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Over-Simplifcation and Tractability in Quasi-Linear NEG Models

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Abstract

An inconsistency is found in the demand side of the NEG models developed in Pflüger (2004) that follows from the absence of a non-negativity constraint on the consumption of agricultural goods. This seriously weakens the results of the original paper and those of ensuing contributions in Pflüger and Südekum (2008a,b). A solution to this problem is developed which imposes severe restrictions on the relative size of two of the core model parameters, the implications of which are examined.

JEL classification: D11; R12; F12.

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1 Introduction

The quasi-linear utility new economic geography (NEG) model developed in Pflüger (2004) aims to provide a framework of analysis in that is both devoid of income effects in manufacturing and analytically tractable. These two properties allow the presence of stable partial agglomeration to be linked to the absence of income effects in manufacturing. The basic framework is then extended in Pflüger and Südekum (2008a), where a housing market is introduced and a welfare analysis is carried out. Additionally, in Pflüger and Südekum (2008b) the approach is compared to existing footloose entrepreneur NEG models, in order to investigate what determines smooth or catastrophic patterns of agglomeration.

However, the use of quasi-linear utility requires the constraining income above the threshold that ensures positive consumption of all goods. This is not done in the Pflüger (2004) framework, leaving the possibility of negative consumption. Indeed, it is shown that negative consumption of the agricultural good occurs for the chosen parameter values, thus leading to unreliable predictions. The solution to this problem is a nonnegativity constraint which carries implications for the extensions in Pflüger and Südekum (2008a,b).

The remainder of the paper is organised as follows: section 2 presents the violation of non-negativity that occurs in the Pflüger (2004) framework. A solution to this problem is presented in section 3. Finally, section 4 discusses the implications of this finding and concludes.

2 Quasi-linear utility in the Pflüger framework

The fundamental assumption of the model developed in Pflüger (2004) is a quasi-linear utility function that eliminates income effects from the agglomerating sector. In this model, C_X is a household's consumption of a Dixit and Stiglitz (1977) aggregate and C_A is the consumption of the agricultural good, which is also the numeraire. Y is the income of the household, which can be either W for labour or R for human capital.

$$\begin{cases} \max \quad U = \alpha \ln C_X + C_A \\ \text{s.t.} \quad P_X C_X + C_A = Y \end{cases} \tag{1}$$

The first order conditions and indirect utility function are the standard result expected from a quasi-linear framework: the level of manufacturing expenditure α is constant, and agents spend what is left of their income on the agricultural good.

$$\begin{cases} P_X C_X = \alpha \\ C_A = Y - \alpha \end{cases} \tag{2}$$

$$V = Y - \alpha \ln P_X + \alpha (\ln \alpha - 1) \tag{3}$$

Quasi-linear utility requires defining a non-negativity constraint on the linear good. Indeed, equation (2) shows that if $Y < \alpha$, there is a negative consumption of the agricultural good. This case technically requires performing the optimisation (1) with $C_A = 0$, leading to the following indirect utility, in which income effects are present:

$$V' = \alpha \ln \frac{Y}{P_X} \tag{4}$$

The Pflüger (2004) model does not use this alternative specification for the $Y < \alpha$ case, and therefore the internal consistency of the model requires formal investigation that agricultural consumption is positive, so that this pathological case of negativity does not occur. Because the model assumes that $W = 1$ in both regions, non-negativity necessarily holds for labour.[1] However, no explicit condition is imposed on R and R^*. These are defined below, using the regional endowments of geographically immobile labour L, L^*, the regional endowment of geographically mobile human capital K, K^* and the freeness of trade $\phi = \tau^{1-\sigma}$.[2]

$$\sigma R = \frac{\alpha (L + K)}{K + \phi K^*} + \frac{\phi \alpha (L^* + K^*)}{\phi K + K^*} \tag{5}$$

$$\sigma R^* = \frac{\phi \alpha (L + K)}{K + \phi K^*} + \frac{\alpha (L^* + K^*)}{\phi K + K^*} \tag{6}$$

While Pflüger (2004) expresses these as a function of the levels of labour and human capital in both regions, it is straightforward to show that they only depend on the following relative distributions of fixed and variable endowments over regions:

$$\rho = \frac{L}{K + K^*} \qquad \rho^* = \frac{L^*}{K + K^*} \qquad \lambda = \frac{K}{K + K^*}$$

[1] The value of alpha is constrained well below 1 by other considerations, explained in footnote 8 of Pflüger (2004)

[2] For a more detailed explanation of how these are obtained, the reader is referred to Pflüger (2004).

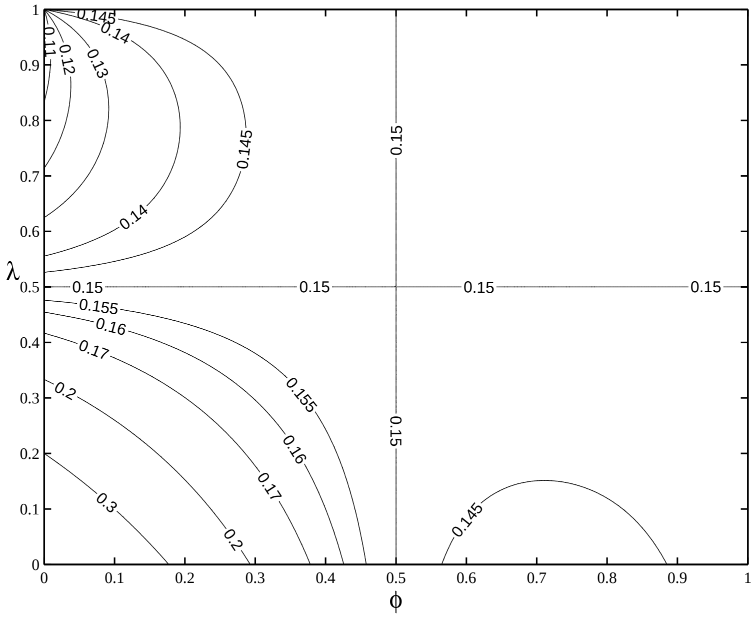

Figure 1: Human capital wages R in the model phase space $\alpha = 0.3$, $\sigma = 6$, $\rho = \rho^* = 1$

Rearranging, one can express all the levels based only on the relative variables and a single input endowment level, for example, the home labour L:

$$K + K^* = \frac{L}{\rho} \qquad L^* = \frac{\rho^*}{\rho} L$$

$$K = \frac{\lambda}{\rho} L \qquad K^* = \frac{1-\lambda}{\rho} L$$

Replacing these in (5) and (6) and rearranging, one sees that R and R^* are a function of the relative variables only.[3]

$$\frac{R}{\alpha} = \frac{1}{\sigma} \left(\frac{(\rho + \lambda)}{\lambda + \phi(1-\lambda)} + \frac{\phi(\rho^* + 1 - \lambda)}{\phi\lambda + 1 - \lambda} \right) \tag{7}$$

$$\frac{R^*}{\alpha} = \frac{1}{\sigma} \left(\frac{\phi(\rho + \lambda)}{\lambda + \phi(1-\lambda)} + \frac{(\rho^* + 1 - \lambda)}{\phi\lambda + 1 - \lambda} \right) \tag{8}$$

Equations (7) and (8) are expressed as the income to manufacturing expenditure ratio, which *must* be greater or equal to 1 for the demand equations to be internally consistent. However, a simple calculation over the $\{\phi, \lambda\}$ phase space shows that this is not the case with the parameters used in Pflüger (2004).[4]

Figure 1 shows that only a small portion of the phase space satisfies $R/\alpha > 1$. Furthermore, because of the symmetry of R and R^* in (7) and (8), nowhere are they *simultaneously* greater than α, implying a violation of non-negativity in all the $\{\phi, \lambda\}$ phase

[3]These specifications are the same as in Pflüger and Südekum (2008a), p 548, although this is presented as the result of a normalisation of the amount of human capital $K + K^*$ to one, without being explicit about the fact that this holds *regardless of the values chosen for* K *and* K^*.

[4]As is the case in Pflüger (2004) and Pflüger and Südekum (2008a,b) we assume symmetry of the regions, so that $\rho^* = \rho$.

space. Human capital implicitly consumes a negative amount of the agricultural good, or alternatively the agricultural sector implicitly subsidises the consumption of human capital owners. The model is not internally consistent for the parameters chosen, and analytical predictions such as welfare functions or the location of breakpoints cannot be relied upon.

3 Imposing the non-negativity constraint

There are two solutions to this problem. The first is to allow for demand curves based on the indirect utility function (4), which is consistent with $R/\alpha < 1$. This would result in two separate models which are 'glued' together in the phase space at the locations where $R/\alpha = 1$. This solution departs, however, from the desired tractability and lack of income effects present in the original framework.

The second solution is to derive a non-negativity condition and restrict the analysis to parameters that satisfy it. Given the tractability of the framework, this is straightforward. Intuitively, Figure 1 suggests that the minimum value of R occurs when $\phi \to 0$ and $\lambda \to 1$. As $\lambda \to 1$ the supply of the fixed input K is maximised while the autarky imposed by $\phi \to 0$ minimises the overall demand for manufacturing goods, and therefore the earnings of the fixed input.[5]

Taking the limit of (7) when $\phi \to 0$ gives the following expression for R on the vertical axis of the phase space. Symmetry of the regions leads to the same equation for R^*, with $1-\lambda$ instead of λ. The λ, $1-\lambda$ denominators for R and R^* imply that $\lambda \in]0,1[$.

$$\lim_{\phi \to 0} \frac{R}{\alpha} = \frac{1}{\sigma}\left(\frac{\rho+\lambda}{\lambda}\right) \tag{9}$$

Taking the limit of this expression as $\lambda \to 1$, the following non-negativity condition is immediately obtained, in the form of a lower bound on the ρ parameter.

$$\frac{R}{\alpha} \geq 1 \Leftrightarrow \rho \geq \sigma - 1 \tag{10}$$

This is shown in Figure 2, which plots R in the $\{\phi, \lambda\}$ phase space with the non-negativity condition (10) and shows that $R/\alpha \geq 1$ everywhere in the phase space.

4 Discussion and conclusion

The lack of a non-negativity constraint on the consumption of the agricultural good in the quasi-linear framework of Pflüger (2004) and Pflüger and Südekum (2008a,b) leads to misleading predictions for certain parameter combinations. The most simple solution to avoiding this inconsistency is simply to impose a lower bound (10) on ρ. This is extremely restrictive, however, and has two central implications.

The first is that condition (10) effectively results in the removal of a parametric degree of freedom. Indeed, one would wish both a low ρ, to ensure that the immobile population L is not too large compared to the mobile population $K+K^*$, and a high σ consistent with the NEG literature. This is the case in Pflüger (2004), which chooses $\rho = 1$ and $\sigma = 6$.

[5] A proof that when $\rho \geq 1/3$ the lowest value of R occurs for $\phi \to 0$ and $\lambda \to 1$ is provided in Appendix A.

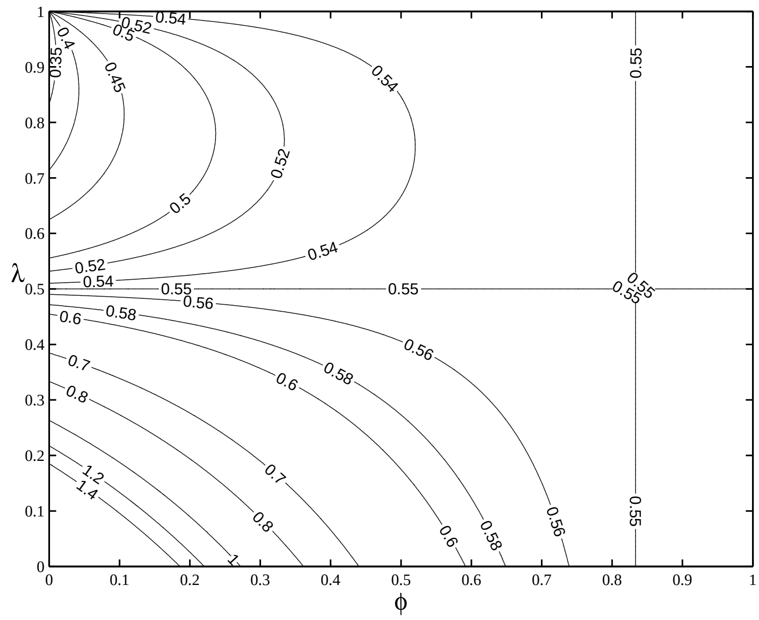

Figure 2: Human capital wages R in the model phase space $\alpha = 0.3$, $\sigma = 6$, $\rho = \rho^* = 5$

However, the inequality in condition (10) works in the other direction. The conservative choice is therefore to choose (10) with equality, so that $\rho = \sigma - 1$.

As a result, one is free to to set either a value for the labour/human capital ratio, or a value for the elasticity of substitution, but not both simultaneously. Furthermore, even if one does not wish to apply (10) with equality, fixing either parameter severely restricts the plausible range of the other parameter.

The second implication is the effect on the model predictions of this high lower bound imposed on ρ. A simple illustration of this issue is the break point of the Pflüger (2004) model, given by $\phi_c = [\sigma(2\rho - 1) - 2\rho] / [\sigma(3 + 2\rho) - 2(1 + \rho)]$. Using the constraint $\rho = \sigma - 1$ and simplifying one obtains $\phi_c = 1 - 2/\sigma$. With the $\sigma = 6$ value used in Pflüger (2004), this implies a true break point of $\tau_c \approx 1.084$ ($\phi_c = 1/3$), instead of the value of $\tau_c \approx 1.454$ obtained with $\rho = 1$ and visible in Figure 2 of Pflüger (2004). Similarly, solving the implicit condition for the full agglomeration point gives $\tau_f \approx 1.083$, instead of the value of $\tau_f \approx 1.337$. These represent minimum thresholds, obtained by using (10) with equality and the inequality amplifies these effects.

These two related effects, i.e. the loss of a parametric degree of freedom and the induced shift in the critical points, imply that both the welfare analysis with housing of Pflüger and Südekum (2008a) and the comparison with other footloose entrepreneur frameworks in Pflüger and Südekum (2008b) need to be investigated. Indeed, because both are based on the R and R^* equations of Pflüger (2004) and re-use results such as the break point equation presented above, their predictions are liable to change significantly once non-negativity is imposed.

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A Location of the lower bound of R in the $\{\phi, \lambda\}$ phase space

Let $\underline{R}$ be the postulated lower bound of R, given by (9) for $\lambda \to 1$:

$$\underline{R} = \frac{\alpha}{\sigma}(\rho + 1) \tag{A-1}$$

Rearranging (7) to allow for a common denominator gives:

$$R = \frac{\alpha}{\sigma}\left(\frac{(\rho+\lambda)(\phi\lambda + 1 - \lambda) + \phi(\rho + 1 - \lambda)(\lambda + \phi(1-\lambda))}{(\lambda + \phi(1-\lambda))(\phi\lambda + 1 - \lambda)}\right) \tag{A-2}$$

Subtracting $\underline{R}$ in (A-1) from the general specification of R in (A-2) gives:

$$R - \underline{R} = \frac{\alpha}{\sigma}\left(\frac{\rho\phi\lambda(2-\lambda) + (1-\lambda)^2(\rho(1-\phi(1-\phi)) - \phi(1-\phi)) + \phi\lambda(1-\lambda)(1-\phi)}{(\lambda + \phi(1-\lambda))(\phi\lambda + 1 - \lambda)}\right) \tag{A-3}$$

The sign of (A-3) depends on the sign of the numerator of the term in brackets, as all the other terms are positive for allowable values of ϕ and λ. Similarly, the first and third additive terms in the numerator are positive or equal to zero for all ϕ and λ.

Given that $\phi \in]0, 1]$ the maximum value of $\phi(1-\phi)$ is 0.25 for $\phi = 0.5$. The second term in the numerator is therefore non-negative for $\rho \geq 1/3$.

Therefore, for $\rho \geq 1/3$, $R - \underline{R}$ is positive for all locations in the phase space, and tends to zero for $\phi \to 0$ and $\lambda \to 1$. $\underline{R}$ is the lower bound of R. ■