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Fernando Restoy
Philippe Weil

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ABSTRACT

This paper reconsiders the determination of asset returns in a model with Kreps-Porteus generalized isoelastic preferences where returns appear governed, on the basis of Euler equations, by a combination of the two most common measures of risk -- covariance with the market return and covariance with consumption. To go beyond Euler equations and to take into account the links that the consumers' optimal behavior establishes, through a budget constraint, between market returns and consumption, we derive an approximate consumption function (obtained, as in Campbell (1994), by log-linear approximation). Arguing that total consumer wealth is unobservable, we use this consumption function to reconstruct from observed consumption data i) the wealth that supports the agents' consumption as an optimal income, and ii) the rate of return on the consumers' wealth portfolio. This procedure enables us to derive formulas that (approximately) price, in the tradition of Lucas (1978), all assets as a function of their payoffs and of consumption. The generalized consumption CAPM that we obtain is derived for both homoskedastic and heteroskedastic consumption processes. We also use our approximate pricing kernel to highlight the crucial role of temporal risk aversion in the determination of the equilibrium term structure of real interest rates.

Fernando Restoy
Research Department
Blanco de España
Alcala 50
28014 Madrid
SPAIN
restoy@bde.es

Philippe Weil
European Center for Advanced Research
in Economics
ECARE
39 av. F.D. Roosevelt
1050 Bruxelles
BELGUIM
and NBER
philippe.weil@ulb.ac.be

1. Introduction

This paper is motivated by two observations.

The first one is empirical. According to Kendrick (1976), non-human wealth represented only 47% of total wealth in the United States in 1969. Moreover, total business wealth amounted at the same date, at market value, to only 21% of total domestic wealth. Moreover, in 1990, the market value of corporate stocks was $b4,165 billions of dollars, while residential mortgages represented $b2,924, corporate bonds $b987, US Federal, state and local securities $b2,705. Even if one excludes human capital, there is therefore much more to consumer wealth than stocks, and much more to the rate of return on wealth than the rate of return on the stock market. The rate of return on the stock market—the measure of the rate of return on wealth used by most of the capital asset pricing literature—is in all likelihood a very poor proxy for the rate of return on wealth.

The second observation pertains to theory. Many authors seem to have forgotten that two of the main contenders in the search for the explanation of excess returns—the static (or market) capital asset pricing model (SCAPM) and the consumption capital asset pricing model (CCAPM)—are not independent and unrelated models. Regardless of the view one takes on the exact degree of rationality of consumers, the length of their economic lifetime, or the completeness of markets, there must be *some* link between consumption and asset returns, between quantities and prices. In the simplest case that we will explore in this paper—the complete markets, representative agent framework—this link has a name: the consumption function. The reason for the neglect of the consumption function is obvious: it is difficult to solve for it in interesting problems. But technical difficulties are no valid reason for sticking with Euler equations when their sole consideration leads one to mistakenly believe that there is no theoretical link between the two measures of risk represented by covariance of asset returns with the wealth return and consumption.

In this paper, we attempt to take these two remarks seriously. We develop an equilibrium capital asset pricing model based on Kreps-Porteus preferences—as exposed in Epstein and Zin (1989), Giovannini and Weil (1989) and Weil (1990)—in which the marginal rate of substitution depends both on the rate of growth of consumption but also on the rate of return on wealth. But, contrary to previous authors with the glaring exception of Campbell (1994), we make explicit (albeit through log-linear approximations) the links between consumption and wealth returns to characterize equilibrium excess returns.

Although our paper conforms to Campbell's philosophy—we go beyond Euler equations by also using the information contained in the consumption function— it takes a radically different perspective on the goals to be achieved. Campbell's objective is to use the consumption function to eliminate consumption from his asset pricing expressions, or, as he puts it, to compute asset prices "without consumption data". His rationale is that aggregate per capita consumption of non-durables and services i) is a poor measure for the consumption of market participants, and ii) is subject to measurement and time-aggregation errors. As a result, he derives expressions for excess returns that look like a generalized version of the market CAPM.

Our view, suggested at the outset, is that, from a data perspective, the difficulties involved with measuring the rate of return on wealth are as large as, if not larger than, those involved with measuring the consumption of market participants:[1] the rate of return on total wealth is not simply mismeasured, it is not measured at all. Reversing Campbell's method, we observe that consumer's total wealth can be reconstructed *from consumption data alone* under the maintained assumption that the consumption data that we observe were generated by (Kreps-Porteus) utility maximizing agents. From these reconstructed total wealth data, we can compute an implied series of rates of return on total consumer wealth—which again is solely a function of consumption data. These reconstructed wealth returns can then be used to calculate an (approximate) pricing kernel which, because it is in turn also solely a function of consumption data, yields a generalized consumption CAPM.

From an equilibrium perspective, what we are doing is simply apply the equilibrium asset pricing methods of Lucas (1978) or Mehra and Prescott (1985) but without the "fruit tree" imagery: we take consumption as given, and we infer back from budget constraints and first-order conditions the wealth and the asset prices that support observed consumption as a utility-maximizing outcome. The (approximate) asset pricing kernel that we compute enables us to price any asset (including wealth) and determine its equilibrium returns solely as a function its payoff and of observed consumption. This procedure allows us to price the stock market as a *subset* of wealth, and to accurately characterize the implications of this class of models for the equity premium (as distinct from the "wealth premium" implicitly computed

[1] In another paper, Campbell (1993) attempts to circumvent the absence of data on the rate of return on human wealth by assuming that human wealth is constant fraction of total wealth, and that its return can be approximated by a linear function of labor income growth. Since human wealth is not the only component of wealth for which no data are available, this is only a partial solution to the data difficulties which motivate us.

by the authors who identify wealth to stocks). It is enables us to highlight, in the spirit of Drèze and Modigliani (1972), the crucial role of temporal risk aversion in the determination of the equilibrium term structure of interest rates.

The paper is organized as follows. We present the model, and the basics of our reconstruction of wealth from observed consumption data, in section 2. We then turn, in section 3, to the determination of asset prices in a world with a homoskedastic consumption process, postponing to section 4 the analysis of equilibrium with heteroskedastic consumption. In section 5, we examine the implications of our model for the term structure of real interest rates. The conclusion offers directions for future research.

2. The Model

The economy consists of many identical infinitely-lived consumers. All wealth is assumed to be tradeable. Let W_t denote wealth at time t, and $R_{w,t}$ the rate of return on the "wealth portfolio" between dates $t-1$ and t. Wealth can be accumulated in many forms, among which money, stocks, bonds, real estate, physical and human capital. The rate of return on wealth will be, in equilibrium, the rate of return on this exhaustive "market portfolio."

A representative consumer faces the following budget constraint:

$$W_{t+1} = R_{w,t+1}(W_t - C_t). \tag{2.1}$$

In addition, our consumer's initial wealth is given, and she faces a solvency constraint to rule out Ponzi games.

Following Epstein and Zin (1989) and Weil (1990), we assume that consumers have Kreps-Porteus *generalized isoelastic preferences* (GIP) with a constant elasticity of substitution, $1/\rho$, and a constant (but in general unrelated) coefficient of relative risk aversion, γ, for timeless gambles. These preferences can be represented recursively as

$$V_t = \left\{(1-\beta)C_t^{1-\rho} + \beta(\mathrm{E}_t\, V_{t+1})^{1/\theta}\right\}^{\theta}, \tag{2.2}$$

where $0 < \beta < 1$, V_t is the agent's utility at time t, C_t denotes consumption, the operator E_t denotes mathematical expectation conditional on information available at t, and the parameter

$$\theta = (1-\gamma)/(1-\rho)$$

measures the departure of the agents' preferences away from the time-additive isoelastic expected utility framework. Thus, when $\theta = 1$, the preferences in (2.2) reduce to the standard time-additive isoelastic expected utility representation.

2.1. **The Euler equation.** Epstein and Zin (1989) have shown that for any asset with gross rate of return $R_{i,t+1}$ between dates t and $t+1$ the following Euler equation must be satisfied:

$$\mathrm{E}_t \left\{ \beta^\theta \left[\frac{C_{t+1}}{C_t} \right]^{-\rho\theta} R_{w,t+1}^{\theta-1} R_{i,t+1} \right\} = 1. \tag{2.3}$$

Assume that consumption and asset returns have a joint conditional lognormal distribution.[2] Then, taking logs on both sides of (2.3) and subtracting the version of (2.3) that holds for a safe one-period bond with gross rate of return $R_{f,t+1}$, we obtain the familiar[3] expression for the excess return on any asset:

$$\mathrm{E}_t\, r_{i,t+1} - r_{f,t+1} = -\frac{\sigma_{ii,t}}{2} + \rho\theta\sigma_{ic,t} + (1-\theta)\sigma_{iw,t}, \tag{2.4}$$

where lowercase letters denote the logarithm of their uppercase counterpart, and where $\sigma_{pq,t}$ denotes the conditional covariance at time t between random variables p_{t+1} and q_{t+1}.

This equation is often interpreted[4] as implying that, for GIP preferences, excess returns are determined by a combination of the CCAPM and of the SCAPM. This is misleading, since consumption and the return on wealth (or $\sigma_{ic,t}$ and $\sigma_{iw,t}$) in general depend on each other, through the behavior of forward-looking, optimizing consumers who must satisfy the budget constraint (2.1).

2.2. **The (approximate) relation between consumption and asset returns.** To make explicit the link between consumption and the rate of return on wealth, and to obtain as a consequence more meaningful asset pricing formulas, one must go beyond the necessary, but not sufficient, Euler equation and use the information provided by the budget constraint the link between consumption and the rate of return on wealth. This objective unfortunately requires that we get around the difficulty that budget constraints are multiplicative in consumption and wealth returns—a fact which in general precludes under uncertainty the analytical derivation of the consumption function.

[2]This assumption will, again, be validated in equilibrium.

[3]See, for instance, Giovannini and Weil (1989).

[4]See, for instance, Epstein and Zin (1989) or Giovannini and Weil (1989).

To circumvent this difficulty, we proceed as in Campbell (1994), and log linearize the budget constraint. Let

$$A_t \equiv C_t / W_t$$

denote the consumption-wealth ratio, and

$$X_{t+1} \equiv C_{t+1} / C_t$$

represent the rate of growth of consumption. The budget constraint (2.1) can then be rewritten as:

$$R_{w,t+1} = \frac{X_{t+1}}{A_{t+1}(A_t^{-1} - 1)}, \tag{2.5}$$

or, in logarithms,

$$r_{w,t+1} = x_{t+1} - a_{t+1} - \log(\mathrm{e}^{-a_t} - 1). \tag{2.6}$$

Taking a first-order Taylor expansion of $\log(\mathrm{e}^{-a_t} - 1)$ around the unconditional mean of the logarithm of the consumption-wealth ratio,[5] we obtain the following approximate log-linear budget constraint:

$$r_{w,t+1} \approx x_{t+1} - a_{t+1} + \frac{1}{\delta} a_t - k \tag{2.7}$$

where k and δ ($0 < \delta < 1$) are two easily computed linearization constants.[6]

Equation (2.7) implies that

$$\mathrm{S}_{t+1}\, r_{w,t+1} = \mathrm{S}_{t+1}(x_{t+1} - a_{t+1}). \tag{2.8}$$

where, for any random variable q_{t+1}, the *surprise operator* S is defined as

$$\mathrm{S}_{t+1}\, q_{t+1} \equiv \mathrm{E}_{t+1}\, q_{t+1} - \mathrm{E}_t\, q_{t+1} = q_{t+1} - \mathrm{E}_t\, q_{t+1}.$$

An implication of (2.8) is that, if the budget constraint is satisfied, the conditional covariance of any asset return with the rate of return on wealth is just the difference between, one the one hand, the conditional covariance of this asset's return with consumption and, on the other hand, the conditional covariance of this asset's return with the (log) propensity to consume. Namely,

$$\sigma_{iw,t} = \sigma_{ic,t} - \sigma_{ia,t}. \tag{2.9}$$

While this equation is not operational (we have not yet said anything about $\sigma_{ia,t}$), it has the merit of pointing out that the covariance between individual returns and the return on the wealth portfolio is endogenous, through its

[5] Throughout, we assume stationarity. The assumption that log consumption-wealth ratio is stationary is validated in equilibrium.

[6] The constant δ is equal to $1 - \exp[E(a)]$, and the constant k is equal to $\log(\delta/1 - \delta) + (1/\delta)E(a)$.

dependence on the covariance $\sigma_{ia,t}$ between individual returns and the still to be computed endogenous propensity to consume.

2.3. **Eliminating the rate of return on wealth.** Since our goal in this paper is to derive asset pricing expressions that do not involve the unobservable rate of return on the wealth portfolio, we can use equation (2.9) to eliminate the terms involving the rate of return on wealth from the excess returns expression (2.4):[7]

$$\mathrm{E}_t\, r_{i,t+1} - r_{f,t+1} = -\frac{\sigma_{ii,t}}{2} + \gamma\sigma_{ic,t} + (\theta - 1)\sigma_{ia,t}. \tag{2.10}$$

Equation (2.10) highlights two important special cases that are explored systematically in Giovannini and Weil (1989):

- *in the expected utility case* ($\theta = 1$), equation (2.10) is the excess return equation characteristic of the CCAPM.
- when *the consumption-wealth ratio is constant* ($a_t = a$ for all t), equation (2.10) implies that asset returns must also conform to the CCAPM, but that model should then be equivalent to the SCAPM, since consumption growth and the rate of return on wealth are then perfectly correlated.[8]

2.4. **The consumption-wealth ratio.** The expression for excess returns in (2.10) is still non-operational, for the extra-term introduced by GIP preferences, $\sigma_{ia,t}$ depends on the propensity to consume that we have not yet calculated. We now take up the task of characterizing the optimal consumption-wealth ratio.

From the version of the Euler equation (2.3) that holds for the wealth return, it follows that, when $\theta \neq 0$,

$$\mathrm{E}_t\, r_{w,t+1} = -\log\beta + \rho\,\mathrm{E}_t\, x_{t+1} - \frac{\theta}{2}\,\mathrm{Var}_t(r_{w,t+1} - \rho x_{t+1}) \tag{2.11}$$

Now, from (2.7),

$$\mathrm{E}_t\, r_{w,t+1} = \mathrm{E}_t\, x_{t+1} - \mathrm{E}_t\, a_{t+1} + \frac{1}{\delta}a_t - k \tag{2.12}$$

$$\mathrm{Var}_t(r_{w,t+1} - \rho x_{t+1}) = \mathrm{Var}_t(a_{t+1} - (1-\rho)x_{t+1}). \tag{2.13}$$

[7] It is at this point that we part ways from Campbell (1994).

[8] See, for instance, the budget constraint (2.6).

Substituting (2.12) and (2.13) into (2.11) yields

$$a_t \approx \delta \left(k - \log \beta + \mathrm{E}_t[a_{t+1} - (1-\rho)x_{t+1}] - \frac{\theta}{2} \mathrm{Var}_t[a_{t+1} - (1-\rho)x_{t+1}] \right). \tag{2.14}$$

Consistent with our approach that seeks to express all variables in terms of consumption, we interpret (2.14) as a difference equation in the a's driven by the x's. Under the transversality condition $\lim_{s\to\infty} \delta^s a_{t+s} = 0$,[9] (2.14) implies that

$$a_t = \frac{\delta k}{1-\delta} - (1-\rho)\, \mathrm{E}_t \sum_{j=1}^{\infty} \delta^j x_{t+j} - \frac{\theta}{2} \mathrm{E}_t \sum_{j=1}^{\infty} \delta^j \, \mathrm{Var}_{t+j-1}\, z_{t+j}, \tag{2.15}$$

where

$$z_t \equiv a_t - (1-\rho)x_t. \tag{2.16}$$

Two remarks are in order. First, (2.15) still does not provide the solution for the consumption wealth ratio a_t as a function of consumption and preferences, since conditional first moments of future conditional second moments appear on the right-hand side. But as we shall see below, (2.15) does provide a clue as to the functional form of the solution. Second, uniqueness of the solution (when the solution exists) is guaranteed by the fact that the transversality condition, the Euler equation and the budget constraint (all of which are imbedded in (2.15)) are jointly necessary and sufficient for a unique solution to the optimal consumption problem we are approximating.

2.5. **The equilibrium concept.** To proceed beyond (2.15) and to solve for the consumption-wealth ratio, we need to make distributional assumptions on the consumption growth process, and more specifically on its conditional second moments. There are two ways to view these distributional assumptions, and two associated interpretations of the results of the model.

One can either take a *general equilibrium* perspective, and imagine as in Lucas (1978) that output is non storable manna falling from a tree. In Lucas' economy, consumption is equal to output, and the stochastic process we assume for consumption is just given by the exogenous stochastic process followed by output. In this perspective, our model provides approximate, but explicit, formulas for general equilibrium asset prices in a Kreps-Porteus version of the Lucas model, and an analytical method to understand the numerical results in Weil (1989).

[9] This condition is also used by Campbell (1994).

Or, alternatively, one can take a *partial equilibrium* perspective, and note that, for any given consumption process, one can always compute the total wealth and the asset prices that are consistent with the hypothesis that consumers behave optimally, and satisfy their budget constraints, their solvency constraint and their Euler equations. In this perspective, our model has obviously nothing to say about the determination of consumption: it just takes the consumption process from the data, and focuses instead on the computation of the wealth process and on determination of prices that support it.[10]

While we prefer the second, partial equilibrium interpretation (it has the merit of not implying that consumption should be equal to output, and of being more forthright as to its conceptual limitations), the reader may choose to adopt instead the first, general equilibrium interpretation. Nothing in our analysis hinges on the view one takes. As a matter of fact, we will refer from now on to "equilibrium" returns: the reader is free to think of them as general or partial equilibrium returns. What matters, however, is that we take consumption as given, and not the equilibrium rate of return on wealth as in Campbell (1994).

With these methodological caveats in mind, we are now ready to turn to the determination of equilibrium returns. We examine two cases in the next two sections. First, a case in which log consumption growth is conditionally homoskedastic. Second, a case in which consumption is conditionally heteroskedastic and follows an AR(1) process with GARCH(1,1) disturbances.

3. Equilibrium Returns: Homoskedastic Consumption

Suppose the log consumption growth and the conditional mean of future log consumption growth are jointly conditionally homoskedastic, so that the conditional variance of consumption growth, its conditional covariance with future expected consumption growth and the conditional variance of future expected consumption growth are constant over time.

3.1. **The propensity to consume.** When conditional second-order moments are constant, it is straightforward to check that the solution to (2.15) is simply

$$a_t = g - (1 - \rho) \operatorname{E}_t \sum_{j=1}^{\infty} \delta^j x_{t+j} \tag{3.1}$$

[10]One can of course reinterpret the models of Lucas (1978) or Mehra and Prescott (1985) in that way.

where the constant g is given by

$$g \equiv \frac{\delta}{1-\delta}\left[k - \log\beta - \frac{\theta(1-\rho)^2}{2}(\sigma_{cc} + \sigma_{hh} + 2\sigma_{ch})\right],$$

and where σ_{hh} and σ_{ch} respectively denote, under the notation

$$h_{t+1} \equiv \mathrm{E}_{t+1}\sum_{i=1}^{\infty}\delta^j x_{t+j+1},$$

the conditional variance of expected discounted future consumption growth, and the conditional covariance between consumption growth and expected discounted future consumption growth.

The interpretation of (3.1) is straightforward and intuitive. High expected future consumption stems, given current wealth, from high expected future returns on wealth. If the elasticity of intertemporal substitution is large ($\rho < 1$)—i.e., if substitution effects are stronger than income effects—our consumer reacts to high expected future returns by consuming less, so that the consumption-wealth ratio declines. If, on the other hand, the elasticity of intertemporal substitution is small ($\rho > 1$), high expected future consumption is associated with an increase in the propensity to consume.

Note that, as a result of (3.1), homoskedasticity of consumption growth and future expected consumption growth implies homoskedasticity of the consumption wealth ratio.

3.2. **The rate of return on wealth.** From (2.7) and (3.1), we can reconstruct the equilibrium rate of return on the (unobservable) wealth portfolio:

$$r_{w,t+1} = u + \rho x_{t+1} + (1-\rho)\,\mathrm{S}_{t+1}\sum_{j=0}^{\infty}\delta^j x_{t+j+1}, \tag{3.2}$$

where

$$u = -\ln\beta - \frac{\theta(1-\rho)^2}{2}(\sigma_{cc} + \sigma_{hh} + 2\sigma_{ch}). \tag{3.3}$$

This equation (3.2) enables us to compute, date by date and state by state, the return on the wealth portfolio from observable consumption data alone. This equilibrium rate of return on wealth has to be understood as the return on wealth which supports, under the assumption that the model is true, the consumption process as an *equilibrium* consumption path. In other terms, equation (3.2) allows us to reconstruct the unobserved return on wealth from observed consumption data.

An implication of equation (3.2) is that:

$$\mathrm{E}_t\, r_{w,t+1} = u + \rho\, \mathrm{E}_t\, x_{t+1}. \tag{3.4}$$

As a consequence, in this homoskedastic world, the conditional expected return on wealth must be higher, for a given (positive) conditional expected rate of growth of consumption, the lower the elasticity of intertemporal substitution—i.e, the higher ρ. As under certainty, this is required to convince consumers to overcome in equilibrium their distaste for intertemporal substitution.

3.3. **The approximate pricing kernel.** The expression (3.2) for the implied rate of return of the wealth portfolio enables us to compute the (approximate) equilibrium pricing kernel for this economy as a function of the consumption process. From the Euler equation (2.3), it follows that the log marginal rate of substitution between periods t and $t+1$ is

$$m_{t+1} = \theta \ln \beta - \rho\theta x_{t+1} + (\theta - 1) r_{w,t+1}.$$

Substituting (3.2) into this expression and rearranging, we find that

$$m_{t+1} = v - \rho x_{t+1} + (\rho - \gamma)\, \mathrm{S}_{t+1} \sum_{j=0}^{\infty} \delta^j x_{t+j+1},$$

where $v = \theta \ln \beta + (\theta - 1)u$. In the standard time and state-additive case ($\gamma = \rho$) and/or in an i.i.d. world ($\mathrm{S}_{t+1} \sum_{j=0}^{\infty} \delta^j x_{t+j+1} = 0$), the (log) pricing kernel is, up to a constant, simply a linear function of the (log) consumption growth rate. In all other cases, it depends on in addition on the news received at time $t+1$ about consumption growth rates in periods $t+1$ and beyond.

3.4. **Excess returns.** It follows from equation (3.1) that surprises in the propensity to consume are given by

$$\mathrm{S}_{t+1}\, a_{t+1} = -(1 - \rho)\, \mathrm{S}_{t+1} \sum_{j=1}^{\infty} \delta^j x_{t+j+1}, \tag{3.5}$$

so that the conditional covariance between the return on any asset and the marginal propensity to consume is

$$\sigma_{ia} = -(1 - \rho)\sigma_{ih} \tag{3.6}$$

where σ_{ih} denotes the conditional covariance between the return on asset i and expectations of future (discounted) consumption growth.

Therefore, substituting (3.6) into (2.4), the equilibrium excess return on any asset satisfies

$$\mathrm{E}_t\, r_{i,t+1} - r_{f,t+1} = -\frac{\sigma_{ii}}{2} + \gamma\sigma_{ic} + (\gamma - \rho)\sigma_{ih}. \tag{3.7}$$

According to this expression, the excess return on any asset depends on its own variance (a Jensen's inequality term), on its conditional covariance with contemporaneous consumption, and on its conditional covariance with future consumption. To understand (3.7), it is best to think of time as consisting of three dates: today, tomorrow, and the day after tomorrow (or future), and to examine separately the three terms $\gamma\sigma_{ic}$, $\gamma\sigma_{ih}$, and $-\rho\sigma_{ih}$ that govern excess returns.

An asset with $\sigma_{ic} > 0$ is an asset whose return between today and tomorrow tends to be high (low) when consumption tomorrow is high (low). Holding such an asset in one's portfolio makes it difficult to smooth consumption over states of nature. Therefore, risk averse investors require a premium over the riskless return to hold this asset. This premium is larger the larger the consumers' aversion to substitution over states of nature, i.e., the larger their coefficient of relative risk aversion γ. The presence of the term $\gamma\sigma_{ic}$ on the right hand-side of (3.7) thus reflects our consumer's *aversion to substitution over states of nature.*

An asset with $\sigma_{ih} > 0$ is an asset whose return between today and tomorrow tends to be high (low) when there are good (bad) news about consumption the day after tomorrow. Such an asset is not attractive, as it provides, say, more wealth tomorrow when good news about future consumption make it less desirable to be able to save for precautionary motives.[11] As a result, our consumers require a premium to hold this asset, and the term $\gamma\sigma_{ih}$ reflects the desire of our consumers' *precautionary saving motive.*

However, an asset with $\sigma_{ih} > 0$ is desirable for consumers who dislike fluctuations of consumption across dates, as holding such an asset smoothes the intertemporal consumption profile. Therefore, the more consumers are averse to intertemporal substitution (the larger ρ), the more willing they are to hold an asset with $\sigma_{ih} > 0$, and the smaller the excess return required in equilibrium to induce consumers to hold this asset. This explains the presence of the $-\rho\sigma_{ih}$ term, which reflects our consumers' *aversion to intertemporal substitution.*

Two special cases of (3.7) are worth noting:

[11]Our consumers, because they have risk preferences with constant relative risk aversion, do save for precautionary motives and have decreasing absolute prudence.

- When $\gamma = \rho$, the precautionary saving and intertemporal substitution effects cancel out. It is thus an unfortunate feature of standard isoelastic preferences that they hide two fundamental determinants of equilibrium excess returns.
- When $\gamma = 0$ and $\rho > 0$, i.e., when consumers have no desire to smooth consumption over states and do not engage in precautionary saving,[12] excess returns can well be negative when $\sigma_{ih} < 0$. There is nothing pathological about this: Drèze and Modigliani (1972) have taught us about the temporal dimension of risk aversion. A zero aversion to atemporal risk ($\gamma = 0$) does not imply a zero risk premium as long as one is not indifferent to intertemporal substitution ($\rho > 0$). It is, therefore, another unfortunate feature of standard isoelastic preferences that they associate zero risk aversion to atemporal gambles with zero aversion to intertemporal substitution and thus to zero risk premia: this is simply not a general result.

Finally, one should also note that, in contrast with the excess return equation derived by Campbell (1994), (3.7) does assign a role to the intertemporal elasticity of substitution in the determination of equilibrium excess returns. This is because the covariances that appear in (3.7) are covariances with consumption. Loosely speaking, expressing excess returns, as Campbell does, as a function of covariances with market returns "hides" the ρ coefficient into the covariance terms since an implication of (3.2) is that, for $j > 1$,

$$\mathrm{S}_{t+1}\, r_{w,t+j} = \rho\, \mathrm{S}_{t+1}\, x_{t+j}. \tag{3.8}$$

Thus, it is misleading to say that aversion to intertemporal substitution plays no role in the determination of excess returns. When one takes an equilibrium perspective, excess returns are not independent of the elasticity of intertemporal substitution because this parameter affects both equilibrium consumption and the equilibrium portfolio return. Furthermore, when one takes consumption and not the return on wealth as given, ρ does appear directly in the excess return expression.

3.5. **The excess return on wealth.** We now turn to the computation of the equilibrium excess return on wealth. From (3.2),

$$\mathrm{S}_{t+1}\, r_{w,t+1} = \mathrm{S}_{t+1}\, x_{t+1} + (1-\rho)\, \mathrm{S}_{t+1} \sum_{j=1}^{\infty} \delta^j x_{t+j+1}. \tag{3.9}$$

[12] Both the second and third derivatives of the risk utility function are zero when $\gamma = 0$.

As a consequence,

$$\sigma_{wc} = \sigma_{cc} + (1-\rho)\sigma_{ch} \tag{3.10}$$
$$\sigma_{wh} = \sigma_{ch} + (1-\rho)\sigma_{hh} \tag{3.11}$$
$$\tag{3.12}$$

Substituting into (3.7), we obtain the following formula for the equilibrium excess return on wealth (up to a Jensen's inequality term brought for clarity to the left-hand side):

$$\begin{aligned} &\mathrm{E}_{t+1} r_{w,t+1} - r_{f,t+1} + \sigma_{ww}/2 \\ &\qquad = \gamma[\sigma_{cc} + (1-\rho)\sigma_{ch}] + (\gamma - \rho)[\sigma_{ch} + (1-\rho)\sigma_{hh}]. \end{aligned} \tag{3.13}$$

When returns are i.i.d., all the terms involving h are zero, and the rate of return on wealth is equal to $\gamma\sigma_{cc}$ regardless of whether the expected utility restriction is satisfied: the excess return on the wealth portfolio is then determined solely by risk aversion and the variance of consumption growth. This is not surprising, as time—and thus the coefficient aversion to intertemporal substitution, ρ—is essentially irrelevant in an i.i.d. world. As soon as we depart from the i.i.d. world, however, the "wealth premium" depends on both aversion to risk and aversion to intertemporal substitution.[13]

Note that the rate of return on wealth is simply, from (2.1), the rate of return on a claim to aggregate consumption—a concept that has, in general, little to do empirically with the rate of return on the equity traded in the stock market.[14]

3.6. **Prices.** One should note that the expression in (2.10) does not provide us with a formula to compute the equilibrium excess return on an asset as a function of its payoff structure, the consumption process and preferences. The reason is, of course, that the endogenous return r_i appears in the conditional second order moments on the right-hand side of (2.10). To find such a *bona fide* asset pricing formula, we first need to consider how the return on an asset depends on its price and the dividends (payoffs) it distributes.

Let $p_{i,t}$ denote the log of the (*cum* dividend) price-dividend ratio of asset i at time t,[15] and $d_{i,t}$ the log rate of growth of the dividends paid off by asset i between dates t and $t+1$. Then, by definition, the log return on asset i

[13] Similar results are noted in Weil (1989).

[14] The two returns are however identical *by construction* in the Lucas (1978) or Mehra and Prescott (1985) models.

[15] That is, the log of the cum dividend price minus the log of the dividend.

satisfies the identity:

$$r_{i,t+1} \equiv d_{i,t+1} + p_{i,t+1} - \log(e^{p_{i,t}} - 1). \tag{3.14}$$

Following Campbell and Shiller (1988), we assume that the log dividend growth process is stationary and use a Taylor expansion similar to the one applied above to the budget constraint to find that

$$r_{i,t+1} \approx d_{i,t+1} + p_{i,t+1} - \frac{1}{\delta_i} p_{i,t} - k_i, \tag{3.15}$$

where k_i and δ_i ($0 < \delta_i < 1$) are two linearization constants. Since wealth is simply an asset that distributes a dividend equal to per capita aggregate consumption, the approximate budget constraint (2.7) is but a special case of (3.15) with $d_{i,t} = x_t$, $p_{i,t} = -a_t$, $\delta_i = \delta$, and $k_i = k$.

Now, it follows from (3.7) that, because of homoskedasticity, the expected rate of return on asset i differs from the expected rate of return on wealth only by a constant, call it π_{iw}:

$$\pi_{iw} = \mathrm{E}_t\, r_{i,t+1} - \mathrm{E}_t\, r_{w,t+1}. \tag{3.16}$$

Therefore, applying conditional expectations to both sides of (3.15), substituting (3.16) into the resulting expression, and iterating (3.15) forward using the property that bubbles are infeasible in this economy, we find that

$$p_{i,t} = -(k_i + \pi_{iw}) \frac{\delta_i}{1 - \delta_i} + \mathrm{E}_t \sum_{s=1}^{\infty} \delta_i^s [d_{i,t+s} - r_{w,t+s}]. \tag{3.17}$$

The only term on the right-hand side of this expression that we do not yet know how to compute from consumption data alone is π_{iw}. Now notice that we can rewrite

$$\pi_{iw} = \mathrm{E}_t\, r_{i,t+1} - \mathrm{E}_t\, r_{w,t+1} = [\mathrm{E}_t\, r_{i,t+1} - r_{f,t+1}] - [\mathrm{E}_t\, r_{w,t+1} - r_{f,t+1}].$$

We have already computed the equilibrium excess return of the wealth portfolio in (3.13), so that the only task left is to characterize the excess return on individual assets.

Notice that (3.17) implies

$$\mathrm{S}_{t+1}\, p_{i,t+1} = \mathrm{S}_{t+1} \sum_{s=1}^{\infty} \delta_i^s [d_{i,t+s+1} - r_{w,t+s+1}], \tag{3.18}$$

so that, using (3.8),

$$\mathrm{S}_{t+1}\, p_{i,t+1} = \mathrm{S}_{t+1} \sum_{s=1}^{\infty} \delta_i^s [d_{i,t+s+1} - \rho x_{w,t+s+1}]. \tag{3.19}$$

Now, from (3.14),

$$\mathrm{S}_{t+1}\, r_{i,t+1} = \mathrm{S}_{t+1}\, d_{i,t+1} + \mathrm{S}_{t+1}\, p_{i,t+1}.$$

Therefore, from (3.19), we find that

$$\mathrm{S}_{t+1}\, r_{i,t+1} = \mathrm{S}_{t+1}\, d_{i,t+1} + \mathrm{S}_{t+1} \sum_{s=1}^{\infty} \delta_i^s d_{i,t+s+1} - \rho\, \mathrm{S}_{t+1} \sum_{s=1}^{\infty} \delta_i^s x_{t+s+1}. \quad (3.20)$$

The interpretation of (3.20) is straightforward. Good news about the rate of return on asset i can come from good news about tomorrow's dividends or future dividends (the first two terms on the right-hand side). Or they can come from news that future consumption growth will be low (the third term on the right-hand side), since, by (3.8), bad news about future consumption growth translate, in equilibrium, into news that future returns will be low, and, therefore, into news that the present discounted value of future dividends is high. The more averse the consumers are to intertemporal substitution (the larger ρ), the more sensitive equilibrium returns are to changes in consumption growth, and the more bad news about future consumption means good news for current returns.

Equation (3.20) immediately implies that

$$\sigma_{ic} = \sigma_{d_ic} + \sigma_{f_ic} - \rho\sigma_{h_ic}, \quad (3.21)$$

$$\sigma_{ih} = \sigma_{d_ih} + \sigma_{f_ih} - \rho\sigma_{h_ih}, \quad (3.22)$$

with the notation

$$f_{i,t+1} \equiv \mathrm{E}_{t+1} \sum_{j=1}^{\infty} \delta_i^j d_{i,t+j+1} \quad \text{and} \quad h_{i,t+1} \equiv \mathrm{E}_{t+1} \sum_{j=1}^{\infty} \delta_i^j x_{t+j+1}.$$

Thus, for instance, σ_{h_ic} measures the conditional covariance between expected discounted future dividend growth of asset i and tomorrow's consumption, while σ_{h_ih} measures the conditional covariance between two differently discounted expectations of future consumption.

Substituting (3.21) and (3.22) into (3.7), and collecting terms, we find that the equilibrium excess return on any asset i is given by

$$\begin{aligned} \mathrm{E}_t\, r_{i,t+1} - r_{f,t+1} &= \gamma\sigma_{d_ic} + (\gamma - \rho)\sigma_{d_ih} \\ &+ \gamma\sigma_{f_ic} + (\gamma - \rho)\sigma_{f_ih} \\ &- \rho[\gamma\sigma_{h_ic} + (\gamma - \rho)\sigma_{h_ih}]. \end{aligned} \quad (3.23)$$

Equation (3.23) computes the equilibrium excess return on asset i solely as a function of the moments of this asset's dividend growth process and of the consumption growth process. The interpretation of (3.23) runs, of course, very much along the lines of the interpretation of (3.7). We showed in (3.7) that there are three behavioral determinants of excess returns: aversion to risk, prudence, and aversion to intertemporal substitution. The excess return equation (3.23) simply shows that each of these behavioral determinants applies to the each of the events, described in (3.20), associated with good news about the return on asset i: news that tomorrow's dividends will be high, that future dividends will be high, or that future consumption growth will be low.

To complete the computation of equilibrium prices, all that remains to be done is to subtract from (3.23) the equilibrium excess return on wealth computed in (3.13). This will yield the constant difference, π_{iw}, between the rate of return on asset i and the rate of return on wealth. Using the expression in (3.2) for the rate of return on wealth, and substituting the just computed π_{iw} into (3.17), would yield the (approximate) equilibrium price of any asset i as a function of consumption and dividend data alone.

4. Equilibrium Returns: Heteroskedastic Consumption

In this section, we extend the results of the previous section by showing how the main result derived in the homoskedastic case—the generalized CCAPM of (3.7)—generalizes to the case where the log of consumption growth is heteroskedastic.

Since solving equation (2.15) when consumption follows an arbitrary heteroskedastic process is a formidable task (it requires computing conditional moments of conditional moments of conditional moments etc.), and rather than attempting the impossible,[16] we parameterize the heteroskedasticity by assuming that log consumption growth follows a simple AR(1) process with GARCH(1,1) disturbances:

$$x_{t+1} = a + bx_t + u_t \tag{4.1}$$

$$u_{t+1} \sim |_t \ \mathcal{N}(0, \sigma_{cc,t}) \tag{4.2}$$

$$\sigma_{cc,t} = \alpha_0 + \alpha_1 u_t^2 + \alpha_2 \sigma_{cc,t-1} \tag{4.3}$$

We will use three properties of GARCH processes that are proved in Restoy (1991).[17] If two random variables have a joint normal conditional distribution whose second order moments follow GARCH processes analogous to (4.3), then:

[16] At least impossible to us.

[17] The straightforward proofs can be found there in Lemmas 1, 2, and 3.

Property 1 : Today's conditional expectation of products of powers of tomorrow's conditional second order moments is a polynomial in today's conditional second order moments.

Property 2 : Today's conditional covariance between products of powers of tomorrow's conditional second order moments is a polynomial in today's conditional second order moments.

Property 3 : Today's conditional covariance between one of these random variables tomorrow and product of powers of tomorrow's conditional second order moments is zero.

4.1. **The consumption-wealth ratio.** Properties 1 and 2 immediately imply that the solution to (2.15) (i.e., the equilibrium consumption wealth ratio) can be written as

$$a_t = n - (1-\rho)\frac{\delta b}{1-\delta b}x_t + \sum_{j=1}^{\infty} \zeta_j \sigma_{cc,t}^j, \tag{4.4}$$

where the constant n and the ζ_j coefficients—which are, as we shall see below, uninstructive and irrelevant for excess returns—can be computed as in Restoy (1991).

To understand this equation, it is best to compare it with (2.15). The term in x_t on the right-hand side of (4.4) represents the expected present discounted value of future consumption, which is just a linear function of current consumption growth because of the AR(1) process followed by consumption growth. The polynomial in the current conditional variance of consumption is present by virtue of Properties 1 and 2, which guarantee that the last term in (2.15) can be expressed in the form, given in (4.4), of a weighted sum of powers of the current conditional variance of consumption.

4.2. **Excess returns.** Property 3 implies

$$\text{Cov}_t\left(r_{i,t+1}, \sum_{j=1}^{\infty} \zeta_j \sigma_{cc,t+1}^j\right) = 0.$$

As a consequence, from (4.4) and (4.2),

$$\sigma_{ia,t} = \frac{\delta b}{1-\delta b}\sigma_{ic,t}. \tag{4.5}$$

This is an important result because it embodies the fundamental insight that, for our AR(1)–GARCH(1,1) process, returns are only able to predict future conditional *means* of consumption growth but carry no information about the future conditional *variances*. Therefore, the ζ_j parameters are irrelevant when it comes to computing excess returns, and the parameters of GARCH process do not matter for excess returns! Indeed, substituting (4.5) into equation (2.10), one obtains

$$\mathrm{E}_t\, r_{i,t+1} - r_{f,t+1} = -\frac{\sigma_{ii}}{2} + \left[\gamma + (\gamma - \rho)\frac{\delta b}{1-\delta b}\right]\sigma_{ic,t}. \qquad (4.6)$$

Because of Properties 1 to 3, this expression is almost identical formally to the one we would have obtained, in (3.7), for an AR(1) process with homoskedastic errors. Because of the autoregressive nature of consumption growth, the only conditional moment that matters for excess returns is the current conditional covariance between asset returns and consumption. But the one crucial distinction is that excess returns now vary over time, reflecting the time variation of the conditional variance of log consumption growth.

While one might be tempted to conclude from (4.6) that this model is observationally equivalent to a standard CCAPM model with coefficient of relative risk aversion (or inverse of the elasticity of intertemporal substitution)

$$\gamma' = \gamma + (\gamma - \rho)\delta b/(1 - \delta b),$$

this would be mistaken. If γ is small relative to ρ and consumption growth is highly persistent, the implied γ' might well be negative, and the excess return on an asset might be negative when the conditional covariance between that asset's return and consumption is positive.

A particular case is when the consumption growth rate is not persistent ($b = 0$), but exhibits conditional heteroskedasticity of the GARCH form. From (4.6), that assumption implies that the CCAPM's excess returns expression holds. Similarly, equations (2.9), (4.4) and Property 3 imply that the SCAPM also holds. This result shows how i.i.d. consumption growth (as in Kocherlakota (1990)) is a sufficient but not necessary distributional assumption to get observational equivalence between SCAPM, CCAPM and the excess return expression associated to the model with GIP preferences. Notice however that, even in this case, it is not true that elasticity of intertemporal substitution is irrelevant to determine asset prices as long as it affects the equilibrium rate of return on wealth.

5. Temporal Risk Aversion and The Term Structure of Real Interest Rates

The previous sections have highlighted in several instances the fact that risk neutrality towards timeless gambles does not imply, as is widely believed, that excess premia should be zero for all assets regardless of their maturity. As we emphasized above, the latter presumption is valid only in the time- and state-additive expected utility case—for, in that case, neutrality towards timeless risks coincides with indifference to the date at which one consumes, and thus to the irrelevance of the time dimension of risk. However, this coincidental result does not carry over to more general setups, and there is no blanket presumption that equilibrium risk premia should be zero at all maturities when consumers are neutral towards timeless risks—which confirms in equilibrium the partial equilibrium analysis of Drèze and Modigliani (1972).

To highlight the role of temporal risk aversion, we now return to the homoskedastic case[18] and characterize the equilibrium term structure of real bond returns under the assumption that the log consumption growth process follows an homoskedastic, AR(1) process:

$$x_{t+1} = a + bx_t + \epsilon_{c,t+1}, \tag{5.1}$$

$$\epsilon_{c,t+1} \sim |_t \; \mathcal{N}(0, \sigma_{cc}). \tag{5.2}$$

We consider pure discount bonds maturing $j \geq 1$ periods from now, i.e., riskfree claims that promise to pay one unit of the consumption good in every state of nature j periods from now. Let $R_t(j)$ denote the gross one-period return at time t on a bond of maturity j.[19] It is straightforward to show that $R_t(j)$ must satisfy the following Euler equation:

$$\mathrm{E}_t \left\{ \beta^{j\theta} \prod_{k=1}^{j} X_{t+k}^{-\rho\theta} \prod_{k=1}^{j} R_{w,t+k}^{\theta-1} \right\} [R_t(j)]^j = 1. \tag{5.3}$$

Similarly, the return on a j-period *rolling over short* strategy must satisfy

$$\mathrm{E}_t \left\{ \beta^{j\theta} \prod_{k=1}^{j} X_{t+k}^{-\rho\theta} \prod_{k=1}^{j} R_{w,t+k}^{\theta-1} \prod_{k=0}^{j-1} R_{t+k}(1) \right\} = 1. \tag{5.4}$$

In the appendix we show that, under the same joint lognormality assumption we used above, the Euler quation corresponding to the j-period bond can be

[18] Computations are more tedious, but the results not more instructive, in the heteroskedastic case.

[19] The one-period rate of return at t on a bond maturing at $t+1$, $R_t(1)$, is simply what we called earlier $R_{f,t+1}$.

written as.

$$r_t(j) = -\log\beta + \rho S(a,b,j) + \rho T(b,j)x_t + \frac{1}{2}\left\{\frac{(\rho-\gamma)(1-\gamma)}{(1-\delta b)^2} - A(b,j)\left[\rho + \frac{\gamma-\rho}{1-\delta b}\right]^2\right\}\sigma_{cc}, \quad (5.5)$$

where

$$S(a,b,j) \equiv \frac{1}{j}\frac{a}{1-b}\left[j - b\frac{1-b^j}{1-b}\right],$$

$$T(b,j) \equiv \frac{b}{j}\frac{1-b^j}{1-b},$$

$$A(b,j) \equiv \frac{1}{j}\sum_{k=1}^{j}\frac{1-b^{2k}}{1-b^2}\left[1 + 2b\frac{1-b^{j-k}}{1-b}\right].$$

Equation (5.5) allows us to draw (approximate) yield curves for pure discount bonds. In this homoskedastic world, those yield curves would be flat if consumption is *i.i.d.* ($b = 0$) and/or agents have an infinite elasticity of intertemporal substitution ($\rho = 0$).

In the appendix we also show that the rolling over short strategy yields a return which can be written as the return on a j- period bond plus a term premium. This term premium has the form:

$$TP(j) = \rho b\left[-\frac{1}{2}\rho b + \rho + \frac{\gamma-\rho}{1-\delta b}\right]A(b,j-1)(j-1) + \rho b\left[\rho + \frac{\gamma-\rho}{1-\delta b}\right]\sum_{k=1}^{j-1} b^{j-k}\frac{1-b^2k}{1-b^2}. \quad (5.6)$$

The term premium is a complex function of the persistence parameter b and the preference parameters γ and ρ. Under the standard time-additive expected utility preferences, the term premium is zero if agents are risk neutral—because zero risk aversion is then associated with zero aversion to intertemporal substitution ($\gamma = \rho = 0$). In general, however, a zero coefficient of relative risk aversion for timeless gambles does *not* imply a zero term premium. By contrast, if agents have an infinite elasticity of intertemporal substitution ($\rho = 0$), the term premium is zero in equilibrium regardless of the value of the coefficient γ: when consumers do not care when they consume, the rate of return on a long bond and on the corresponding rolling over short strategy must be identical. Finally, note that the term premium is, of course, always zero if consumption is *i.i.d.*

6. Conclusion

We have shown in this paper that the equilibrium capital asset pricing model that emerges from Kreps-Porteus GIP preferences can be written—both in the case of homoskedastic and in the case of AR(1)-GARCH(1,1) consumption—as a generalized CCAPM in which both aversion to risk and to intertemporal substitution matter for excess returns. This generalized CCAPM features, relative to the standard CCAPM, an extra term that captures the effects on excess returns of a possible correlation between an asset return and news about future consumption, and that reflects the interaction between precautionary saving and consumption smoothing. Because of the presence of this extra term, the predictions of this generalized CCAPM can be quite different from and richer than those of the standard CCAPM. For instance, the equilibrium excess return on an asset whose return is positively correlated with consumption might well be negative . . . .

A second contribution is that we have derived approximate equilibrium asset pricing formulas that can be used to price explicitly any asset solely as a function of the conditional moments of its dividend process and of consumption. In particular, these formulas make it possible to compute, albeit approximately, the predicted excess on equity—as distinct from the rate of return of a claim to aggregate consumption that is computed in most of the asset pricing literature. This should help shed new light on the long-standing debate on the equity premium and riskless rate puzzles. These formulas also show how to compute the otherwise unobservable rate of return on wealth from consumption data alone. This method could be applied empirically to characterize the true implications of the SCAPM when the rate of return on wealth is inferred from consumption data instead of being measured as the rate of return on the stock market.

Third, our paper clarifies the often forgotten role of temporal risk aversion for equilibrium asset prices: excess returns are in general not zero, and the yield curve for real bond returns is not flat, when the consumers are neutral towards timeless risks.

Finally, this paper should be viewed as our contribution to a budding branch of literature[20] that attempts, through approximations, to provide an analytic understanding of the workings of models that usually must be solved numerically. This approach makes it possible to unify theoretical results and numerical insights.

[20]See Kimball [1992], Campbell [1992].

APPENDIX: COMPUTING THE RETURN ON A j-PERIOD BOND AND THE J-PERIOD TERM PREMIUM

Using the lognormality assumption, we can write

$$\begin{aligned} jr_t(j) &= -j\theta\log\beta + \rho\theta\,\mathrm{E}_t\sum_{k=1}^{j}x_{t+k} - (\theta-1)\,\mathrm{E}_t\sum_{k=1}^{j}r_{w,t+k} \\ &- \frac{1}{2}\left[\rho\theta^2\,\mathrm{Var}_t\left(\sum_{k=1}^{j}x_{t+k}\right) + (\theta-1)^2\,\mathrm{Var}_t\left(\sum_{k=1}^{j}r_{w,t+k}\right)\right. \\ &+ \left. 2\rho(\theta-1)\,\mathrm{Cov}_t\left(\sum_{k=1}^{j}x_{t+k}, \sum_{k=1}^{j}r_{w,t+k}\right)\right]. \end{aligned} \tag{A.1}$$

Similarly, under the lognormality assumption, equations (A.1) and (5.4) yields

$$\sum_{k=0}^{j-1}E_t r_{t+k}(1) = jr_t(j) + TP(j), \tag{A.2}$$

where

$$\begin{aligned} TP(j) &= \frac{1}{2}Var_t\left[\sum_{k=0}^{j-1}r_{t+k}(1)\right] + \rho\theta\,\mathrm{Cov}_t\left[\sum_{k=0}^{j-1}r_{t+k}(1), \sum_{k=1}^{j}x_{t+k}\right] \\ &- (\theta-1)\,\mathrm{Cov}_t\left[\sum_{k=0}^{j-1}r_{t+k}(1), \sum_{k=1}^{j}r_{w,t+k}\right] \end{aligned} \tag{A.3}$$

is the j-period term premium.

For the homoskedastic AR(1) process given in (5.1), (3.2) and (3.3) specialize to

$$r_{w,t+1} = u + \rho x_{t+1} + \frac{1-\rho}{1-\delta b}\,\mathrm{S}_{t+1}\,x_{t+1}, \tag{A.4}$$

where

$$u = -\log\beta - \frac{\theta(1-\rho)^2}{2}\frac{1}{(1-\delta b)^2}\sigma_{cc}. \tag{A.5}$$

Equation (A.4) implies that

$$r_{w,t+j} - \mathrm{E}_t\,r_{w,t+j} = \left(\rho + \frac{1-\rho}{1-\delta b}\right)(x_{t+j} - \mathrm{E}_t\,x_{t+j}). \tag{A.6}$$

Therefore,

$$\mathrm{E}_t \sum_{k=1}^{j} r_{w,t+k} = uj + \rho\, \mathrm{E}_t \sum_{k=1}^{j} x_{t+k}, \tag{A.7}$$

$$\mathrm{Var}_t \left(\sum_{k=1}^{j} r_{w,t+k} \right) = \left(\rho + \frac{1-\rho}{1-\delta b} \right)^2 \mathrm{Var}_t \left(\sum_{k=1}^{j} x_{t+k} \right), \tag{A.8}$$

$$\mathrm{Cov}_t \left(\sum_{k=1}^{j} x_{t+k}, \sum_{k=1}^{j} r_{w,t+k} \right) = \left(\rho + \frac{1-\rho}{1-\delta b} \right) \mathrm{Var}_t \left(\sum_{k=1}^{j} x_{t+k} \right). \tag{A.9}$$

Now,

$$\begin{aligned} \mathrm{E}_t \sum_{k=1}^{j} x_{t+k} &= \mathrm{E}_t \sum_{k=1}^{j} \left[a(1 + b + \cdots + b^{k-1}) + b^k x_t \right] \\ &= j[S(a,b,j) + T(b,j)x_t], \end{aligned} \tag{A.10}$$

where

$$S(a,b,j) \equiv \frac{1}{j} \frac{a}{1-b} \left[j - b \frac{1-b^j}{1-b} \right] \quad \text{and} \quad T(b,j) \equiv \frac{b}{j} \frac{1-b^j}{1-b}.$$

When consumption growth is i.i.d. ($b = 0$), $S(a, 0, j) = a$ and $T(0, j) = 0$, while for one-period bonds ($j = 1$), $S(a, b, 1) = a$ and $T(b, 1) = b$.

Moreover,

$$\mathrm{Var}_t \left(\sum_{k=1}^{j} x_{t+k} \right) = \sum_{k=1}^{j} \mathrm{Var}_t(x_{t+k}) + 2 \sum_{k=1}^{j-1} \sum_{l=k+1}^{j} \mathrm{Cov}_t(x_{t+k}, x_{t+l}). \tag{A.11}$$

But

$$\mathrm{Var}_t(x_{t+k}) = \mathrm{Var}_t \left(b^k x_t + \sum_{s=0}^{k-1} b^s \epsilon_{c,t+k-s} \right) = \frac{1-b^{2k}}{1-b^2} \sigma_{cc}, \tag{A.12}$$

and, for $l > k$,

$$\begin{aligned} \mathrm{Cov}_t(x_{t+k}, x_{t+l}) &= \mathrm{Cov}_t \left(x_{t+k}, b^{l-k} x_{t+k} \sum_{s=0}^{l-k-1} b^s \epsilon_{c,t+l-s} \right) \\ &= b^{l-k}\, \mathrm{Var}_t(x_{t+k}) \\ &= b^{l-k} \frac{1-b^{2k}}{1-b^2} \sigma_{cc}. \end{aligned} \tag{A.13}$$

Therefore,

$$\begin{aligned}\mathrm{Var}_t\left(\sum_{k=1}^{j} x_{t+k}\right) &= \sum_{k=1}^{j}\frac{1-b^{2k}}{1-b^2}\sigma_{cc} + 2\sum_{k=1}^{j-1}\sum_{l=k+1}^{j} b^{l-k}\frac{1-b^{2k}}{1-b^2}\sigma_{cc} \\ &= \sum_{k=1}^{j}\frac{1-b^{2k}}{1-b^2}\sigma_{cc} + 2\sum_{k=1}^{j-1}\frac{1-b^{2k}}{1-b^2}\sum_{l=k+1}^{j} b^{l-k}\sigma_{cc} \\ &= jA(b,j)\sigma_{cc}, \qquad \text{(A.14)}\end{aligned}$$

where

$$A(b,j) \equiv \frac{1}{j}\sum_{k=1}^{j}\frac{1-b^{2k}}{1-b^2}\left[1+2b\frac{1-b^{j-k}}{1-b}\right].$$

When consumption growth is i.i.d. $(b = 0)$, $A(0, j) = 1$, while for one-period bonds $(j = 1)$, $A(b, 1) = 1$.

Substituting (A.7), (A.8), (A.9), (A.10) and (A.14) into the Euler equation for j-period bonds (A.1), using (A.5) and rearranging, one obtains

$$\begin{aligned}r_t(j) &= -\log\beta + \rho S(a,b,j) + \rho T(b,j)x_t \\ &+ \frac{1}{2}\left\{\frac{(\rho-\gamma)(1-\gamma)}{(1-\delta b)^2} - A(b,j)\left[\rho+\frac{\gamma-\rho}{1-\delta b}\right]^2\right\}\sigma_{cc} \qquad \text{(A.15)}\end{aligned}$$

which is the expression for the return on a j-period bond given in (5.5).

Now, from equation (5.5) the return on a 1-period bond is

$$r_t(1) = -log\beta + \rho a + \rho b x_t + M, \qquad \text{(A.16)}$$

where

$$M = \frac{1}{2}\left\{\frac{(\rho-\gamma)(1-\gamma)}{(1-\delta b)^2} - \left(\rho+\frac{\gamma-\rho}{1-\delta b}\right)^2\right\}\sigma_{cc}. \qquad \text{(A.17)}$$

Then, from equations (A.3), (A.9), and (A.16) the j-period term premium can be written as

$$\begin{aligned}TP(j) &= \rho b\left\{-\frac{1}{2}\rho b + \rho\theta - (\theta-1)\left(\rho+\frac{1-\rho}{1-\delta b}\right)\right\}\mathrm{Var}_t\left(\sum_{k=1}^{j-1} x_{t+k}\right) \\ &+ \rho b\left\{\rho\theta - (\theta-1)\left(\rho+\frac{1-\rho}{1-\delta b}\right)\right\}\mathrm{Cov}_t\left(x_{t+j}, \sum_{k=1}^{j-1} x_{t+k}\right) \qquad \text{(A.18)}\end{aligned}$$

Then, using equations (A.13) and (A.14) and rearranging,

$$\begin{aligned} TP(j) &= \rho b\left[-\frac{1}{2}\rho b + \rho + \frac{\gamma - \rho}{1 - \delta b}\right] A(b, j-1)(j-1) \\ &+ \rho b\left[\rho + \frac{\gamma - \rho}{1 - \delta b}\right] \sum_{k=1}^{j-1} b^{j-k} \frac{1 - b^{2k}}{1 - b^2}, \end{aligned} \tag{A.19}$$

which coincides with expression (5.6) in the text.

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