$ converge at the steady state to individual Lagrange multipliers. This holds for any replacement rate.

We now turn to the FOC with respect to τ_t:

$$\begin{aligned}&\frac{S_{t,e}}{S_{t,u}}\left(\frac{1}{\varphi}+1-\frac{1-\alpha}{1-\tau_t}\right)\int_i \psi_t^i\frac{l_t^i y_t^i}{L_t}1_{e_t^i=u}\ell(di)= \\ &\frac{\alpha}{(1-\tau_t)K_{t-1}}\int_i\left(\lambda_{t-1}^i U_c(c_{t+1}^i,\hat{l}_{t+1}^i)+\psi_{t+1}^i a_{t-1}^i\right)+\frac{1}{\varphi}\int_i\psi_t^i\frac{l_t^i y_t^i}{L_t}1_{e_t^i=e}\ell(di),\end{aligned} \tag{61}$$

whose steady-state formulation is straightforward to write. This implies that the optimal replacement rate ϕ_* is characterized by $f_\phi(\phi_*)=0$ (assuming existence of the full-fledged Ramsey equilibrium), where the function f_ϕ is the limit of $(f_{\tau,N})$ for large N (since these functions solely depend on steady-state allocations, the limit of which is the one of the full-fledged Bewley model). With a continuity argument stating that $f_{\tau,N}$ and f_ϕ are continuous, we can deduce that $\phi_*=\lim_N \phi_N$. It remains to prove the continuity of the functions $f_{\tau,N}$ and f_ϕ. This is a direct consequence of the Berge theorem – which requires the compactness of the choice set. This is guaranteed by the upper bound $a_{\max}$ on saving choices and the Tychonoff theorem. This concludes the proof and states the truncated Ramsey equilibria at the steady state converge to the (steady-state) full-fledged Ramsey equilibria.

A.5 Deriving the Ramsey program

A.5.1 Rewriting the Ramsey program

Let us recall that: $\hat{l}_{t,s^N}=l_{t,s^N}1_{e_0^N=e}+\zeta_{y_0^N}1_{e_0^N=u}$. The planner's program can be written as:

$$\max_{\left((a_{t,s^N},c_{t,s^N},l_{t,s^N})_{s^N},\phi_t,\tau_t,r_t,w_t\right)_{t\geq 0}} \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\sum_{s^N\in\mathcal{S}^N}S_{t,s^N}\xi_{t,s^N}U(c_{t,s^N},\hat{l}_{t,s^N})\right], \tag{62}$$

$$\xi_{t,s^N}U_c(c_{t,s^N},\hat{l}_{t,s^N})-\nu_{t,s^N}=\beta\mathbb{E}_t\Big[(1+r_{t+1})\sum_{\tilde{s}^N\in\mathcal{S}^N}\Pi_{t+1,s^N\tilde{s}^N}\xi_{t+1,\tilde{s}^N}U_c(c_{t+1,\tilde{s}^N},\hat{l}_{t+1,\tilde{s}^N})\Big], \tag{63}$$

and subject to (37), (24), (14), and (3). Let $\beta^t S_{t,s^N}\lambda_{t,s^N}$ be the Lagrange multiplier on the Euler equation. With $\nu_{t,s^N}\lambda_{t,s^N}=0$, the planner's objective, denoted by J, becomes:

$$\begin{aligned}J=\mathbb{E}_0\Big[\sum_{t=0}^{\infty}\beta^t\sum_{s^N\in\mathcal{S}^N}S_{t,s^N}\xi_{s^N}U(c_{t,s^N},\hat{l}_{t,s^N})\Big]-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_{s^N\in\mathcal{S}^N}S_{t,s^N}\lambda_{t,s^N}\\ \times\bigg(\xi_{s^N}U_c(c_{t,s^N},\hat{l}_{t,s^N})-\beta\mathbb{E}_t\bigg((1+r_{t+1})\sum_{\tilde{s}^N\in\mathcal{S}^N}\Pi_{t+1,s^N\tilde{s}^N}\xi_{s^N}U_c(c_{t+1,\tilde{s}^N},\hat{l}_{t+1,\tilde{s}^N})\bigg)\bigg),\end{aligned}$$

After some rewriting and using the definition (39) of $\Lambda_{t+1,\tilde{s}^N}$, we get:

$$J=\mathbb{E}_0\sum_{t,s^N}\beta^t\left(S_{t,s^N}\xi_{t,s^N}U(c_{t,s^N},\hat{l}_{t,s^N})-\Big(\lambda_{t,s^N}-(1+r_t)\Lambda_{t,s^N}\Big)\xi_{t,s^N}U_c(c_{t,s^N},\hat{l}_{t,s^N})\right). \tag{64}$$

A.5.2 Solving the Ramsey program

The Ramsey program then consists in maximizing J in (64) over $((a_{t,s^N}, c_{t,s^N}, l_{t,s^N})_{s^N}, \phi_t, \tau_t)_{t\geq 0}$ subject to the relevant constraints. Substituting for ϕ_t, r_t, w_t yields the Lagrangian:

$$\mathcal{L} = \mathbb{E}_0 \sum_{t,s^N} \beta^t S_{t,s^N} \xi_{s^N} \left(U(c_{t,s^N}, \hat{l}_{t,s^N}) - (\lambda_{t,s^N} - (1+r_t)\Lambda_{t,s^N}) U_c(c_{t,s^N}, \hat{l}_{t,s^N}) \right), \tag{65}$$

subject to equations (37), (24), (14), and using (3) to substitute for the real interest rate r_t and the real wage w_t. The Lagrangian (65) can be seen as depending only on saving choices (a_{t,s^N}) and the replacement rate ϕ_t. We now compute FOCs.

A.5.3 FOC with respect to saving choices a_{t,s^N}

We compute the FOC of the Lagrangian (65) with respect to a_{t,s^N}. We use the notation $U_{c,t,s^N} = u'(c_{t,s^N} - \chi^{-1} \frac{\hat{l}_{t,s^N}^{1+1/\varphi}}{1+1/\varphi})$, and similarly for U_{cc,t,s^N}, U_{cl,t,s^N}, and U_{ll,t,s^N}.

Partial derivatives. Using equations (24), we can show for aggregate quantities that:

$$\frac{\partial K_t}{\partial a_{t,s^N}} = S_{t,s^N}, \; \frac{\partial K_{t-1}}{\partial a_{t,s^N}} = 0, \; \frac{\partial L_{t+1}}{\partial a_{t,s^N}} = \frac{\varphi L_{t+1} \frac{F_{KL,t+1}}{F_{L,t+1}}}{1 - \varphi L_{t+1} \frac{F_{LL,t+1}}{F_{L,t+1}}} S_{t,s^N}, \; \frac{\partial L_t}{\partial a_{t,s^N}} = 0.$$

After some algebraic manipulations, we obtain for individual choices (labor and consumption):

$$\frac{\partial l_{t,\tilde{s}^N}}{\partial a_{t,s^N}} = 0, \; \frac{\partial l_{t+1,\tilde{s}^N}}{\partial a_{t,s^N}} = \varphi \frac{\frac{F_{KL,t+1}}{F_{L,t+1}}}{1 - \varphi L_{t+1} \frac{F_{LL,t+1}}{F_{L,t+1}}} S_{t,s^N} l_{\tilde{s}^N,e,t+1} 1_{e_{\tilde{s}^N}=e}, \; \frac{\partial c_{t,\tilde{s}^N}}{\partial a_{t,s^N}} = -1_{s^N=\tilde{s}^N},$$

$$\begin{aligned}\frac{\partial c_{t+1,\tilde{s}^N}}{\partial a_{t,s^N}} &= (1+F_{K,t+1})\Pi_{t+1,s^N\tilde{s}^N} + \frac{F_{KK,t+1} + \varphi L_{t+1} \frac{F_{KL,t+1}^2 - F_{KK,t+1}F_{LL,t+1}}{F_{L,t+1}}}{1 - \varphi L_{t+1} \frac{F_{LL,t+1}}{F_{L,t+1}}} S_{t,s^N} \tilde{a}_{t,s^N} \\ &+ \left((1-\tau_{t+1}) 1_{e_{\tilde{s}^N}=e} + \frac{S_{t+1,e}}{S_{t+1,u}} \tau_{t+1} 1_{e_{\tilde{s}^N}=u} \right) l_{\tilde{s}^N,e,t+1} \tilde{y} (1+\varphi) \frac{F_{KL,t+1}}{1 - \varphi L_{t+1} \frac{F_{LL,t+1}}{F_{L,t+1}}} S_{t,s^N}.\end{aligned}$$

We deduce for $\frac{\partial C_{t+1,\tilde{s}^N}}{\partial a_{t,s^N}} = \frac{\partial c_{t+1,\tilde{s}^N}}{\partial a_{t,s^N}} - \chi^{-1} l_{t+1,\tilde{s}^N}^{\frac{1}{\varphi}} \frac{\partial l_{t+1,\tilde{s}^N}}{\partial a_{t,s^N}}$

$$\begin{aligned}\frac{\partial C_{t+1,\tilde{s}^N}}{\partial a_{t,s^N}} &= (1+F_{K,t+1})\Pi_{t+1,s^N\tilde{s}^N} + \frac{F_{KK,t+1} + \varphi L_{t+1} \frac{F_{KL,t+1}^2 - F_{KK,t+1}F_{LL,t+1}}{F_{L,t+1}}}{1 - \varphi L_{t+1} \frac{F_{LL,t+1}}{F_{L,t+1}}} S_{t,s^N} \tilde{a}_{t,s^N} \\ &+ \left((1-\tau_{t+1}) 1_{e_{\tilde{s}^N}=e} + \frac{S_{t+1,e}}{S_{t+1,u}} \tau_{t+1} 1_{e_{\tilde{s}^N}=u} \right) l_{\tilde{s}^N,e,t+1} \tilde{y} (1+\varphi) \frac{F_{KL,t+1}}{1 - \varphi L_{t+1} \frac{F_{LL,t+1}}{F_{L,t+1}}} S_{t,s^N} \\ &- (1-\tau_{t+1}) y_{\tilde{s}^N} \varphi \frac{F_{KL,t+1}}{1 - \varphi L_{t+1} \frac{F_{LL,t+1}}{F_{L,t+1}}} S_{t,s^N} l_{\tilde{s}^N,e,t+1}\end{aligned} \tag{66}$$

Lagrangian simplification. The derivative of (65) implies – with (42) – that $\frac{\partial \mathcal{L}}{\partial a_{t,s^N}} = 0$ if:

$$\Psi_{t,s^N} = \beta \sum_{\tilde{s}^N} \mathbb{E}_t \left[\frac{S_{t+1,\tilde{s}^N}}{S_{t,s^N}} \Psi_{t,s^N} \left(\frac{\partial c_{t+1,\tilde{s}^N}}{\partial a_{t,s^N}} - \chi^{-1} l_{t+1,\tilde{s}^N}^{\frac{1}{\varphi}} \frac{\partial l_{t+1,\tilde{s}^N}}{\partial a_{t,s^N}} \right) \right] \tag{67}$$

$$+\beta \sum_{\tilde{s}^N} \mathbb{E}_t \left[S_{t+1,\tilde{s}^N} \left(F_{KK,t+1} + F_{KL,t+1} \frac{\varphi L_{t+1} \frac{F_{KL,t+1}}{F_{L,t+1}}}{1 - \varphi L_{t+1} \frac{F_{LL,t+1}}{F_{L,t+1}}} \right) \Lambda_{t+1,\tilde{s}^N} \xi_{\tilde{s}^N} U_{c,\tilde{s}^N,t+1} \right].$$

A.5.4 FOC with respect to replacement rate ϕ_t

Rather than computing the derivative with respect to ϕ_t, we do so with respect to τ_t. As for the derivative with respect to a_{t,s^N}, we start with partial derivatives.

Partial derivatives. For aggregate quantities, we obtain quite directly: $\frac{\partial L_t}{\partial \tau_t} = -\frac{\varphi \frac{L_t}{1-\tau_t}}{1-\varphi L_t \frac{F_{LL,t}}{F_{L,t}}}$, $\frac{\partial K_{t-1}}{\partial \tau_t} = 0$. The computation for individual choices is lengthier, and yields:

$$\frac{\partial l_{t,\tilde{s}^N}}{\partial \tau_t} = -\frac{\frac{\varphi}{1-\tau_t} l_{t,\tilde{s}^N}}{1 - \varphi L_t \frac{F_{LL,t}}{F_{L,t}}} 1_{e_{\tilde{s}^N}=e},$$

$$\frac{\partial c_{t,\tilde{s}^N}}{\partial \tau_t} = -\frac{\varphi \frac{L_t}{1-\tau_t}}{1 - \varphi L_t \frac{F_{LL,t}}{F_{L,t}}} F_{KL,t} \tilde{a}_{t,\tilde{s}^N} + (1-\tau_t)^{\varphi-1} \frac{S_{t,e}}{S_{t,u}} \chi^{\varphi} \tilde{y}^{\varphi+1} F_{L,t}^{1+\varphi} 1_{e_{\tilde{s}^N}=u}$$

$$- \left((1-\tau_t) 1_{e_{\tilde{s}^N}=e} + \frac{S_{t,e}}{S_{t,u}} \tau_t 1_{e_{\tilde{s}^N}=u} \right) \chi^{\varphi} \tilde{y}^{\varphi+1} (1+\varphi) \frac{(1-\tau_t)^{\varphi-1} F_{L,t}^{1+\varphi}}{1 - \varphi L_t \frac{F_{LL,t}}{F_{L,t}}}.$$

$l_{t,s^N} = \chi^{\varphi}(1-\tau_t)^{\varphi} y_0^{N\varphi} F_L(K_{t-1}, L_t)^{\varphi} 1_{e_0^N=e}$,

Back to the Lagrangian expression. Using (65), we obtain:

$$\frac{\partial \mathcal{L}}{\partial \tau_t} = \sum_{\tilde{s}^N} S_{t,\tilde{s}^N} \left(\xi_{t,\tilde{s}^N} U_{c,\tilde{s}^N,t} - \left(\lambda_{t,\tilde{s}^N} - (1+F_{K,t}) \Lambda_{t,\tilde{s}^N} \right) \xi_{t,\tilde{s}^N} U_{cc,\tilde{s}^N,t} \right) \tag{68}$$

$$\times \left(\frac{\partial c_{t,\tilde{s}^N}}{\partial \tau_t} - (1-\tau_t) \tilde{y} F_{L,t} \frac{\partial l_{t,\tilde{s}^N}}{\partial \tau_t} \right) - \frac{\varphi \frac{L_t}{1-\tau_t} F_{KL,t}}{1 - \varphi L_t \frac{F_{LL,t}}{F_{L,t}}} \sum_{\tilde{s}^N} S_{t,\tilde{s}^N} \Lambda_{t,\tilde{s}^N} \xi_{t,\tilde{s}^N} U_{c,\tilde{s}^N,t}.$$

or using the expression of partial derivatives:

$$\sum_{\tilde{s}^N} S_{t,\tilde{s}^N} \Lambda_{t,\tilde{s}^N} \xi_{t,\tilde{s}^N} U_{c,\tilde{s}^N,t} = \sum_{\tilde{s}^N} S_{t,\tilde{s}^N} \Psi_{t,\tilde{s}^N} \times \tag{69}$$

$$\left(-\tilde{a}_{t,\tilde{s}^N} + \frac{l_{\tilde{s}^N,e,t}}{L_t} \tilde{y} \frac{F_{L,t}}{\varphi F_{KL,t}} \left(-(1-\tau_t) 1_{e_{\tilde{s}^N}=e} + \frac{S_{t,e}}{S_{t,u}} (1 - \varphi L_t \frac{F_{LL,t}}{F_{L,t}} - (1+\varphi)\tau_t) 1_{e_{\tilde{s}^N}=u} \right) \right).$$

A.5.5 Conclusion

We can further simplify FOCs (67) and (69) using $F(K,L) = K^\alpha L^{1-\alpha} - \delta K$.

$$F_L = (1-\alpha)\left(\tfrac{K}{L}\right)^\alpha, \qquad F_K = \alpha\left(\tfrac{K}{L}\right)^{\alpha-1} - \delta\ ,$$
$$F_{KK} = -\alpha(1-\alpha)\tfrac{1}{L}\left(\tfrac{K}{L}\right)^{\alpha-2}, \quad F_{LL} = -\alpha(1-\alpha)\tfrac{1}{L}\left(\tfrac{K}{L}\right)^{\alpha}, \quad F_{KL} = \alpha(1-\alpha)\tfrac{1}{L}\left(\tfrac{K}{L}\right)^{\alpha-1}.$$

This implies: $\frac{F_{KL}^2 - F_{KK}F_{LL}}{F_L} = 0$; $L\frac{F_{LL}}{F_L} = -\alpha$; $\frac{F_L}{F_{KL}L} = \frac{K}{\alpha L}$. We thus deduce for (67) and (69):

$$\sum_{\tilde{s}^N} S_{t,\tilde{s}^N}\left(\Lambda_{t,\tilde{s}^N}\xi_{t,\tilde{s}^N}U_{c,\tilde{s}^N,t} + \Psi_{t,\tilde{s}^N}\tilde{a}_{t,\tilde{s}^N}\right) = \frac{K_{t-1}}{\alpha\varphi L_t}\sum_{\tilde{s}^N} S_{t,\tilde{s}^N}\Psi_{t,\tilde{s}^N}l_{\tilde{s}^N,e,t}\tilde{y} \tag{70}$$
$$\times\left(-(1-\tau_t)1_{e_{\tilde{s}^N}=e} + \frac{S_{t,e}}{S_{t,u}}\left(1+\alpha\varphi-(1+\varphi)\tau_t\right)1_{e_{\tilde{s}^N}=u}\right),$$
$$\Psi_{t,s^N} = \beta\sum_{\tilde{s}^N}\mathbb{E}_t\left[\Psi_{t+1,\tilde{s}^N}(1+F_{K,t+1})\Pi_{t+1,s^N\tilde{s}^N}\right] + \frac{\beta\alpha(1-\alpha)}{\alpha\varphi}\sum_{\tilde{s}^N}\mathbb{E}_t\Big[S_{t+1,\tilde{s}^N}\Psi_{t+1,\tilde{s}^N}\times \tag{71}$$
$$\left[\left((1-\tau_{t+1})1_{e_{\tilde{s}^N}=e} + \frac{S_{t+1,e}}{S_{t+1,u}}((1+\varphi)\tau_{t+1}-1)1_{e_{\tilde{s}^N}=u}\right)l_{\tilde{s}^N,e,t+1}\tilde{y}\frac{1}{L_{t+1}}\left(\frac{K_t}{L_{t+1}}\right)^{\alpha-1}\right]\Big]$$

B Details of the implementation of the truncation method

B.1 Summary of the model dynamics

The system characterizing the dynamics of the model in the presence of an optimal time-varying replacement rate can be written as follows. We start with equations valid for all $s^N \in \mathcal{S}^N$:

$$\begin{aligned}
l_{t,s^N} &= \chi^\varphi(1-\tau_t)^\varphi w_t^\varphi y_0^{N,\varphi}1_{e_0^N=e},\\
c_{t,s^N} + a_{t,s^N} &\leq (1+r_t)\tilde{a}_{t,s^N} + \left((1-\tau_t)1_{e_0^N=e} + \phi_t 1_{e_0^N=u}\right)l_{t,s^N}y_0^N w_t,\\
\tilde{a}_{t,s^N} &= \sum_{\tilde{s}^N\in\mathcal{S}^N}\Pi_{t,s^N\tilde{s}^N}\frac{S_{\tilde{s}^N}}{S_{s^N}}a_{t-1,\tilde{s}^N},\\
\Psi_{t,s^N} &= U_{c,t,s^N} - \left(\lambda_{t,s^N} - (1+r_t)\Lambda_{t,s^N}\right)\xi_{s^N}U_{cc,t,s^N},
\end{aligned} \tag{72}$$

Then, the equations valid for unconstrained histories only ($s^N \notin \mathcal{C}$) are:

$$\xi_{s^N}U_{c,t,s^N} = \beta(1+r)\sum_{\tilde{s}^N\in\mathcal{S}^N}\Pi_{t+1,s^N\tilde{s}^N}\xi_{\tilde{s}^N}U_{c,t+1,\tilde{s}^N}$$
$$\Psi_{t,s^N} = \beta\sum_{\tilde{s}^N\in\mathcal{S}^N,}\Pi_{t+1,s^N\tilde{s}^N}\mathbb{E}_t\left[(1+r_{t+1})\Psi_{t+1,\tilde{s}^N}\right] + \beta\frac{1-\alpha}{\varphi}\mathbb{E}_t\Big[\frac{1}{L_{t+1}}\left(\frac{K_t}{L_{t+1}}\right)^{\alpha-1}$$
$$\times\sum_{\tilde{s}^N\in\mathcal{S}^N}\left((1-\tau_{t+1})1_{\tilde{e}_0^N=e} + \frac{S_{t+1,e}}{S_{t+1,u}}(\tau_{t+1}(1+\varphi)-1)1_{\tilde{e}_0^N=u}\right)\Psi_{t+1,\tilde{s}^N}S_{t+1,\tilde{s}^N}l_{t+1,\tilde{s}^N}\tilde{y}_0^N\Big],$$

For constrained histories ($s^N \in \mathcal{C}$), we have $a_{t,s^N} = -\bar{a}$ and $\lambda_{t,s^N} = 0$. Aggregate equations are:

$$\sum_{s^N \in \mathcal{S}^N} S_{t,s^N} \Lambda_{t,s^N} \xi_{s^N} U_{c,t,s^N} = - \sum_{s^N \in \mathcal{S}^N} S_{t,s^N} \Psi_{t,s^N} \tilde{a}_{t,s^N} + \frac{1}{\alpha\varphi} \frac{K_{t-1}}{L_t} \tag{73}$$

$$\times \sum_{s^N \in \mathcal{S}^N} \left(-(1-\tau_t) 1_{e_0^N = e} + \frac{S_{t,e}}{S_{t,u}} \left(1 + \alpha\varphi - (1+\varphi)\tau_t\right) 1_{e_0^N = u} \right) S_{t,s^N} \Psi_{t,s^N} l_{t,s^N} \tilde{y}_0^N,$$

$$K_t = \sum_{s^N \in \mathcal{S}^N} S_{t,s^N} a_{t,s^N}, \; L_t = \sum_{s^N \in \mathcal{S}^N} S_{t,s^N} y_0^N l_{t,s^N}, \text{ and } \sum_{s^N \in \mathcal{S}^N} S_{t,s^N} c_{t,s^N} + K_t = Y_t + K_{t-1},$$

$$r_t = \alpha Z_t \left(\frac{K_{t-1}}{L_t} \right)^{\alpha - 1} - \delta \text{ and } w_t = (1-\alpha) Z_t \left(\frac{K_{t-1}}{L_t} \right)^{\alpha}.$$

For the sake of simplicity, we define the beginning-of-period wealth as $\tilde{a}_{t,s^N}$ in equation (72). The optimal replacement rate is given by equation (73), corresponding to the first-order condition of the Ramsey program laid out in Lemma 4. Note that the dynamics of the model with an exogenous replacement rate can be deduced from the previous set of equations, in which ϕ (and τ) has to be set to its exogenous value and equation (73) is discarded.

B.2 Matrix representation at the steady state to compute ξ and λ

Before turning to the matrix representation, we introduce the following notation:

$\circ$ is the Hadamard product, $\otimes$ is the Kronecker product, $\times$ is the usual matrix product.

For any vector V, we denote by $diag\,(V)$ the diagonal matrix with V on the diagonal.

Matrix notations have been introduced in Appendix A.4. Recall that each history is identified by the integer $k_{s^N} = 1, \ldots, N_{tot}$ defined in equation (52), which corresponds to an enumeration in base S, and where $N_{tot} = S^N$.

B.2.1 Computing the ξs: Proof of Proposition 2

Let $\mathbf{S}$ be the N_{tot}-vector of steady-state history sizes. Similarly, let $\mathbf{a}$, $\mathbf{c}$, $\boldsymbol{\ell}$, $\boldsymbol{\nu}$, $\mathbf{U_c}$, $\mathbf{U_{cc}}$ be the N_{tot}-vectors of end-of-period wealth, consumption, labor supply, Lagrange multipliers, marginal utilities, and derivatives of the marginal utility, respectively. These vectors are known from the steady-state equilibrium of the Bewley model. Each element is defined as the truncation of the relevant variable as defined in Section 4 and is computed using equation (29). We also define:

$$\mathbf{W} = w \begin{bmatrix} \phi \\ 1-\tau \end{bmatrix} \otimes \begin{bmatrix} y_1 \\ \vdots \\ y_Y \end{bmatrix} \otimes 1_B, \, \mathbf{L}_e = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} y_1 \\ \vdots \\ y_Y \end{bmatrix} \otimes 1_B, \, \mathbf{L}_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} y_1 \\ \vdots \\ y_Y \end{bmatrix} \otimes 1_B,$$

Let $\mathbb{P}$ be the diagonal matrix having 1 on the diagonal at s^N if and only if the history s^N is not credit constrained (i.e., $\nu_{s^N} = 0$), and 0 otherwise. Similarly, define $\mathbb{P}^c = \mathbf{I} - \mathbb{P}$, where $\mathbf{I}$ is the

$(N_{tot} \times N_{tot})$-identity matrix. Let Π be the transition matrix across histories. In the steady state:

$$\mathbf{S} = \Pi \mathbf{S}, \tag{74}$$

$$\mathbf{S} \circ \mathbf{c} + \mathbf{S} \circ \mathbf{a} = (1+r)\Pi\left(\mathbf{S} \circ \mathbf{a}\right) + \left(\mathbf{S} \circ \mathbf{W} \circ \boldsymbol{\ell}\right),$$

$$\mathbb{P}^c \mathbf{a} = -\bar{a} 1_{N_{tot}\times 1},$$

$$\left(\frac{r+\delta}{\alpha}\right)^{\frac{1}{\alpha-1}} {\mathbf{L_e}}^\top \times \mathbf{S} = \mathbf{S}^\top \times \mathbf{a}, \tag{75}$$

$$\tau = \phi \frac{{\mathbf{L_u}}^\top \times \mathbf{S}}{{\mathbf{L_e}}^\top \times \mathbf{S}}, \text{ and } w = (1-\alpha)\left(\frac{r+\delta}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}.$$

(The previous equalities can be double-checked numerically). We define the vector $\tilde{\boldsymbol{\xi}}$ as:

$$\tilde{\boldsymbol{\xi}} = \left[\mathbb{P}\left(diag\left(u'\left(\mathbf{c}\right)\right) - \beta(1+r)\Pi \times diag\left(u'\left(\mathbf{c}\right)\right)\right) + \mathbb{P}^c\right]^{-1} \boldsymbol{\nu}, \tag{76}$$

which is well-defined since the matrix $\mathbb{P}\left(diag\left(u'\left(\mathbf{c}\right)\right) - \beta(1+r)\Pi \times diag\left(u'\left(\mathbf{c}\right)\right)\right) + \mathbb{P}^c$ is invertible (because $\beta\left(1+r\right) < 1$, and $\mathbb{P} + \mathbb{P}^c = \mathbf{I}$), and since the vector $\boldsymbol{\nu}$ is not zero (because some histories are credit constrained due to a credit limit above the natural borrowing limit). With definition (76), we can check that we have the following for unconstrained histories:

$$\tilde{\boldsymbol{\xi}} \circ u'\left(\mathbf{c}\right) = \beta(1+r)\Pi\left(\tilde{\boldsymbol{\xi}} \circ u'\left(\mathbf{c}\right)\right),$$

and for constrained histories, $\tilde{\boldsymbol{\xi}} \circ u'\left(\mathbf{c}\right) = \beta(1+r)\Pi\left(\tilde{\boldsymbol{\xi}} \circ u'\left(\mathbf{c}\right)\right) + \boldsymbol{\nu}$, where we use $\tilde{\boldsymbol{\xi}} \circ u'\left(\mathbf{c}\right) = diag\left(u'\left(\mathbf{c}\right)\right)\tilde{\boldsymbol{\xi}}$. These two properties being invariant after a positive rescaling, we finally define $\boldsymbol{\xi} = \tilde{\boldsymbol{\xi}}/sum\left(\mathbf{S} \circ \tilde{\boldsymbol{\xi}}\right)$ with $sum\left(\mathbf{S} \circ \boldsymbol{\xi}\right) = 1$ ($sum(\boldsymbol{x}) = \sum_{x \in \boldsymbol{x}} x$ for any vector $\boldsymbol{x}$).

B.2.2 Computing equilibrium Lagrange multipliers λ

We derive here the Lagrange multipliers of the Ramsey program as a function of the steady-state solution (i.e., allocations and prices), which is assumed to be known. Denoting the vectors associated with the Lagrange multipliers by $\boldsymbol{\lambda}$, $\boldsymbol{\Lambda}$, and $\boldsymbol{\Psi}$, we have: $\boldsymbol{\Psi} = \boldsymbol{\xi} \circ \mathbf{U_c} - (\boldsymbol{\lambda} - (1+r)\boldsymbol{\Lambda}) \circ \boldsymbol{\xi} \circ \mathbf{U_{cc}}$, from (42). We define the matrix Π^Λ by $\Pi^\Lambda_{\tilde{k}k} = \frac{S_k \Pi_{k\tilde{k}}}{S_{\tilde{k}}}$, such that $\boldsymbol{\Lambda} = \Pi^\Lambda \boldsymbol{\lambda}$, and the matrix Π^Ψ by:

$$\Pi^\Psi_{k\tilde{k}} = \beta(1+F_K)\Pi_{k\tilde{k}} + \beta\frac{1-\alpha}{\alpha\varphi}\frac{1}{L}\left(r+\delta\right)$$
$$\times\left((1-\tau)1_{\tilde{k}>\frac{N_{tot}}{2}} + \frac{S_e}{S_u}\left(\tau\left(1+\varphi\right)-1\right)1_{\tilde{k}\leq\frac{N_{tot}}{2}}\right) S_{\tilde{k}} l_{\tilde{k}} \left(\mathbf{L}^e + \mathbf{L}^u\right)_{\tilde{k}}.$$

Note that $1_{\tilde{k}>\frac{N_{tot}}{2}}$ represents employed agents and $1_{\tilde{k}\leq\frac{N_{tot}}{2}}$ unemployed agents. It can be checked that $\mathbb{P}\boldsymbol{\Psi} = \mathbb{P}\Pi^\Psi\boldsymbol{\Psi}$, from (43), and that the vector of the Lagrange multipliers, $\boldsymbol{\lambda}$, verifies:

$$\boldsymbol{\lambda} = \left[\mathbb{P}^c + \mathbb{P}(\mathbf{I} - \Pi^\Psi)\left(diag\left(\boldsymbol{\xi} \circ \mathbf{U_{cc}}\right)\left(\mathbf{I} - (1+F_K)\Pi^\Lambda\right)\right)\right]^{-1} \mathbb{P}(\mathbf{I} - \Pi^\Psi)(\boldsymbol{\xi} \circ \mathbf{U_c}). \tag{77}$$

Importantly, the right-hand side can be deduced from the Bewley allocations, which makes the computation of $\boldsymbol{\lambda}$ straightforward. We then deduce $(\boldsymbol{\Lambda}, \boldsymbol{\Psi})$ with:

$$\boldsymbol{\Lambda} = \Pi^{\Lambda}\lambda, \text{ and } \boldsymbol{\Psi} = \xi \circ \mathbf{U}_c - \xi \circ \mathbf{U_{cc}} \circ \left(\mathbf{I} - (1+F_K)\Pi^{\Lambda}\right)\lambda. \tag{78}$$

The condition (73) can be simplified into:

$$\mathbf{V}^{\top} \times 1_{N_{tot}\times 1} = 0, \tag{79}$$

$$\text{where: } \mathbf{V} \equiv \frac{\varphi\alpha}{K}(\mathbf{S}\circ\Psi\circ\tilde{\mathbf{a}} + \mathbf{S}\circ\Lambda\circ\xi\circ\mathbf{U}_c) + \frac{1-\tau}{L}\mathbf{S}\circ\Psi\circ\mathbf{L}^e - (1-\tau+\varphi(\alpha-\tau))\frac{S_e}{S_u}\mathbf{S}\circ\Psi\circ\mathbf{L}^u,$$

which can be used to check the optimality of the planner's instruments.

B.3 Algorithm for solving the Ramsey problem

The algorithm for computing Ramsey policies is as follows.

1. Choose a length of the truncation N.
2. Set a reasonable initial value for the replacement rate ϕ (we start with $\phi = 50\%$).
   (a) Solve the general Bewley model for the given value of ϕ.
   (b) Use the steady-state outcome to compute the truncated allocation: $\mathbf{a}$, $\mathbf{c}$, $\boldsymbol{\ell}$, $\boldsymbol{\nu}$, $\mathbf{U_c}$, $\mathbf{U_{cc}}$. Then deduce $\tilde{\boldsymbol{\xi}}$ and $\boldsymbol{\xi}$ from equations (76).
   (c) Determine the steady-state values of the multipliers $\boldsymbol{\lambda}, \boldsymbol{\Lambda}$, and $\boldsymbol{\Psi}$ from Section B.2.2.
3. Iterate on ϕ and repeat Step 2 until equality (79) holds.

Once the steady state and the partition have been determined, it is easy to simulate the model using standard perturbation techniques with existing software such as Dynare (see Adjemian et al., 2011). Simulating the whole optimal allocation for the calibrated economy with aggregate shocks takes less than 1 minute on a standard laptop.

C Supplemental numerical results

C.1 Computational methods

For all simulation methods, we first solve for the steady-state allocation of the full-fledged Bewley model. The consumer's problem is solved on a grid using the Endogenous Grid Point Method (EGM) of Carrol (2006). See also Den Haan (2010) for a presentation. For the decision rules, the asset grid has 50 points, non-linearly spaced, as in Boppart et al. (2018), and households can choose points off the grid by linear interpolation. There are 7 different productivity levels and 2 employment status. We thus solve for 14 policy rules. The Euler equation errors reported in Section 6.5 show that the accuracy is satisfactory.

Truncated model. To derive the truncated model for a given N, we first use the steady-state distribution of the full-fledged Bewley model to compute average consumption and savings in each of the 14 idiosyncratic states. We then use the policy rules $g_i(a)$ for savings and $g_i^c(a)$ for consumption, together with the transition probabilities, $(\Pi_{i,j})_{i,j=1,\dots,14}$, to compute the average consumption and saving levels for each idiosyncratic history. For example, when we know the steady-state beginning-of-period distribution of wealth, $\Lambda_i(a)$, of agents in states $i = 1,\dots,14$, we can compute the steady-state distribution of wealth of agents with history (i,j), $\Lambda_{(i,j)}(a)$ by computing:

$$\Lambda_{(i,j)}(a') = \Pi_{i,j} \int_{a, g_i(a)=a'} \Lambda_i(a)\, da, \tag{80}$$

Once we have the distributions $\Lambda_{(i,j)}$, the average consumption levels and savings by history are simply constructed by $c_{i,j} = \int_a g_j^c(a)\Lambda_{(i,j)}(a)\,da$. and $a_{i,j} = \int_a a\Lambda_{(i,j)}(a)\,da$ (note that we use the policy rule of the last states). For histories (i,j,k) with $i,j,k = 1,\dots,14$, we start from $\Lambda_{(i,j)}$ and construct $\Lambda_{(i,j,k)}$ as in (80). This strategy allows us to construct recursively the steady-state distribution of wealth for any idiosyncratic history of arbitrary length. Once the average values are computed, we deduce the ξs using equation (76). Note that the Dynare solver can be used to double-check that the steady state computed in Section B.2 is indeed a steady state of the dynamic equations. Finally, we then compute a first-order approximation for the aggregate shock of the whole system of equations, given in Section B.1, using Dynare again.

Reiter model. We implement the algorithm described in Reiter (2009), which is now standard. For each asset level and for each idiosyncratic state, we perform a first-order approximation of the policy rule for the aggregate states. We use these approximated policy rules to simulate the dynamics of the model for 10,000 periods.

Boppart, Krusell, and Mitman. Following Boppart et al. (2018), we first simulate an unexpected shock (MIT shock) to the innovation ε_t^z, to compute the IRFs for the various variables of interest. The transition path is then solved by iteration on the capital path, assuming that the economy comes back to its steady state after 400 periods. These IRFs are then used as numerical partial derivatives for any variable x_t under consideration according to the aggregate shock (which is continuous), at different time-horizons, namely $\frac{\partial x_{t+k}}{\partial \varepsilon_t^z}$ for $k = 0,\dots,T$ (where T is chosen high enough for the derivative to be negligible for $k > T$). These derivatives are next used to simulate the economy with aggregate shocks, using a Taylor-expansion: $x_t = \bar{x}^{ss} + \sum_{k=0}^{T} \frac{\partial x_t}{\partial \varepsilon_{t-k}^z} \varepsilon_{t-k}^z$ for the simulated history of the innovation ε_t^z.

C.2 Comparison with other solution methods

We compare here three different computational solutions: the Reiter method, the truncated method, and the method of Boppart et al. (2018) – described in Section C.1.

We simulate the same economy as in Section 6.3, with the same parameters. However, we focus on TFP shocks only ($\sigma_{ue} = 0$), as in Boppart et al. (2018) to simplify the comparison. The results are reported in Table 7. The first column describes the computed statistics (using simulations with 10,000 periods). Column BKM reports the results for the Boppart et al. (2018) methodology, and the subsequent columns correspond (in this order) to the Reiter method, the truncation method, and the representative-agent (RA) economy. The three methods (Reiter, BKM, truncation) yield very similar results – and are very different from the RA economy. For instance the normalized standard deviation of GDP is between 1.75% and 1.79% for the first three methods, whereas it is 2.10% in the RA case.

Methods		BKM	Reiter	Truncated	RA
Simulations		(1)	(2)	(3)	(4)
Y	mean	1.17	1.17	1.17	1.08
	std/mean (%)	1.75	1.77	1.79	2.10
C	mean	0.84	0.84	0.84	0.80
	std/mean (%)	1.45	1.45	1.45	1.73
L	mean	0.30	0.30	0.30	0.30
	std/mean (%)	0.58	0.59	0.60	.70
K	mean	13.14	13.14	13.14	11.09
	std/mean (%)	1.61	1.62	1.69	2.18
$corr(C, C_{-1})$	(in %)	99.13	99.13	99.17	99.66
$corr(Y, Y_{-1})$		97.48	97.49	97.55	98.19
$corr(C, Y)$		96.17	96.21	96.07	94.59

Table 7: Moments of the simulated model for different resolution techniques.

C.3 Additional numerical checks: IRFs and alternative rules

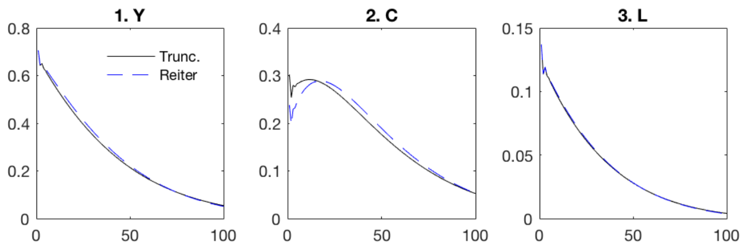

Figure 1: IRFs after a positive shock with the replacement rate following the optimal rule. Simulations are provided for bot the truncated economy and the Reiter economy.

Figure 1 plots the IRFs after a one-standard-deviation TFP shock for the calibrated model of

Section 6.3 (with a fixed replacement rate), computed with the Reiter and truncation methods. These simulations correspond to Economies (4) and (5) of Table 4. The IRFs are very close for both simulation techniques.

We also investigate the optimality of the replacement rule (46). We implement two variations of the rule and quantify their impact on aggregate welfare, which is computed using the Reiter method. We simulate all economies for 10,000 periods and then compute relevant moments and welfare. The first variation is a procyclical replacement rate (with similar variances). The coefficients of the rule (46) are $(\phi^{ss}, a_1^{\phi}, a_2^{\phi}, a_0^{\varepsilon}, a_1^{\varepsilon}, a_2^{\varepsilon}, a^K) = (0.4200, -24.4772, 0.0104, -0.6048, 0.3317, -0.3669, 0.0326)$. The results are reported in column (2) of Table 8. Second, we implement the rule

Simulations		(1)	(2)	(3)
		Bench.	Procyclical	High variance
Y	mean	1.17	1.17	1.17
	std/mean (%)	2.71	1.49	2.99
C	mean	0.84	0.84	0.84
	std/mean (%)	2.27	1.20	2.52
L	mean	0.30	0.30	0.30
	std/mean (%)	1.70	0.31	2.02
K	mean	13.14	13.14	13.14
	std/mean (%)	2.45	2.27	2.67
ϕ	mean(ϕ)(%)	42	42	42
	std(ϕ)(%)	23	25	35
$corr(C, C_{-1})$	(%)	99.18	98.50	99.17
$corr(Y, Y_{-1})$	(%)	97.52	97.28	97.47
$corr(C, Y)$	(%)	95.30	98.13	94.87
$corr(Y, \Phi)$	(%)	-96.65	96.87	-0.9628
Equ. cons. increase Δ^c	(%)	-	2.0	1.7

Table 8: Impact of different rules for the replacement rate.

with higher variances. The coefficients are $(\phi^{ss}, a_1^{\phi}, a_2^{\phi}, a_0^{\varepsilon}, a_1^{\varepsilon}, a_2^{\varepsilon}, a^K) = (0.4200, -24.4772, 0.0104, 0.9048, -0.4517, 0.5469, -0.04726)$. The results are reported in column (3). For each rule, we report the same statistics as in Table 4, as well as the welfare difference with the benchmark rule (column (1) and equation (46)). The welfare is computed in period 0 using an intertemporal utilitarian welfare criterion. We report the increase in consumption Δ^c of all agents in all periods (such that the agents have a period utility $u((1+\Delta^c)c_t^i - \chi^{-1}\frac{(l_t^i)^{1+1/\varphi}}{1+1/\varphi}))$ for the total intertemporal welfare to be the same in economies (2) and (3) as in the benchmark economy (1). Both rules imply a decrease in welfare compared to the benchmark economy, by around 1% or 2% of consumption equivalent, thereby confirming the optimality of the replacement rate implied by the Ramsey program.

D Supplemental theoretical results

D.1 Understanding Lagrange multipliers on Euler equations and penalty functions

The analysis of the main text uses Lagrange multipliers on Euler equations and claims that these multipliers can be either positive or negative and that their sign is related to the distortions on the saving incentives (from the planner's point of view). This section provides a very simple example (textbook style) to illustrate this statement. In addition, it clarifies some properties of exterior penalty functions, which are used in Section D.2 below.

Consider an economy where the planner has an instrument τ, and the agent has a choice variable a. The agent maximizes a concave objective with a constraint $a \geq -\bar{a}$. Her program is:

$$\max_{a\in\mathbb{R}} \; -(a-\tau)^2 - \tau^2, \quad \text{s.t. } a \geq -\bar{a},$$

which yields the FOC (ν being the credit-constraint Lagrange multiplier):

$$a - \tau = \nu. \tag{81}$$

This corresponds to $a = \tau$ if $\tau \geq -\bar{a}$ or $a = -\bar{a}$ if $\tau < -\bar{a}$.

The planner's program can now be written as:

$$\max_{a,\tau\in\mathbb{R}} \; -(a-\tau)^2 - \tau^2$$
$$\text{s.t. } a - \tau = \nu \text{ and } a \geq -\bar{a}.$$

The Lagrangian is (λ is the Lagrange multiplier on the agent's FOC and μ the multiplier on the credit-constraint): $\mathcal{L} = -(a-\tau)^2 - \tau^2 - 2\lambda(a-\tau) - 2\mu(a+\bar{a})$, which yields the two FOCs:

$$a - \tau + \lambda + \mu = 0 \tag{82}$$

$$-a + 2\tau - \lambda = 0 \tag{83}$$

Note that we have $\lambda\mu = 0$ and $\lambda\nu = 0$. There are two possible cases, depending on whether the constraint $a \geq -\bar{a}$ is binding or not (i.e., whether $\lambda \neq 0$ or not):

1. $\lambda \neq 0$. So, we have $\mu = \nu = 0$. Equation (81) implies $a = \tau$ and equations (82) and (83) become: $\lambda = 0$ and $\tau - \lambda = 0$. So the solution is:

$$\lambda = \nu = \mu = 0, \text{ and } a = \tau = 0. \tag{84}$$

2. $\lambda = 0$. So $a = -\bar{a}$ and equations (82) and (83) become:

$$-\mu = \nu = -\frac{\bar{a}}{2}, \text{ and } \tau = -\frac{\bar{a}}{2}. \tag{85}$$

Penalty functions

We define the penalty function g as follows:

$$g(a) = \frac{1}{2}\max(-\bar{a} - a, 0)^2, \quad g'(a) = -\max(-\bar{a} - a, 0), \quad g''(a) = 1_{a \leq -\bar{a}}.$$

Agent's program. Denoting by γ the weight of the penalty function, the agent's program is:

$$\max_{a \in \mathbb{R}} \; -(a - \tau)^2 - \tau^2 - 2\gamma g(a).$$

The FOC yields:

$$a - \tau - \gamma \max(-\bar{a} - a, 0) = 0 \tag{86}$$

So, there are two solutions:

$$a = \begin{cases} \tau & \text{if } -\bar{a} - \tau \leq 0, \\ \frac{\tau}{1+\gamma} + \frac{\gamma}{1+\gamma}(-\bar{a}) & \text{if } -\bar{a} - \tau > 0, \end{cases}$$

We can observe than when $\gamma \to \infty$, the solutions of the problem with penalty functions converge to the solutions (84) and (85) of the Ramsey problem.

Planner's program. For the sake of generality, we consider a penalty function with a different coefficient $\tilde{\gamma} = \kappa\gamma$, where we can have $0 < \kappa \leq 1$ or $\kappa > 1$, depending on who (the agent of the planner) gives the constraint the highest value. The planner's program is:

$$\max_{a, \tau \in \mathbb{R}} \; -(a - \tau)^2 - \tau^2 - 2\tilde{\gamma} g(a),$$
$$\text{s.t. } a - \tau = -\gamma g'(a).$$

The Lagrangian is: $\mathcal{L} = -(a - \tau)^2 - \tau^2 - 2\lambda(a - \tau + \gamma g'(a)) + 2\tilde{\gamma} g(a)$, with the FOCs:

$$0 = \frac{a + \lambda}{2} + \lambda\gamma 1_{a \leq -\bar{a}} - \tilde{\gamma}\max(-\bar{a} - a, 0), \tag{87}$$
$$\tau = \frac{a + \lambda}{2}. \tag{88}$$

Again, there are two cases:

1. $a > -\bar{a}$. Then (86) implies $a = \tau$ and FOCs (87) and (88) become:

$$a = \tau = \frac{a + \lambda}{2} = 0.$$

2. $a \leq -\bar{a}$. Then (86) implies $\tau = (\gamma+1)a + \gamma\bar{a}$. We obtain from (88):

$$\lambda = (2\gamma+1)a + 2\gamma\bar{a},$$

and from (87):

$$(\gamma + 1 + \gamma(2\gamma+1) + \tilde{\gamma})\, a = -\left(2\gamma^2 + \gamma + \tilde{\gamma}\right)\bar{a}.$$

After some algebraic manipulations, we finally obtain:

$$a = \frac{2\gamma^2+\gamma+\tilde{\gamma}}{2\gamma^2+2\gamma+\tilde{\gamma}+1}\bar{a},\ \lambda = -\frac{\tilde{\gamma}-\gamma}{2\gamma^2+2\gamma+\tilde{\gamma}+1}\bar{a},\ \tau = -\frac{\gamma^2+\tilde{\gamma}}{2\gamma^2+2\gamma+\tilde{\gamma}+1}\bar{a}.$$

We obtain (as $\tilde{\gamma} = \kappa\gamma$):

$$\lambda = -\frac{(\kappa-1)\gamma}{2\gamma^2+(\kappa+2)\gamma+1}\bar{a}. \tag{89}$$

Remark 1 (Sign of λ) *The sign of λ can be positive or negative, depending on whether $\kappa > 1$ or $\kappa < 1$, which indicates who (between the planner and the agent) gives the credit constraint the highest value.*

At the limit: $a \to -\bar{a},\ \lambda \to 0,\ \tau \to -\frac{\bar{a}}{2}$, which is the solution of the initial program.

D.2 Penalty functions

In this section, we replace the credit constraint with a penalty function denoted by $g(a)$ for a saving choice a. The functional form of g is standard for exterior penalty function (see Luenberger and Ye, 2016 for a textbook treatment of penalty functions). For a saving $a < -\bar{a}$, the distance to the credit limit is $-\bar{a} - a$. The baseline penalty function is thus $g(a) = \max(-\bar{a} - \underline{a}, 0)^2$. We parametrize this penalty function by a scalar $\gamma > 0$ such that all agents face a penalty function $\gamma g(a_{t,s^N})$. The goal of this section is to show that the first-order conditions (21) and (22) are limits of first-order conditions for infinitely concave penalty functions, i.e., for $\gamma \to \infty$.

D.2.1 The "island" program

In the presence of the penalty function, the island program can be written as:

$$\max_{(c^\gamma_{t,s^N}, l^\gamma_{t,s^N}, a^\gamma_{t,s^N}, \tilde{a}^\gamma_{t,s^N})_{t\geq 0, s^N\in\mathcal{S}^N}} \tilde{J}_\gamma = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\sum_{s^N \in \mathcal{S}^N} S_{t,s^N} \left(\xi_{s^N} U(c^\gamma_{t,s^N}, \hat{l}^\gamma_{t,s^N}) - \gamma g(a^\gamma_{t,s^N}) \right) \right], \tag{90}$$

$$a^\gamma_{t,s^N} + c^\gamma_{t,s^N} = ((1-\tau_t) l^\gamma_{t,s^N} 1_{e^N_0 = e} + \phi_t l^\gamma_{t,s^N,e} 1_{e^N_0 = u}) y^N_0 w^\gamma_t + (1+r_t)\tilde{a}^\gamma_{t,s^N} \quad (s^N \in \mathcal{S}^N), \tag{91}$$

$$c^\gamma_{t,s^N},\, l^\gamma_{t,s^N} \geq 0 \quad (s^N \in \mathcal{S}^N), \tag{92}$$

with given initial conditions. Compared to the initial program (18)–(20), the objective function (90) includes penalty functions, while the credit constraint has been removed from the program.

We now derive all first-order equations of the agents and the planner and then investigate their convergence when $\gamma \to \infty$. From the program (90), we deduce the following Euler equations:

$$l^{\gamma}_{t,s^N} = \chi(1-\tau_t)y_0^N w_t U_c(c^{\gamma}_{t,s^N}, l^{\gamma}_{t,s^N}), \tag{93}$$

$$\xi_{s^N} U_c(c^{\gamma}_{t,s^N}, \hat{l}^{\gamma}_{t,s^N}) = \gamma g'(a^{\gamma}_{t,s^N}) + \beta \mathbb{E}_t \Big[\sum_{\tilde{s}^N} \Pi_{t+1,s^N,\tilde{s}^N} (1+r_{t+1}) \xi_{\tilde{s}^N} U_c(c^{\gamma}_{t+1,\tilde{s}^N}, \hat{l}^{\gamma}_{t+1,\tilde{s}^N}) \Big]. \tag{94}$$

Compared to the initial Euler equations (21) and (22), the labor Euler equation (93) remains unchanged, while the consumption Euler equation (94) does not feature a Lagrange multiplier for the credit constraint but instead the derivative of the penalty function. All other equations characterizing the equilibrium are unchanged (factor prices, market clearing conditions, etc.).

As the replacement rate is exogenous, we make the following additional assumption.

Assumption B *We assume that for all $t \geq 0$: $1-\tau_t > \frac{\varphi}{1+\varphi}$ and $\phi_t > \frac{\varphi}{1+\varphi}$.*

Assumption B is purely technical: it guarantees that the utility – with a GHH utility function – of employed and unemployed agents in autarky is well-defined and finite. This assumption is obviously verified in our numerical exercise of Section 6.

Lemma 2 *For any $\gamma > 0$, we have $\tilde{J}_\gamma > -\infty$, where $\tilde{J}_\gamma$ is defined in equation (90).*

Lemma 2 states that the individual welfare of the economy is well-defined and finite for any values of γ. It is a direct consequence of Assumption B.

Proof. We prove that the autarky allocation (i.e., null savings at all dates) is feasible. We assume that $a^{aut}_{t,s^N} = 0$, for all $t \geq 0$ and $s^N \in \mathcal{S}^N$, and thus that $\tilde{a}^{aut}_{t,s^N} = 0$ (from (17)). The consumption of an agent when employed and with productivity y is $c^{aut}_{t,s^N} = (1-\tau) l^{aut}_{t,s^N}(y)\, y w_t$. Since her labor supply is $l^{aut}_{t,s^N}(y) = (\chi w_t y)^{\varphi}$, we have

$$c^{aut}_{t,s^N} - \frac{1}{\chi} \frac{l^{aut}_{t,s^N}(y)^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}} = \left(1-\tau - \frac{\varphi}{1+\varphi}\right) \frac{(\chi w_t y)^{1+\varphi}}{\chi}, \tag{95}$$

Similarly for the unemployment case, from which $\zeta(y) = l(y)$:

$$c^{aut}_{t,s^N} - \frac{1}{\chi} \frac{l^{aut}_{t,s^N}(y)^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}} = \left(\phi - \frac{\varphi}{1+\varphi}\right) \frac{(\chi w_t y)^{1+\varphi}}{\chi}, \tag{96}$$

Assumption B ensures that quantities (95) and (96) are bounded away from zero for all γ. The autarky allocation is thus feasible (independently of γ), and has a finite welfare, which concludes the proof. ■ We can now state our result regarding the limit Euler equation.

Lemma 3 (Limit penalty) *When $\gamma \to \infty$, the solution of (18) is such that:* $\lim_{\gamma\to\infty} a^{\gamma}_{t,s^N} = -\bar{a}$ *or, defining* $c^{\infty}_{t,s^N} = \lim_{\gamma\to\infty} c^{\gamma}_{t,s^N}$ *and* $\hat{l}^{\infty}_{t,s^N} = \lim_{\gamma\to\infty} \hat{l}^{\gamma}_{t,s^N}$*:*

$$\xi_{s^N} U_c(c^{\infty}_{t,s^N}, \hat{l}^{\infty}_{t,s^N}) = \beta \mathbb{E}_t\Big[\sum_{\tilde{s}^N} \Pi_{t+1,s^N,\tilde{s}^N}(1+r_{t+1})\xi_{\tilde{s}^N} U_c(c^{\infty}_{t+1,s^N}, \hat{l}^{\infty}_{t+1,s^N})\Big]. \tag{97}$$

Lemma 3 states that, when the penalty function becomes infinitely concave, then either the borrowing of agents facing a positive penalty tends toward the borrowing limit, or their limit allocation verifies Euler equation (97), which is the same as in our baseline island economy (8). This property is easy to prove here because we have a finite number of agents for which $S_{s^N} > 0$.
Proof. The proof is performed by contradiction. Assume that there exists $s^N \in \mathcal{S}^N$, such that $\lim_{\gamma\to\infty} a_{t,s^N} < -\bar{a}$. Then, there exists $\varepsilon > 0$, such that $a_{t,s^N} \le -\bar{a} - \varepsilon$, for γ high enough, which implies that $\lim_{\gamma\to\infty} \gamma g(a_{t,s^N}) = \lim_{\gamma\to\infty} \gamma\varepsilon^2 = \infty$. Hence, $\lim_{\gamma\to\infty} \tilde{J}_{\gamma} = -\infty$, which contradicts Lemma 2. The second part stems from (94), as $\gamma g'(a^{\gamma}_{t,s^N}) = 0$ if $a^{\gamma}_{t,s^N} > -\bar{a}$. ■

D.2.2 The Ramsey program

We now rewrite the Ramsey program in the presence of penalty functions. The planner's program can be written as follows – we drop the dependence in γ to lighten the notation:[19]

$$\max_{\left((a_{t,s^N}, c_{t,s^N}, l_{t,s^N})_{s^N}, \phi_t, \tau_t\right)_{t\ge 0}} \mathbb{E}_0\Big[\sum_{t=0}^{\infty} \beta^t \sum_{s^N \in \mathcal{S}^N} S_{t,s^N}\Big(\xi_{s^N} U(c_{t,s^N}, \hat{l}_{t,s^N}) - \gamma g(a_{t,s^N})\Big)\Big], \tag{98}$$

$$\xi_{s^N} U_c(c_{t,s^N}, \hat{l}_{t,s^N}) = \gamma g'(a_{t,s^N}) + \beta \mathbb{E}_t\Big[(1+r_{t+1}) \sum_{\tilde{s}^N \in \mathcal{S}^N} \Pi_{t+1,s^N\tilde{s}^N} \xi_{\tilde{s}^N} U_c(c_{t+1,\tilde{s}^N}, \hat{l}_{t+1,\tilde{s}^N})\Big], \tag{99}$$

and subject to the same equations as in the main text: (37), (24), (14), and (3). There are only two differences compared with the Ramsey program in the main text: (i) the presence of penalty functions in the planner's objective; and (ii) penalty functions in the Euler equations (99). We can now state our main equivalence result.

Proposition 6 (Equivalence result) *The solution of program (40) is a solution of the program (98) when the penalty function become infinitely concave.*

D.2.3 Proof of Proposition 6

Using Lemma 3, it only remains to be proven that the FOCs of the Ramsey program (98) converge to those of the Ramsey program of Lemma 1 when penalty costs become infinitely large.

[19] As illustrated in Section D.1, we can choose the same penalty γ for the agents and the planner.

Rewriting the Ramsey program. The planner's objective, denoted by J_γ, is:

$$J_\gamma = \mathbb{E}_0\Big[\sum_{t,s^N}\beta^t S_{t,s^N}\left(\xi_{s^N}U_{t,s^N} - \gamma g(a_{t,s^N})\right)\Big] - \mathbb{E}_0\sum_{t,s^N}\beta^t S_{t,s^N}\lambda_{t,s^N}$$
$$\times\Big(\xi_{s^N}U_{c,t,s^N} - \gamma g'(a_{t,s^N}) - \beta\mathbb{E}_t\Big[(1+r_{t+1})\sum_{\tilde{s}^N\in\mathcal{S}^N}\Pi_{t+1,s^N\tilde{s}^N}\xi_{\tilde{s}^N}U_{c,t+1,s^N}\Big]\Big),$$

where the Lagrange multiplier on (99) is $\beta^t S_{t,s^N}\lambda_{t,s^N}$. After some rewriting, and using $\Lambda_{t+1,\tilde{s}^N} = \frac{\sum_{s^N\in\mathcal{S}^N} S_{t,s^N}\lambda_{t,s^N}\Pi_{t+1,s^N\tilde{s}^N}}{S_{t+1,\tilde{s}^N}}$, with $\Lambda_{\tilde{s}^N,0}=0$, we obtain:

$$J_\gamma = \mathbb{E}_0\sum_{t,s^N}\beta^t S_{t,s^N}\xi_{s^N}\left(U_{t,s^N} - \left(\lambda_{t,s^N} - (1+r_t)\Lambda_{t,s^N}\right)U_{c,t,s^N}\right) \tag{100}$$
$$+\mathbb{E}_0\sum_{t,s^N}\beta^t S_{t,s^N}\left(-\gamma g(a_{t,s^N}) + \lambda_{t,s^N}\gamma g'(a_{t,s^N})\right).$$

Solving the Ramsey program. The Ramsey program then consists in maximizing J_γ in (100) over $((a_{t,s^N}, c_{t,s^N}, l_{t,s^N})_{s^N}, \phi_t, \tau_t)_{t\geq 0}$ subject to the relevant constraints. The Lagrangian is:

$$\mathcal{L} = \mathbb{E}_0\sum_{t,s^N}\beta^t S_{t,s^N}\xi_{s^N}\left(U_{t,s^N} - \left(\lambda_{t,s^N} - (1+F_K(K_{t-1},L_t))\Lambda_{t,s^N}\right)U_{c,t,s^N}\right) \tag{101}$$
$$+\mathbb{E}_0\sum_{t,s^N}\beta^t S_{t,s^N}\left(-\gamma g(a_{t,s^N}) + \lambda_{t,s^N}\gamma g'(a_{t,s^N})\right),$$

As in the baseline case, this Lagrangian problem can thus be seen as depending only on saving choices (a_{t,s^N}) and the replacement rate ϕ_t. The FOC for ϕ_t is independent of penalty functions and is identical to (44). We only focus here on the FOC with respect to (a_{t,s^N}).

We obtain (using the definition of $\hat{l}_{t,s^N}$ and defining $\frac{\partial C_{t,\tilde{s}^N}}{\partial a_{t,s^N}} = \frac{\partial c_{t,\tilde{s}^N}}{\partial a_{t,s^N}} - \chi^{-1} l_{t,\tilde{s}^N}^{\frac{1}{\varphi}}\frac{\partial l_{t,\tilde{s}^N}}{\partial a_{t,s^N}}$):

$$\frac{\partial\mathcal{L}}{\beta^t\partial a_{t,s^N}} = \sum_{\tilde{s}^N} S_{t,\tilde{s}^N}\xi_{\tilde{s}^N}U_{c,\tilde{s}^N,t}\frac{\partial C_{t,\tilde{s}^N}}{\partial a_{t,s^N}} + \beta\sum_{\tilde{s}^N}\mathbb{E}_t\left[S_{t+1,\tilde{s}^N}\xi_{\tilde{s}^N}U_{c,\tilde{s}^N,t+1}\frac{\partial C_{t+1,\tilde{s}^N}}{\partial a_{t,s^N}}\right] \tag{102}$$
$$-\sum_{\tilde{s}^N} S_{t,\tilde{s}^N}\left(\lambda_{t,\tilde{s}^N} - (1+F_{K,t})\Lambda_{t,\tilde{s}^N}\right)\xi_{\tilde{s}^N}U_{cc,\tilde{s}^N,t}\frac{\partial C_{t,\tilde{s}^N}}{\partial a_{t,s^N}}$$
$$+\sum_{\tilde{s}^N} S_{t,\tilde{s}^N}\left(\left(F_{KK,t}\frac{\partial K_{t-1}}{\partial a_{t,s^N}} + F_{KL,t}\frac{\partial L_t}{\partial a_{t,s^N}}\right)\right)\Lambda_{t,\tilde{s}^N}\xi_{\tilde{s}^N}U_{c,\tilde{s}^N,t}$$
$$-\beta\sum_{\tilde{s}^N}\mathbb{E}_t\left[S_{t+1,\tilde{s}^N}\left(\lambda_{t+1,\tilde{s}^N} - (1+F_{K,t+1})\Lambda_{t+1,\tilde{s}^N}\right)\xi_{\tilde{s}^N}U_{cc,\tilde{s}^N,t+1}\frac{\partial C_{t+1,\tilde{s}^N}}{\partial a_{t,s^N}}\right]$$
$$+\beta\sum_{\tilde{s}^N}\mathbb{E}_t\left[S_{t+1,\tilde{s}^N}\left(F_{KK,t+1}\frac{\partial K_t}{\partial a_{t,s^N}} + F_{KL,t+1}\frac{\partial L_{t+1}}{\partial a_{t,s^N}}\right)\Lambda_{t+1,\tilde{s}^N}\xi_{\tilde{s}^N}U_{c,\tilde{s}^N,t+1}\right]$$
$$+S_{t,s^N}\left(-\gamma g'(a_{t,s^N}) + \lambda_{t,s^N}\gamma g''(a_{t,s^N})\right).$$

After some algebraic manipulations, we obtain from (102) the result that $\frac{\partial \mathcal{L}}{\partial a_{t,s^N}} = 0$ if:

$$\frac{\Psi_{t,s^N}}{\beta} = \sum_{\tilde{s}^N} \mathbb{E}_t \Bigg[\Psi_{t,\tilde{s}^N} \Bigg((1+r_{t+1})\Pi_{t+1,s^N\tilde{s}^N} + S_{t+1,\tilde{s}^N} (F_{KK,t+1} + \frac{\varphi L_{t+1} \frac{F_{KL,t+1}^2}{F_{L,t+1}}}{1-\varphi L_{t+1} \frac{F_{LL,t+1}}{F_{L,t+1}}}) \tilde{a}_{t,s^N}$$

$$+ (1+\varphi) S_{t+1,\tilde{s}^N} \tilde{\tau}_{t+1} l_{\tilde{s}^N,e,t+1} \tilde{y}_0^N \frac{F_{KL,t+1}}{1-\varphi L_{t+1} \frac{F_{LL,t+1}}{F_{L,t+1}}} \Bigg) \Bigg]$$

$$+ \mathbb{E}_t \sum_{\tilde{s}^N} S_{t+1,\tilde{s}^N} \left(F_{KK,t+1} + \frac{\varphi L_{t+1} \frac{F_{KL,t+1}^2}{F_{L,t+1}}}{1-\varphi L_{t+1} \frac{F_{LL,t+1}}{F_{L,t+1}}} \right) \Lambda_{t+1,\tilde{s}^N} \xi_{\tilde{s}^N} U_{c,\tilde{s}^N,t+1}$$

$$- \gamma g'(a_{t,s^N}) + \lambda_{t,s^N} \gamma g''(a_{t,s^N}).$$

Due to the exterior penalty function, for histories for which $a_{t,s^N} \leq -\bar{a}$, $a_{t,s^N} \to -\bar{a}$ as $\gamma \to \infty$ (Assumption B has to be fulfilled at any optimal equilibrium). In addition, the previous equality implies that $\lambda_{t,s^N} \to 0$ when $\gamma \to \infty$ and that the constraints "disappear", as was shown in Section D.1. See Luenberger and Ye (2016) for a proof in a more general case. Finally, for histories for which $a_{t,s^N} > \bar{a}$, the previous constraint converges to the same constraint as in the initial truncated program (given in (71)). In our equilibrium allocation without penalty function, we have $\lambda_{t,s^N} = 0$ and $a_{t,s^N} = -\bar{a}$ for the credit-constrained history. As the FOC for ϕ_t is the same in our initial problem and in the problem with a penalty function, our allocation is therefore a solution of the limit of program with infinitely concave penalty functions. This concludes the proof.

D.3 Generalizing the truncation theory to non-GHH utility functions

In this section, we generalize the truncation method to a separable instantaneous utility function, $U(c,l) = u(c) - v(l)$, instead of the GHH utility function we defined in (1). The functions u and v are supposed to be continuous, twice differentiable, increasing, and concave in both arguments. For the sake of clarity, the presentation follows the same structure as in the main text.

D.3.1 The set-up

Besides this more general utility function, the economy, including the risk structure, production sector, and UI scheme, among others, is strictly similar to the economy presented in Section 2.

The agent's program can now be written as:

$$\max_{\{c_t^i, l_t^i, a_t^i\}_{t=0}^\infty} \mathbb{E}_0 \sum_{t=0}^\infty \beta^t (u(c_t^i) - v(l_t^i 1_{e_t^i=e} + \zeta_{y_t^i} 1_{e_t^i=u})) \tag{103}$$

$$c_t^i + a_t^i = (1+r_t)a_{t-1}^i + ((1-\tau_t)l_t^i 1_{e_t^i=e} + \phi_t \bar{l}_t \left(y_t^i\right) 1_{e_t^i=u}) y_t^i w_t, \tag{104}$$

$$a_t^i \geq -\bar{a},\ c_t^i > 0,\ l_t^i > 0. \tag{105}$$

Compared to (5)–(7), the objective function reflects the different utility function, but the constraints are unchanged. The agents' first-order conditions become:

$$u'(c_t^i) = \beta \mathbb{E}_t \left[(1 + r_{t+1})u'(c_{t+1}^i)\right] + \nu_t^i, \tag{106}$$

$$v'(l_t^i) = (1 - \tau_t)w_t y_t^i u'(c_t^i), \tag{107}$$

where the Euler equation (107) obviously only holds for employed agents.

Market clearing conditions (12) are unchanged, and the only difference is that Euler equations for consumption and labor are more involved than in the GHH case. In particular, the individual labor supply is no longer a linear function of productivity.

D.3.2 The truncation

The island metaphor. The island metaphor is very similar to that of Section 3.1. The only difference is that we need to introduce a preference shifter for the labor supply, denoted by $\xi_{s^N}^l$ for island s^N. The preference shifter for consumption is still ξ_{s^N}, such that the island planner values the utility of island s^N by $\xi_{s^N} u(c_{t,s^N}) - \xi_{s^N}^l v(\hat{l}_{t,s^N})$. The agent's program becomes:

$$\max_{(c_{t,s^N}, l_{t,s^N}, a_{t,s^N}, \tilde{a}_{t,s^N})_{t\geq 0}^{s^N \in \mathcal{S}^N}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big[\sum_{s^N \in \mathcal{S}^N} S_{t,s^N} (\xi_{s^N} u(c_{t,s^N}) - \xi_{s^N}^l v(\hat{l}_{t,s^N})) \Big], \tag{108}$$

$$a_{t,s^N} + c_{t,s^N} = ((1 - \tau_t) l_{t,s^N} 1_{e_0^N = e} + \phi_t l_{t,s^N,e} 1_{e_0^N = u}) y_0^N w_t + (1 + r_t)\tilde{a}_{t,s^N}, \tag{109}$$

$$c_{t,s^N},\ l_{t,s^N} \geq 0, a_{t,s^N} \geq -\bar{a} \tag{110}$$

where $(S_{-1,s^N}, a_{-1,s^N})_{s^N \in \mathcal{S}^N}$ are given and $\hat{l}_{t,s^N} = l_{t,s^N} 1_{e_0^N = e} + \zeta_{y_0^N} 1_{e_0^N = u}$. The first-order conditions of the island-planner can be written as follows:

$$\xi_{s^N} u'(c_{t,s^N}) = \beta \mathbb{E}_t \Big[\sum_{\tilde{s}^N \succeq s^N} \Pi_{t+1,s^N,\tilde{s}^N} \xi_{\tilde{s}^N} u'(c_{t+1,\tilde{s}^N})(1 + r_{t+1}) \Big] + \nu_{t,s^N}, \tag{111}$$

$$\xi_{s^N}^l v'(l_{t,s^N}) = (1 - \tau_t) w_t y_0^N \xi_{s^N} u'(c_{t,s^N}), \tag{112}$$

$$\nu_{t,s^N}(a_{t,s^N} + \bar{a}) = 0 \text{ and } \nu_{t,s^N} \geq 0. \tag{113}$$

The rest is unchanged compared to the island economy of Section 3.1.

A decentralization mechanism. A very similar decentralization mechanism to that in Section 3.2 can be developed. The lump-sum tax retains the same formulation and remains unchanged.

D.3.3 Constructing an approximated model economy

Steady-state economy. Prices and allocations in the truncated economy are characterized by the Euler equations for consumption and labor (111) and (112), the collection of (unchanged) budget constraints (109), and the (unchanged) dynamics of history sizes (16). At the steady state, we have the following for the Euler equations:

$$\xi_{s^N} u'(c_{s^N}) = \nu_{s^N} + \beta(1+r) \sum_{\tilde{s}^N \succeq s^N} \Pi_{s^N,\tilde{s}^N} \xi_{\tilde{s}^N} u'(c_{\tilde{s}^N}), \tag{114}$$

$$\xi^l_{s^N} v'(l_{s^N}) = (1-\tau_t) w_t y_0^N \xi_{s^N} u'(c_{s^N}), \tag{115}$$

As in the GHH economy, the steady-state equilibrium is further characterized by some unchanged equations: market clearing equations (24), UI scheme budget balance (14), and factor prices (3).

Computing the ξs and ξ^ls. Determining the ξs and ξ^ls follows exactly the same logic as in the GHH economy, using the Bewley allocations, in particular of consumption and labor. The ξs (for consumption) are determined such that the truncated consumption levels (for each history) verify the steady-state consumption Euler equation (114). The difference compared with the GHH case is that, due to the separability of the instantaneous utility function in consumption and labor, this operation only requires consumption allocations of the Bewley model. However, this separability also enables us to determine the ξ^ls. This computation is straightforward and for each history, the ξ_{s^N} for each history s^N is computed using the Euler equation for labor (115).

This result can be formalized as in Proposition 2 for the GHH economy.

Proposition 7 (Constructing the ξs) *The preference shifters $(\xi_{s^N}, \xi^l_{s^N})_{s^N \in \mathcal{S}^N}$ can be computed at the steady state, such that the truncation, as in equation (29), of the steady-state allocations of the full-fledged model solves the Euler equations (114) and (115).*

We also have a similar convergence result to that of Proposition 3.

Proposition 8 (Convergence of allocations) *With similar notation to Proposition 3, we have the following convergence result for allocations:*

$$(c_{t,s^N}, a_{t,s^N}, l_{t,s^N})_{s^N} \longrightarrow_N (c_t(s^\infty), a_t(s^\infty), l_t(s^\infty))_{s^\infty \in \mathcal{S}^\infty}, \text{ almost surely.}$$

Similarly, for preference shifters: $\xi_{s^N} \longrightarrow_N 1$ *and* $\xi^l_{s^N} \longrightarrow_N 1$, *almost surely.*

In other words, consideration of a separable utility function does not affect the ability of the truncated model to replicate Bewley allocations. When the ξs and ξ^ls are chosen appropriately (Proposition 7) and truncation length becomes longer, the truncated allocations converge to the Bewley ones (Proposition 8).

The dynamics. As in Section 4.2, we make the assumption that in the presence of aggregate risk, the ξs and ξ^ls remain constant and equal to their steady-state values. Similarly, the set of credit-constrained histories is assumed to be time-invariant.

The truncated model in the presence of aggregate shocks is then characterized by Euler equations (111) and (112), as well as by the budget constraint (109), the market clearing conditions (24), the factor prices (3), and the UI budget constraint (14) (which remain unchanged).

The algorithm for computing the model with aggregate shocks remains the same as in the GHH case, except that it involves computing the ξs and ξ^ls using the steady-state allocations.

D.3.4 The Ramsey program

Formulation. We now formulate the Ramsey program in the presence of the separable utility function. The Ramsey program of equation (38) becomes:

$$\max_{((a_{t,s^N},c_{t,s^N},l_{t,s^N})_{s\in\mathcal{S}^N},\phi_t,\tau_t,r_t,w_t)_{t\geq 0}} \mathbb{E}_0\Big[\sum_{t,s^N}\beta^t S_{t,s^N}(\xi_{s^N}u(c_{t,s^N})-\xi^l_{s^N}v(\hat{l}_{t,s^N}))\Big], \tag{116}$$

subject to the new Euler equations for consumption (111) and labor (112), as well as the same set of equations as in the GHH case: (i) the budget constraints (37), (ii) the UI scheme budget balance (14), (iii) the market clearing constraints (24), and finally (iv) the factor prices (3).

The Ramsey program can be rewritten as in the GHH case, with one twist: the labor supply of employed agents now depends on their consumption choice. Indeed, after some algebraic manipulations, we obtain the following lemma (which parallels Lemma 1 in the GHH case).

Lemma 4 (Simplified Ramsey problem) *The Ramsey problem can be simplified into:*

$$\max_{\left((a_{t,s^N},c_{t,s^N},l_{t,s^N})_{s^N\in\mathcal{S}^N},\phi_t,\tau_t\right)_{t\geq 0}} \mathbb{E}_0\sum_{t,s^N}\beta^t S_{t,s^N}\Big(\xi_{s^N}u(c_{t,s^N})-\xi^l_{s^N}v(\hat{l}_{t,s^N}) \tag{117}$$

$$-\Big(\lambda_{t,s^N}-(1+r_t)\Lambda_{t,s^N}\Big)\xi_{s^N}u'(c_{t,s^N})\Big)$$

$$s.t.\ a_{t,s^N}\geq -\overline{a}\ and\ \lambda_{t,s^N}=0\ if\ a_{t,s^N}=-\overline{a}, \tag{118}$$

and subject to equations (111)–(112), as well as (37), (24), (14), and (3).

Ramsey first-order conditions. To simplify the derivation of first-order conditions, we will also assume that the labor supply $(l_{t,s^N})_{s^N\in\mathcal{S}^N}$ is a choice variable, in addition to savings choices $(a_{t,s^N})_{s^N\in\mathcal{S}^N}$ and the replacement rate ϕ_t that were already present in the GHH case. The non-trivial Euler equation for labor choice in fact complicates the derivation of first-order conditions.

The Lagrange multiplier for the consumption Euler equation is still denoted by $\beta^t S_{t,s^N}\lambda_{t,s^N}$, while the multiplier for the labor Euler equation is $\beta^t S_{t,s^N}\mu_{t,s^N}$. We introduce $\tilde{\Psi}_{t,s^N}$, defined as:

$$\tilde{\Psi}_{t,s^N}=\Psi_{t,s^N}+(1-\tau_t)F_{L,t}\mu_{t,s^N}\xi_{s^N}u''(c_{t,s^N}),$$

which reflects the value of liquidity. In addition to the GHH case, it accounts for the fact that the labor supply diminishes if consumption increases. The FOC for saving choices is:

$$\begin{aligned}\tilde{\Psi}_{t,s^N} &= \beta \sum_{\tilde{s}^N} \mathbb{E}_t \left[\left((1+r_{t+1})\Pi_{t+1,s^N,\tilde{s}^N} + F_{KK,t+1} S_{t+1,\tilde{s}^N} \tilde{a}_{\tilde{s}^N,t} \right) \tilde{\Psi}_{t+1,\tilde{s}^N} \right] \\ &+ \beta F_{KL,t+1} \sum_{\tilde{s}^N} \mathbb{E}_t \left[S_{t+1,\tilde{s}^N} \tilde{\Psi}_{t+1,\tilde{s}^N} ((1-\tau_{t+1}) 1_{\tilde{e}_0^N = e} + \frac{S_{t+1,e}}{S_{t+1,u}} \tau_{t+1} 1_{\tilde{e}_0^N = u}) \tilde{y}_0^N l_{t+1,\tilde{s}^N} \right] \\ &+ \beta \sum_{\tilde{s}^N} \mathbb{E}_t \left[S_{t+1,\tilde{s}^N} \left(F_{KK,t+1} \Lambda_{t+1,\tilde{s}^N} + (1-\tau_{t+1}) F_{KL,t+1} \mu_{t+1,\tilde{s}^N} \xi_{\tilde{s}^N} \right) \xi_{\tilde{s}^N} u'_{t+1,\tilde{s}^N} \right]. \end{aligned} \tag{119}$$

The two first lines are very similar to the GHH case, while the last one includes a term that is specific to our separable utility function. Second, the FOC with respect to labor supply is:

$$\begin{aligned} &\frac{\xi^l_{s^N}}{y_0} \left(v'(l_{t,s^N}) + \mu_{t,s^N} v''(l_{t,s^N}) \right) - \left((1-\tau_t) 1_{\tilde{e}_0 = e} + \frac{S_{t,e}}{S_{t,u}} \tau_t 1_{\tilde{e}_0 = u} \right) F_{L,t} \xi_{s^N} \tilde{\Psi}_{t,s^N} \\ &= \sum_{\tilde{s}^N \in \mathcal{S}^N} S_{t,\tilde{s}^N} \xi_{\tilde{s}^N} \tilde{\Psi}_{t,\tilde{s}^N} \left(F_{KL,t} \tilde{a}_{s^N,t-1} + ((1-\tau_t) 1_{\tilde{e}_0 = e} + \frac{S_{t,e}}{S_{t,u}} \tau_t 1_{\tilde{e}_0 = u} y_0) F_{LL,t} l_{t,\tilde{s}^N} \right) \\ &+ \sum_{\tilde{s}^N} S_{t,\tilde{s}^N} \left(F_{KL,t} \Lambda_{t,\tilde{s}^N} + (1-\tau_t) F_{LL,t} \mu_{t,\tilde{s}^N} \right) \xi_{\tilde{s}^N} u'_{t,\tilde{s}^N}. \end{aligned} \tag{120}$$

Finally, the first-order equation with respect to the replacement rate is:

$$\sum_{s^N \in \mathcal{S}^N} S_{t,s^N} \mu_{t,s^N} \xi_{s^N} u'(c_{t,s^N}) = \sum_{s^N \in \mathcal{S}^N} S_{t,s^N} \xi_{s^N} \tilde{\Psi}_{t,s^N} (-1_{e_0^N = e} + \frac{S_{t,e}}{S_{t,u}} 1_{e_0^N = u}) y_0^N l_{t,\tilde{s}^N}, \tag{121}$$

which balances the benefits of a higher replacement rate with its cost.

D.3.5 Conclusion

This supplemental material has shown that our truncation method can be extended quite easily to a more general utility function. The main steps of the method remain similar: (i) derive a limited-heterogeneity representation based on preference shifters; (ii) calibrate the preference shifters from the steady-state distribution of consumption and labor supply; and (iii) use the preference shifters to simulate the economy in the presence of aggregate shocks. The only difference is that due to the non-linearity of the Euler equation for labor supply, we need to include a second preference shifter that is specific to labor. The Ramsey problem can similarly be computed using the FOCs (119)–(121).

D.3.6 Proof: Derivation of Ramsey first-order conditions

This section solely contains the derivation of first-order conditions (119)–(121). The Lagrangian associated with the Ramsey program laid out in Lemma 4 can be expressed as:

$$
\begin{aligned}
\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{s^N \in \mathcal{S}^N} S_{t,s^N} \Big(& \xi_{s^N} u(c_{t,s^N}) - \xi^l_{s^N} v(\hat{l}_{t,s^N}) \\
& - \Big(\lambda_{t,s^N} - (1 + F_K(K_{t-1}, L_t))\Lambda_{t,s^N}\Big) \xi_{s^N} u'(c_{t,s^N})\Big) \\
& - \sum_{t=0}^{\infty} \beta^t \sum_{s^N \in \mathcal{S}^N} S_{t,s^N} \mu_{t,s^N} \left(\xi^l_{s^N} l_{t,s^N}^{\frac{1}{\varphi}} - \xi_{s^N}(1-\tau_t) F_L(K_{t-1}, L_t) u'(c_{t,s^N}) \right),
\end{aligned} \tag{122}
$$

Derivative with respect to a_{t,s^N}. We follow the same steps as in the main text of the paper. The derivative of the Lagrangian (122) according to a_{t,s^N} is:

$$
\begin{aligned}
\tilde{\Psi}_{t,s^N} = & \beta \sum_{\tilde{s}^N} \mathbb{E}_t \left[\Big((1 + r_{t+1}) \Pi_{t+1,s^N,\tilde{s}^N} + F_{KK,t+1} S_{t+1,\tilde{s}^N} \tilde{a}_{\tilde{s}^N,t} \Big) \tilde{\Psi}_{t+1,\tilde{s}^N} \right] \\
& + \beta F_{KL,t+1} \sum_{\tilde{s}^N} \mathbb{E}_t \left[S_{t+1,\tilde{s}^N} \tilde{\Psi}_{t+1,\tilde{s}^N} \left((1-\tau_{t+1}) 1_{\tilde{e}_0 = e} + \frac{S_{t+1,e}}{S_{t+1,u}} \tau_{t+1} 1_{\tilde{e}_0 = u} \right) y_0 l_{t+1,\tilde{s}^N} \right] \\
& + \beta \sum_{\tilde{s}^N} \mathbb{E}_t \left[S_{t+1,\tilde{s}^N} \Big(F_{KK,t+1} \Lambda_{t+1,\tilde{s}^N} + (1-\tau_{t+1}) F_{KL,t+1} \mu_{t+1,\tilde{s}^N} \xi_{\tilde{s}^N} \Big) \xi_{\tilde{s}^N} u'_{t+1,\tilde{s}^N} \right].
\end{aligned}
$$

Derivative with respect to l_{t,s^N}. We find:

$$
\begin{aligned}
& \frac{\xi^l_{s^N}}{y_0} \Big(v'(l_{t,s^N}) + \mu_{t,s^N} v''(l_{t,s^N}) \Big) - \left((1-\tau_t) 1_{\tilde{e}_0 = e} + \frac{S_{t,e}}{S_{t,u}} \tau_t 1_{\tilde{e}_0 = u} \right) F_{L,t} \xi_{s^N} \tilde{\Psi}_{t,s^N} \\
& = \sum_{\tilde{s}^N \in \mathcal{S}^N} S_{t,\tilde{s}^N} \xi_{\tilde{s}^N} \tilde{\Psi}_{t,\tilde{s}^N} \left(F_{KL,t} \tilde{a}_{s^N,t-1} + ((1-\tau_t) 1_{\tilde{e}_0 = e} + \frac{S_{t,e}}{S_{t,u}} \tau_t 1_{\tilde{e}_0 = u} y_0) F_{LL,t} l_{t,\tilde{s}^N} \right) \\
& + \sum_{\tilde{s}^N} S_{t,\tilde{s}^N} \Big(F_{KL,t} \Lambda_{t,\tilde{s}^N} + (1-\tau_t) F_{LL,t} \mu_{t,\tilde{s}^N} \Big) \xi_{\tilde{s}^N} u'_{t,\tilde{s}^N}.
\end{aligned} \tag{123}
$$

Derivative with respect to ϕ_t. We have:

$$
0 = \sum_{s^N \in \mathcal{S}^N} S_{t,s^N} \mu_{t,s^N} \xi_{s^N} u'(c_{t,s^N}) + \sum_{s^N \in \mathcal{S}^N} S_{t,s^N} \xi_{s^N} \tilde{\Psi}_{t,s^N} \left(-1_{\tilde{e}_0 = e} + \frac{S_{t,e}}{S_{t,u}} 1_{\tilde{e}_0 = u} \right) y_0 l_{t,\tilde{s}^N}. \tag{124}
$$

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