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Résumé: Dans ce papier, nous énonçons les conditions de détermination, c'est-à-dire les conditions assurant l'existence et l'unicité d'une solution bornée, dans les modèles linéaires, à anticipations rationnelles, tournés vers le futur et incorporant des changements de régime. Nous mettons ainsi un terme au débat entre Davig et Leeper (2007) et Farmer et al. (2010). Les conditions de détermination dérivées par les premiers sont uniquement valides dans le sous-espace des solutions bornées ne dépendant que d'un nombre fini de régimes passés, que nous appelons markoviennes. Dans l'espace des solutions bornées, les nouvelles conditions de détermination que nous dérivons sont plus restrictives. Néanmoins, lorsqu'elle est unique, la solution coïncide avec la solution markovienne de Davig et Leeper (2007). Finalement, nous illustrons nos résultats dans un modèle néo-keynesien standard étudié dans les deux articles sus-cités.

Classification JEL: E31, E43 et E52.

Mots-clés: changements de régime, DSGE, indétermination.

Abstract: In this paper, we provide determinacy conditions, i.e. conditions ensuring the existence and uniqueness of a bounded solution, in a purely forward-looking linear Markov switching rational expectations model. We thus settle the debate between Davig and Leeper (2007) and Farmer et al. (2010). The conditions derived by the former are valid in a subset of bounded solutions only depending on a finite number of past regimes, that we call Markovian. However, in the complete bounded solution space, the new determinacy conditions we derive are tighter. Nevertheless, when unique, the solution coincides with the Markovian solution of Davig and Leeper (2007). We finally illustrate our results in the standard new-Keynesian model studied by Davig and Leeper (2007) and Farmer et al. (2010).

JEL classification: E31, E43 and E52.

Keywords Markov switching, DSGE, indeterminacy.

1 The controversy

In this paper, we provide determinacy conditions, i.e. conditions ensuring the existence and uniqueness of a bounded solution, for forward-looking linear rational expectations models with Markov switching parameters. We therefore settle the debate between Davig and Leeper (2007) and Farmer et al. (2010).

Davig and Leeper (2007) provide determinacy conditions for the above class of models. Their approach consists in rewriting the Markov switching model as a linear model by expanding the number of state variables.

However, Farmer et al. (2010) exhibit a counter-example. For a particular set of parameters verifying Davig and Leeper's determinacy conditions at least two different bounded solutions of the standard new-Keynesian model exist.

We show that the debate origin is the type of solutions the authors consider. Davig and Leeper (2007) implicitly restrict the solution space to Markovian solutions, i.e. to solutions which only depend on a finite number of past regimes. Under this restriction, we prove in Proposition 1 that the transformation of the Markov switching model into a linear model with twice as many state variables is valid and that the correct determinacy conditions are those given by Davig and Leeper (2007).

In Proposition 2, we provide a necessary and sufficient determinacy condition for Markovian and non-Markovian solutions. This condition, which is tighter than that of Davig and Leeper (2007), depends on the stability of multiple product matrices involving the different regimes. If a unique bounded solution among all possible bounded policy functions exists, it is Markovian and coincides with the one Davig and Leeper construct.

In their reply, Davig and Leeper argue that the existence of Farmer et al.'s counter-example stems from mainly two reasons. First, they study different models. Indeed, assuming that solutions have to be Markovian allows transforming the Markov-switching problem into a linear one which can be considered as a different model. It is, however, more natural to recognize that the definitions of the solution space are different. Second, they dismiss Farmer et al.'s counterexample based on a non-Markovian solution as contrived and of little economic sense. This second argument relates to the broader issue of the solution space's proper definition in rational expectations models. Settling this general debate is beyond the scope of this paper, here we rather establish the determinacy conditions for both Markovian solution space and for bounded solution space.

From a practical point of view, checking determinacy can be highly time-consuming. This computational cost reflects the intrinsic (numerical) complexity of determinacy conditions of

Markov switching rational expectations models compared to linear models (Blanchard and Kahn (1980)). We provide three cases for which we can easily check determinacy conditions. Nevertheless, given the structure of the problem, there is little - if any - chance of being able to check determinacy conditions in the neighborhood of the indeterminacy frontier.

By building a comprehensive methodology to solve and estimate Markov switching rational expectations models, Farmer et al. (2009b) have greatly fostered applied research on Markov Switching in rational expectations models (e.g. Davig and Doh, 2008; Bianchi, 2012). However, in the absence of a consensus on determinacy conditions, uniqueness is rarely discussed. The potential existence of multiple equilibria could nevertheless modify the policy conclusions.

The remainder of the paper is organized as follows. In Section 2, we expose the problem and recall the counter-example put forward by Farmer et al. (2010). We show that Davig and Leeper's determinacy conditions are valid when considering Markovian bounded solutions only in Section 3. Further, in Section 4, we provide necessary and sufficient conditions for all bounded solutions. We discuss computational issues in Section 5. Finally, we illustrate these new conditions in the New Keynesian model, subject of the debate between Davig and Leeper (2007) and Farmer et al. (2010) and compare these determinacy conditions to those obtained for Markovian solutions in Section 6.

2 The class of models

The new Keynesian model analyzed by Davig and Leeper (2007) and for which Farmer et al. (2010) find two bounded solutions in the Davig and Leeper's determinacy region can be reduced to these three equations model:

$$\mathbb{E}_t y_{t+1} + \sigma \mathbb{E}_t \pi_{t+1} = y_t + \sigma R_t + \epsilon_t^d \tag{1}$$

$$\beta \mathbb{E}_t \pi_{t+1} = \pi_t - \kappa y_t + \epsilon_t^s \tag{2}$$

$$R_t = \alpha_{s_t} \pi_t + \epsilon_t^r \tag{3}$$

where y_t, π_t and R_t are the output-gap, inflation (in log) and the nominal interest rate (in deviation around a certain steady state). Equation (1) is an IS curve linking the output-gap to all the future ex-ante real interest rates and future and current demand shocks, ϵ_t^d. σ is the risk aversion. Equation (2) is a New-Keynesian Philips Curve linking inflation to all the future marginal costs summarized by the output-gap where κ measures the degree of nominal rigidities while β stands for the discount factor. ϵ_t^s denotes a cost-push shock translating the Philips Curve. Equation (3) is a simplified Taylor rule with a potential shift in the reaction to inflation (α_1 or α_2). $s_t \in \{1,2\}$ stands for the current regime of monetary policy. Finally ϵ_t^r is a disturbance measuring the unsystematic part of monetary policy.

When there is no regime switching, i.e. $\alpha_1 = \alpha_2 = \alpha$, the model admits a unique stable solution (whatever the definition of stability considered) if and only if the so-called Taylor principle is satisfied ($\alpha > 1$).

More generally, models of this class can be rewritten in the following form by defining $z_t = [\pi_t \quad y_t \quad R_t]$ and $\epsilon_t = [\epsilon_t^d \quad \epsilon_t^s \quad \epsilon_t^r]$:

$$A_{s_t}\mathbb{E}_t z_{t+1} + B_{s_t} z_t + C_{s_t}\epsilon_t = 0 \tag{4}$$

where z_t is a vector of endogenous variables, and ε_t is a vector of exogenous shocks. We assume that ε_t is bounded, independent of s_t and satisfies : $\mathbb{E}_t\varepsilon_{t+1} = \Lambda\varepsilon_t$. In addition, we assume that the regimes s_t take values $\{1,\cdots,N\}$ and follow a Markov-chain with constant transition probabilities[1]

$$p_{ij} = p(s_t = j | s_{t-1} = i) \tag{5}$$

The vectors z and ε are respectively in $\mathbb{R}^n$ and $\mathbb{R}^p$; for any $i \in \{1,\cdots,N\}$, the matrices A_i and B_i are in $\mathcal{M}_n(\mathbb{R})$, C_i is in $\mathcal{M}_{n,p}(\mathbb{R})$ and Λ is in $\mathcal{M}_p(\mathbb{R})$. Even if this class of models is quite general, it rules out models including backward-looking components. Solving such hybrid models remains a challenge even if recent advances have been accomplished (Foerster et al., 2011).

3 Determinacy conditions for bounded Markovian solutions

Farmer et al. (2010) provide two bounded solutions of model (4) in a region of parameters where Davig and Leeper (2007) claim that there should be a unique bounded solution. The key of this apparent contradiction is that one of the two solutions depend on the sequence of all past regimes while Equation 8 in Davig and Leeper (2007) is true only for solutions depending on current but not past regimes. In this section, we amend Proposition 1 of Davig and Leeper (2007) by restricting the solution space to bounded Markovian solutions.

Definition 1. *Let us denote by ϕ a measurable function*[2] *mapping $\{1,\cdots,N\}^\infty \times (\mathbb{R}^p)^\infty$ into $\mathbb{R}^n$. ϕ is Markovian of order p if ϕ only depends on ε^t and $\{s_t, s_{t-1},\cdots,s_{t-p}\}$.*

[1]See Barthélemy and Marx (2011) for state-dependent transition probabilities

[2]We follow the formalism developed in Woodford (1986).

ϕ is Markovian if there exists p such that ϕ is Markovian of order p.

We denote by $\mathcal{M}_p$ the set of Markovian functions of order p, and by $\mathcal{M} = \bigcup_{p \in \mathbb{N}} \mathcal{M}_p$ the Markovian space.

As in Davig and Leeper (2007), we introduce the matrix in $\mathcal{M}_{nN}(\mathbb{R})$, diagonal by blocks, $\text{diag}(B_1^{-1}A_1, \cdots, B_N^{-1}A_N)$ and define $\mathbf{M} = (P \otimes \mathbb{1}_n) \times \text{diag}(B_1^{-1}A_1, \cdots, B_N^{-1}A_N)$, where $\otimes$ denotes the standard Kronecker product. The main result of Davig and Leeper (2007) can then be rewritten in the following way.

Proposition 1. ***[Davig and Leeper (2007)]***
There exists a unique Markovian bounded solution ϕ_M of finite order of model (4) if and only if $\rho(\mathbf{M})$, the spectral radius of $\mathbf{M}$, i.e. the largest eigenvalue in absolute value, is strictly less than one.

Proof. The proof proceeds in two steps:

- If $\phi \in \mathcal{M}$ is a solution of (4), then $\phi \in \mathcal{M}_0$.
- If $\phi \in \mathcal{M}_0$, then Equation 8 of Davig and Leeper (2007) is valid and we can apply their strategy of expanding the number of state variables. More precisely we define $\mathbf{\Phi}$ such that:

$$\mathbf{\Phi}(\varepsilon^t) = \begin{bmatrix} \phi(1s^{t-1}, \varepsilon^t) \\ \vdots \\ \phi(Ns^{t-1}, \varepsilon^t) \end{bmatrix}. \tag{6}$$

  The stacked vector function, $\mathbf{\Phi}$, is then a solution of a linear rational expectations model (with constant parameters) and we thus can apply Blanchard and Kahn (1980).

See section A in appendix for the detailed proof. □

This Proposition proves that the determinacy condition of Davig and Leeper (2007) is valid in the particular set of bounded Markovian solutions. Thus, if there are multiple bounded solutions in Davig and Leeper's determinacy region, only one of them is Markovian the others are not.

4 Determinacy conditions for bounded solutions

In this part, we provide necessary and sufficient conditions for the model (4) in the space of bounded solutions (Markovian or not). For a given matricial norm on $\mathcal{M}_n(\mathbb{R})$, we introduce the following sequence of matrices, for $p \geq 2$:

$$S_p = \left(\sum_{(i_1,\cdots,i_{p-1})\in\{1,\cdots,N\}^{p-1}} p_{ii_1}\cdots p_{i_{p-1}j}\|B_i^{-1}A_iB_{i_1}^{-1}A_{i_1}\cdots B_{i_{p-1}}^{-1}A_{i_{p-1}}\| \right)_{ij}$$

A similar matrix has already been introduced in Barthélemy and Marx (2011). For any p, an (i, j) element of the matrix S_p corresponds to an upper bound of the expected impact (in norm) of the future endogenous variables along trajectories starting from regime i to regime j in p steps weighted by the probability of each trajectory. When there is only one regime this matrix comes down to a scalar measuring the importance of expected endogenous variables p periods ahead.

First, we prove that the spectral radius of S_p behaves as an exponential sequence asymptotically.

Lemma 1. *The sequence* $\left(\rho(S_p)^{1/p}\right)$ *is convergent and its limit does not depend on the chosen norm. We denote the limit by* ν*:*

$$\nu = \lim_{p\to+\infty} \rho(S_p)^{1/p} \tag{7}$$

Proof. The proof of the convergence is intensively based on the sub-multiplicativity of matricial norms, and the equivalence of norms. The details are in section B of the appendix. □

Second, Proposition 2 gives determinacy conditions for model 4 in the space of bounded solutions. This result is an improvement of Proposition 1 in Barthélemy and Marx (2011) as this result gives necessary and sufficient conditions while Barthélemy and Marx (2011) only give sufficient conditions.

Proposition 2. *There exists a unique bounded solution for model* (4) *if and only if* $\nu < 1$*. In this case, this solution is the Markovian solution of* 0 *order* ϕ_M *given in Proposition 1.*

Furthermore, ν *is smaller than* $\rho(\mathbf{M})$*.*

Proof. We base our proof on the formalism introduced by Woodford (1986) and recently used in Barthélemy and Marx (2011). We show that the model can be reformulated as a functional equation $(\mathbb{1} - \mathcal{R})\phi = \psi_0$ where $\mathcal{R}$ is an operator, ϕ the solution and ψ_0 a function depending on shocks and regimes. Then we show that $(\mathbb{1} - \mathcal{R})$ is invertible if and only if $\nu < 1$. All the details are in appendix, section C.

□

These determinacy conditions coincide with Blanchard and Kahn (1980) when there is no regime switching. Indeed, in this special case, $\rho(S_p)^{1/p} = 2^{1/p}||(B^{-1}A)^p||^{1/p}$ and hence ν is equal to the spectral radius of $B^{-1}A$.

Furthermore, in all the univariate models ($n = 1$), the sequence S_p is a geometric sequence of the form: S^p where S is defined as follows:

$$S = \left(p_{ij}\|B_i^{-1}A_i\|\right)_{ij}.$$

Indeed, when matrices B_i and A_i are scalars, they are commutative. Thus, for univariate models, $\rho(S_p)^{1/p}$ equals $\rho(S)$ and hence ν equals $\rho(S)$. In this case, ν is particularly easy to compute. Elements of S are absolute values of elements of $\mathbf{M}$, as shown in Farmer et al. (2009a).

5 Computational issues

To check determinacy, the computation of ν is thus at the core. This is however a challenging issue. One way to approximate ν is to compute $\rho(S_p)^{1/p}$ for p large enough. This computation is however time-consuming[3]. In addition, the sequence is not necessarily monotonous and the speed of convergence is unknown.

This numerical problem is very similar to the computation of the joint spectral radius (e.g. Theys, 2005). The joint spectral radius is costly to compute and to approximate. The question whether a joint spectral radius is greater than 1 is undecidable (Blondel and Tsitsiklis, 1997, 2000), i.e. it cannot be algorithmically settled in a finite number of steps.

Against this backdrop, we identify three situations in which we can answer whether ν is greater than 1 or not. The first case is when the Davig and Leeper's determinacy condition (see 1) fails. Then ν is larger than 1 as we already know that there exist multiple Markovian solutions. The two other cases are summed up in the two following lemmas.

Lemma 2. *If there exists p such that $\rho(S_p)^{1/p} < 1$, then $\nu < 1$.*

Lemma 2 stems from the fact that ν is the infimum of $\{\rho(S_p)^{1/p}, p \in \mathbb{N}\}$. Indeed, if $\rho(S_p)^{1/p}$ is smaller than one for a given p it ensures that ν is smaller than one also and hence there exists a unique stable equilibrium. This case is the only case for which one can conclude that the equilibrium is determinate. If there exists a unique stable equilibrium, there always exists such p. However, finding it can be impossible in a finite amount of time. When we do not succeed in finding such a p, one may suspect indeterminacy. The following result gives a necessary condition for indeterminacy.

[3]The complexity of the algorithm is of the order of factorial p.

Lemma 3. *If there exists* $i_0 \in \{1,2\}$*,* $p_0 \in \mathbb{N}$*, and a sequence of indexes* $(i_1, i_2, \cdots, i_{p_0}) \in \{1,2\}^{p_0}$ *such that*

$$\rho(p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{p_0} i_0} B_{i_0}^{-1} A_{i_0} \cdots B_{i_{p_0}}^{-1} A_{i_{p_0}}) > 1$$

Then $\nu > 1$*.*

Roughly speaking, when such a sequence is identified, the infinite repetition of this sequence is explosive. In such a case, one may construct a non-Markovian solution in addition to the Markovian solution. In this sense, this Lemma 3 is a generalization of the counterexample of Farmer et al. (2010).

We give the proofs of these results in appendix, section D.

6 Numerical illustration

In this part, we represent the determinacy region for bounded solutions in the case studied in Davig and Leeper (2007), and Farmer et al. (2010). We calibrate the parameters consistently with the baseline case of Davig and Leeper (2007) and the counterexample of Farmer et al. (2010): $\beta = 0.99$, $\sigma = 1$, $\kappa = 0.17$, $\gamma_1 = \gamma_2 = 0$, $p_{11} = 0.8$ and $p_{22} = 0.95$. The spectral radius of $\mathbf{M}$, $\rho(\mathbf{M})$, equals 0.98 when we fix the policy parameters α_1 and α_2 to 3 and 0.92 respectively as in the counter-example of Farmer et al. (2010). There thus exists a unique bounded Markovian solution, due to Proposition 1. However, Proposition 2, and Lemma 3 show that there exists other bounded solutions.

Figure 1 displays the region of determinacy for this choice of parameters and for different values of α_1 and α_2. For each set of parameters, we compute $\rho(S_p)^{1/p}$ for p smaller than 15. If, for a given p, the sequence $\rho(S_p)^{1/p}$ is smaller than 1, a unique stable solution exists by applying Lemma 2. This is the white area. Moreover, when a sequence of regimes $(s_0, s_1, \cdots, s_p)$ satisfies the condition of Lemma 3, there is a unique Markovian solution but many bounded solutions. This is the dark grey area. The pale grey area between the two previous areas represents parameter combinations for which we are not able to compare ν and 1 in a reasonable amount of time. Finally, the dark region displays the Davig and Leeper's indeterminacy region in which multiple stable Markovian solutions exist.

As we have already remarked, the determinacy region suggested by Davig and Leeper and valid for Markovian solutions is always larger than the general determinacy region. Contrasting to Davig and Leeper (2007), we can neither prove nor reject that the determinacy region is monotonously decreasing with respect to the response to inflation in one regime (the light-shaded area is not decreasing after a certain threshold). Nonetheless, an economy with

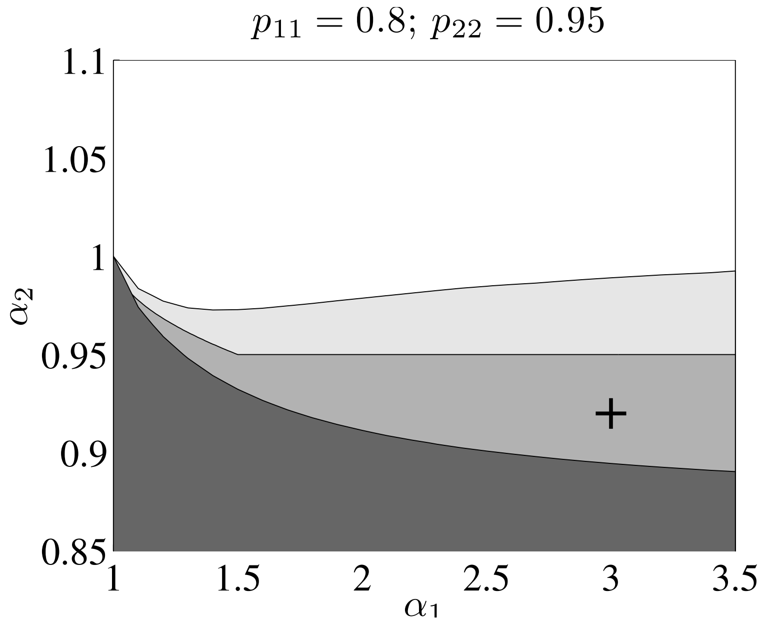

Figure 1: Determinacy regions: new Keynesian model with Markov switching monetary policy

Note: the white area displays the region where we are sure that there exists a unique bounded equilibrium; the light-shaded area represents a region in which we cannot decide whether there is a unique bounded solution or not; the dark-shaded area represents a region in which we know that there exists a unique Markovian solution but several bounded solutions; finally, parameter combinations in the black region imply multiple bounded Markovian equilibria. The cross marks the parameters set for which Farmer et al. (2010) build two stable solutions.

switches among passive and active regimes may be determinate for certain parameter combinations. This confirms one of the main findings of Davig and Leeper (2007) that a passive monetary regime is not necessarily subject to indeterminacy if economic agents expect switch toward an active monetary policy regime.

7 Conclusion

In this paper, we give determinacy conditions for purely forward-looking rational expectations models with Markov-Switching. We thus clarify the debate between Davig and Leeper (2007, 2010) and Farmer et al. (2010). This condition depends on all possible matrix products and probabilities in a manner very close to joint spectral radius of multiple matrices. This condition is thus difficult to assess especially for parameters close to the determinacy frontiers. It reflects the non commutativity of matrix products compared to power matrices that are key in the linear rational expectations model (see Blanchard and Kahn, 1980). We propose three simple cases for which one can decide whether the model is determinate or not. This complexity however raises new and challenging computational issues.

In addition, we highlight the key role of the definition of the solution space in the determinacy conditions. This suggests that researchers on Markov switching rational expectations should always be careful about the class of the solutions they consider as it can substantially modify the number of stable equilibria.

APPENDIX

A Proof of Proposition 1

In this part, we prove Proposition 1. The proof is undertaken in two steps:

- If $\phi \in \mathcal{M}$ is solution of Equation (4), then $\phi \in \mathcal{M}_0$
- Furthermore if $\phi \in \mathcal{M}$, then defining $\mathbf{\Phi}$ by:

$$\mathbf{\Phi}(\varepsilon^t) = \begin{bmatrix} \phi(1s^{t-1}, \varepsilon^t) \\ \vdots \\ \phi(Ns^{t-1}, \varepsilon^t) \end{bmatrix}$$

  $\mathbf{\Phi}$ is solution of a linear rational expectations model with regime-independent parameters. We thus can apply Blanchard and Kahn (1980).

Assume that there exists a p-order Markovian solution of (4), ϕ, we define $\mathcal{P}(q)$ the statement that the solution only depends on the last q regimes:

$$\mathcal{P}(q): \qquad \phi(is_1 \cdots s_q w, \varepsilon^t) = \phi(is_1 \cdots s_q w', \varepsilon^t)$$

$$\forall (s_1, \cdots, s_q) \in \{1, \cdots, N\}^q, \qquad \forall w \in \{1, \cdots, N\}^\infty, \qquad \forall w' \in \{1, \cdots, N\}^\infty, \qquad \forall \varepsilon^t \in V^\infty$$

$\mathcal{P}(p)$ is satisfied by assumption. Let us assume that $\mathcal{P}(q)$ is satisfied for $q \in \{1, \cdots, p\}$. Since ϕ is a solution of (4), for any w, we compute:

$$\phi(s_t s_1 \cdots s_{q-1} w, \varepsilon^t) = -B_{s_t}^{-1} A_{s_t} (\sum_i p_{s_t i} \int \phi(is_t s_1 \cdots s_{q-1} w, \varepsilon \varepsilon^t) d\varepsilon - B_{s_t}^{-1} A_{s_t} \varepsilon_t$$

Due to $\mathcal{P}(q)$, we know that:

$$-B_{s_t}^{-1} A_{s_t} (\sum_i p_{s_t i} \int \phi(is_t s_1 \cdots s_{q-1} w, \varepsilon \varepsilon^t) d\varepsilon = -B_{s_t}^{-1} A_{s_t} (\sum_i p_{s_t i} \int \phi(is_t s_1 \cdots s_{q-1} w', \varepsilon \varepsilon^t) d\varepsilon$$

for any w', and hence ϕ does not depend on w. $\mathcal{P}(q-1)$ is thus satisfied. By decreasing induction we eventually show that ϕ is Markovian of order 0.

More generally if the solution is Markovian, its order is the same than ψ_0. Here, ψ_0 is Markovian of order 0, thus ϕ is also Markovian of order 0.

If $\phi \in \mathcal{M}_0$ is a solution of 4, ϕ is a solution of:

$$\forall i \in \{1, \cdots, N\} \qquad \phi(i, \varepsilon^t) + B_i^{-1} A_i \left(p_{i1} \int \phi(1, \varepsilon \varepsilon^t) d\varepsilon + p_{i2} \int \phi(2, \varepsilon \varepsilon^t) d\varepsilon \right) = -B_i^{-1} C_i \varepsilon_t$$

Thus by introducing $\mathbf{\Psi_0}(\varepsilon^t) = -\begin{bmatrix} B_1^{-1}C_1 \\ \vdots \\ B_N^{-1}C_N \end{bmatrix} \varepsilon_t = \mathbf{C}\varepsilon_t$, this system can be rewritten as:

$$\mathbf{\Phi}(\varepsilon^t) + \mathbf{M}\int \Phi(\varepsilon\varepsilon^t)d\varepsilon = \mathbf{\Psi_0}(\varepsilon^t) \tag{8}$$

where $\mathbf{\Phi}$ is defined in Equation (6). Model (8) is a standard linear rational expectations model with constant parameters. We hence easily prove Proposition 1 by applying Blanchard and Kahn (1980).

We denote by $\mathcal{B}_0$ the set of bounded functions on V^∞, and by $\mathcal{F}$ the bounded operator acting in $\mathcal{B}_0$:

$$\mathcal{F} : \mathbf{\Phi} \mapsto \left((\varepsilon^t) \mapsto \int \mathbf{\Phi}(\varepsilon\varepsilon^t)d\varepsilon \right)$$

We rewrite equation (8) as: $[\mathbb{1}+\mathbf{M}\mathcal{F}]\mathbf{\Phi} = \mathbf{\Psi_0}$. The solution $\mathbf{\Phi}$ is then: $\mathbf{\Phi} = \sum_{k=0}^{\infty} [-\mathbf{M}\mathcal{F}]^k \mathbf{\Psi_0}$. Knowing that:

$$(\mathcal{F}^k\mathbf{\Psi_0})(\varepsilon_t) = \mathbf{C}\Lambda^k\varepsilon_t$$

We get that $\Phi(\varepsilon^t) = \mathbf{R}\varepsilon_t$ with $\mathrm{Vect}(\mathbf{R}) = [1 + \Lambda' \otimes \mathbf{M}]^{-1}\mathrm{Vect}(\mathbf{C})$.

Splitting $\mathbf{R} = \begin{bmatrix} R_1 \\ \vdots \\ R_N \end{bmatrix}$, the solution ϕ is given by: $\phi(s^t, \varepsilon^t) = R_{s_t}\varepsilon_t$

B Proof of Lemma 1

In this part, we prove Lemma 1.

We introduce the real sequence (u_k) defined for $k \geq 2$ by:

$$u_k = \left(\sum_{(i_1,\cdots,i_k)\in\{1,\cdots,N\}^k} p_{i_1 i_2}\cdots p_{i_{k-1}i_k} \|B_{i_1}^{-1}A_{i_1}\cdots B_{i_k}^{-1}A_{i_k}\| \right)^{1/k} \tag{9}$$

We will show that:

- The sequence $(u_k)^k$ is sub-multiplicative ($(u_{m+n})^{m+n} \leq u_m^m u_n^n$), and thus convergent.
- The sequence $\left(\rho(S_p)^{1/p}\right)$ is equivalent to (u_p) when p tends to ∞.
- Their limit, ν, does not depend on the chosen norm.

We first show that (u_k^k) is sub-multiplicative. By sub-multiplicativity of a matricial norm, u_{m+n}^{m+n} satisfies:

$$\sum_{(i_1,\cdots,i_m,i_{m+1},\cdots,i_{m+n})\in\{1,\cdots,N\}^{m+n}} p_{i_1 i_2}\cdots p_{i_{m-1}i_m} \times$$

$$p_{i_m i_{m+1}} \cdots p_{i_{m+n-1} i_{m+n}} \| B_{i_1}^{-1} A_{i_1} \cdots B_{i_m}^{-1} A_{i_m} B_{i_{m+1}}^{-1} A_{i_{m+1}} \cdots B_{i_{m+n}}^{-1} A_{i_{m+n}} \|$$
$$\leq \sum_{(i_1, \cdots, i_m, i_{m+1}) \in \{1, \cdots, N\}^{m+1}} p_{i_1 i_2} \cdots p_{i_{m-1} i_m} p_{i_m i_{m+1}} \| B_{i_1}^{-1} A_{i_1} \cdots B_{i_m}^{-1} A_{i_m} \|$$
$$\times \left(\sum_{(i_{m+2}, \cdots, i_{m+n}) \in \{1, \cdots, N\}^{n-1}} p_{i_{m+1} i_{m+2}} \cdots p_{i_{m+n-1} i_{m+n}} \| B_{i_{m+1}}^{-1} A_{i_{m+1}} \cdots B_{i_{m+n}}^{-1} A_{i_{m+n}} \| \right)$$

We find an upper bound for the second term by summing on i_{m+1}, as all the terms are positive:

$$\sum_{(i_{m+2}, \cdots, i_{m+n}) \in \{1, \cdots, N\}^{n-1}} p_{i_{m+1} i_{m+2}} \cdots p_{i_{m+n-1} i_{m+n}} \| B_{i_{m+1}}^{-1} A_{i_{m+1}} \cdots B_{i_{m+n}}^{-1} A_{i_{m+n}} \|$$

$$\leq \sum_{(i_{m+1}, i_{m+2}, \cdots, i_{m+n}) \in \{1, \cdots, N\}^{n-1}} p_{i_{m+1} i_{m+2}} \cdots p_{i_{m+n-1} i_{m+n}} \| B_{i_{m+1}}^{-1} A_{i_{m+1}} \cdots B_{i_{m+n}}^{-1} A_{i_{m+n}} \| = (u_n)^n$$

Thus,

$$(u_{n+m})^{n+m} \leq u_n^n \sum_{(i_1, \cdots, i_m, i_{m+1}) \in \{1, \cdots, N\}^{m+1}} p_{i_1 i_2} \cdots p_{i_{m-1} i_m} p_{i_m i_{m+1}} \| B_{i_1}^{-1} A_{i_1} \cdots B_{i_m}^{-1} A_{i_m} \| = u_n^n \times u_m^m$$

since $\sum_{i_{m+1} \in \{1, \cdots, N\}} p_{i_m i_{m+1}} = 1$.

This shows that (u_k^k) is sub-multiplicative.

Besides, if a sequence of non-negative real numbers (v_k) is sub-multiplicative, then $v_k^{1/k}$ is converging and $\lim_{k \to +\infty} v_k^{1/k} = \inf_k v_k^{1/k}$, see for instance Lemma 21 p.8 in Müller (2003). Thus (u_k) is convergent.

Now, we consider the norm $|\cdot|_\infty$ on $\mathcal{M}_2(\mathbb{R})$ defined by $|M|_\infty = \sum_{i,j} |m_{ij}|$. One can observe that:

$$|S_p|_\infty = \sum_{i, i_1, \cdots, i_{p-1}, j} p_{i i_1} \cdots p_{i_{p-1} j} \| B_i^{-1} A_i B_{i_1}^{-1} A_{i_1} \cdots B_{i_{p-1}}^{-1} A_{i_{p-1}} \| = u_{p-1}^{p-1} \tag{10}$$

As the spectral radius is the infimum of matricial norms, Equation 10 leads to:

$$\rho(S_p) \leq u_{p-1}^{p-1} \tag{11}$$

Furthermore,

$$\left(S_p^q\right)_{ij} = \sum_{i_1, \cdots, i_{p-1}, i_p, i_{p+1}, \cdots, i_{2p}, \cdots, i_{p(q-1)+1}, \cdots, i_{pq-1}} p_{i i_1} \cdots p_{i_{p-1} i_p} p_{i_{(q-1)p} i_{(q-1)p+1}} \cdots p_{i_{qp-1} j} \times$$
$$\| B_i^{-1} A_i \cdots B_{i_{p-1}}^{-1} A_{i_{p-1}} \| \cdots \| B_{i_{(q-1)p}}^{-1} A_{i_{(q-1)p}} \cdots B_{i_{qp-1}}^{-1} A_{i_{qp-1}} \|$$

And by sub-multiplicativity of matricial norms:

$$\left(S_p^q\right)_{ij} \geq \sum_{i_1, \cdots, i_{p-1}, i_p, i_{p+1}, \cdots, i_{2p}, \cdots, i_{p(q-1)+1}, \cdots, i_{pq-1}} p_{i i_1} \cdots p_{i_{p-1} i_p} p_{i_{(q-1)p} i_{(q-1)p+1}} \cdots p_{i_{qp-1} j} \times$$

$$\|B_i^{-1}A_i \cdots B_{i_{p-1}}^{-1}A_{i_{p-1}} \cdots B_{i_{(q-1)p}}^{-1}A_{i_{(q-1)p}} \cdots B_{i_{qp-1}}^{-1}A_{i_{qp-1}}\|$$

and hence,

$$|S_p^q|_\infty \geq u_{pq-1}^{pq-1} \tag{12}$$

Equation 11 can be rewritten as follows:

$$|S_p^q|_\infty^{(1/q)} \geq (u_{pq-1})^{p-1/q}$$

As for any norm, Gelfand's Theorem shows that $\lim_{q\to\infty} ||X^q||_\infty^{(1/q)} = \rho(X)$, thus when q tends to infinity, 11 leads to:

$$\lim_{k\to\infty} u_k^p \leq \rho(S_p)$$

Thus, as $p > 1$,

$$\lim_{k\to\infty} u_k \leq \rho(S_p)^{1/p} \tag{13}$$

Combining Equations (11) and (13), we find the following upper and lower bounds:

$$\lim_{k\to\infty} u_k \leq \rho(S_p)^{1/p} \leq u_{p-1}^{1-1/p}$$

and thus, $\left(\rho(S_p)^{1/p}\right)$ is convergent and has the same limit as (u_k).

Finally, by equivalence of the norms in $\mathcal{M}_n(\mathbb{R})$, it is immediate that ν does not depend on the chosen norm. This ends the proof of Lemma 1.

C Proof of Proposition 2

C.1 Prolegomenon

Assuming that B_i is invertible for any $i \in \{1, \cdots, N\}$, we rewrite (4) as:

$$z_t + B_{s_t}^{-1}A_{s_t}\mathbb{E}_t z_{t+1} = -B_{s_t}^{-1}C_{s_t}\varepsilon_t \tag{14}$$

Then, considering $z_t = z(s^t, \varepsilon^t)$ as a function of all the past shocks $\{\varepsilon_t, \cdots, \varepsilon_{-\infty}\}$ and regimes $\{s_t, \cdots, s_{-\infty}\}$, introducing ψ_0 such that $\psi_0(s^t, \varepsilon^t) = -B_{s_t}^{-1}C_{s_t}\varepsilon_t$ and defining the operator $\mathcal{R}$ as

$$\mathcal{R} : z \mapsto ((s^t, \varepsilon^t) \mapsto -B_{s_t}^{-1}A_{s_t}\mathbb{E}_t z(s^{t+1}, \varepsilon^{t+1})) \tag{15}$$

Equation (14) is equivalent to the functional equation:

$$(\mathbb{1} - \mathcal{R})z = \psi_0 \tag{16}$$

This equation admits a unique solution if the operator $\mathbb{1} - \mathcal{R}$ is invertible, and thus if $1 \notin \sigma(\mathcal{R})$. As a consequence, conditions of existence and uniqueness of a solution of (4) rely on the spectrum of $\mathcal{R}$, this spectrum depending on the space of solutions we consider.

C.2 Characterization of the spectral radius of $\mathcal{R}$

We will prove the following lemma, describing the spectrum of $\mathcal{R}$ in $\mathcal{B}$.

Lemma 4. *The operator $\mathcal{R}$ is bounded in $\mathcal{B}$ and its spectrum is given by:*

$$\sigma(\mathcal{R}) = [-\nu, \nu]$$

First, $\mathcal{R}$ is bounded as the expectation operator is a bounded operator. The rest of the proof is based on two main arguments:

- The spectrum of $\mathcal{R}$ is symmetric convex.
- $$\lim_{k \to +\infty} \|\mathcal{R}^k\|^{1/k} = \nu$$

The second point ensures that $\rho(\mathcal{R}) = \nu$ by applying the Gelfand characterization of the spectral radius for an operator, see for instance Theorem 22 p.8 in Müller (2003), while the first point leads to the equality $\sigma(\mathcal{R}) = [-\nu, \nu]$.

First, we introduce the operators $\mathcal{F}_i$, for $i \in \{1, \cdots N\}$, $\mathcal{F}$ and $\mathcal{L}$ on $\mathcal{B}$ defined by:

$$\mathcal{F}_i : \phi \mapsto ((s^t, \varepsilon^t) \mapsto \int_V \phi(is^t, \varepsilon\varepsilon^t) d\varepsilon$$

$$\mathcal{L} : \phi \mapsto ((s^t, \varepsilon^t) \mapsto \phi(s^{t-1}, \varepsilon^{t-1})$$

$$\mathcal{F}(\phi)(s^t, \varepsilon^t) = (p_{s_t 1}\mathcal{F}_1 + p_{s_t 2}\mathcal{F}_2)(\phi)(s^t, \varepsilon^t)$$

The operators $\mathcal{F}_i$ and $\mathcal{L}$ have the following straightforward properties.

1. $\mathcal{F}_i\mathcal{L} = \mathbb{1}$, and $\mathcal{F}\mathcal{L} = \mathbb{1}$
2. $|||\mathcal{F}_i||| = 1$ and $|||\mathcal{L}||| = 1$

where $|||\cdot|||$ is the triple norm associated with the infinite norm $\|\cdot\|_\infty$ on $\mathcal{B}$. Then $\mathcal{R}$ can be rewritten as:

$$\mathcal{R}(\phi)(s^t, \varepsilon^t) = B_{s_t}^{-1} A_{s_t} (p_{s_t 1}\mathcal{F}_1 + p_{s_t 2}\mathcal{F}_2)(\phi)(s^t, \varepsilon^t)$$

We define $\tilde{\mathcal{R}}$ by

$$\tilde{\mathcal{R}} : \phi \mapsto A_{s_{t-1}}^{-1} B_{s_{t-1}} \mathcal{L}(\phi)(s^t, \varepsilon^t)$$

We have that:

$$\tilde{\mathcal{R}}\mathcal{R} = \mathcal{L}\mathcal{F}, \qquad \mathcal{R}\tilde{\mathcal{R}} = \mathbb{1}$$

We mimic techniques used to study the spectrum of isometries in Banach spaces as for instance in Conway (1990). We refer to this book and to Müller (2003) for the different type of spectrum. We know that the spectrum of $\mathcal{R}$ is a closed subset of $[-\|\mathcal{R}\|, \|\mathcal{R}\|]$, and that the boundary $\partial\sigma(\mathcal{R})$ of $\sigma(\mathcal{R})$ is included in the point approximate spectrum, i.e. the set of values λ such that $\mathcal{R} - \lambda\mathbb{1}$ is not injective or not bounded below. We assume that $\sigma(\mathcal{R})$ is not convex, and that there exists $\lambda_0 \in (0, \nu)$ such that $\lambda \in \partial\sigma(\mathcal{R})$. Then, we prove that λ_0 is an eigenvalue. Actually, $\mathcal{R} - \lambda\mathbb{1}$ is bounded below for any $\lambda < \|\mathcal{R}\|$. $\mathcal{R}$ is the composition of an invertible operator and an isometry, and thus is bounded below. Moreover, we notice that:

$$\|\mathcal{R}\| = \sup_{v \in \mathrm{Im}(\tilde{\mathcal{R}})} \frac{\|\mathcal{R}v\|}{\|v\|} = \left(\inf_u \frac{\|\tilde{\mathcal{R}}u\|}{\|u\|} \right)^{-1}$$

which implies that:

$$\|u - \lambda\tilde{\mathcal{R}}u\| \leq (1 - \frac{\lambda}{\|\mathcal{R}\|})\|u\|$$

We show now that for any α such that $|\alpha| < 1$, then $\lambda\alpha$ belongs to $\sigma(\mathcal{R})$. We know that λ is an eigenvalue of $\mathcal{R}$, let $\phi_0 \in \mathcal{B}$ an eigenvector of $\mathcal{R}$ associated with λ,

$$\mathcal{R}\phi_0 = \lambda\phi_0$$

We define f by:

$$f = \phi_0 - \lambda\tilde{\mathcal{R}}\phi_0$$

We notice that $\mathcal{R}(f) = o$, and that $\|(\lambda\tilde{\mathcal{R}})^k(f)\| \leq \|\phi_0\|$. Fix α such that $|\alpha| < 1$. We define $\tilde{\phi}_0$ by :

$$\tilde{\phi}_0 = \sum_{k=0}^{\infty} \alpha^k (\lambda\tilde{\mathcal{R}})^k(f)$$

We compute:

$$\mathcal{R}(\tilde{\phi}_0) = \sum_{k=0}^{\infty} \alpha^k \mathcal{R}(\lambda\tilde{\mathcal{R}})^k(f)$$

$$\mathcal{R}(\tilde{\phi}_0) = \alpha\lambda \sum_{k=0}^{\infty} \alpha^k (\lambda\tilde{\mathcal{R}})^k(f) = \alpha\lambda\tilde{\phi}_0$$

Thus $\alpha\lambda$ is an eigenvalue of $\mathcal{R}$, which contradicts $\lambda \in \partial\sigma(\mathcal{R})$, and $\partial\sigma(\mathcal{R}) = \nu$.

Concerning the second point, we first prove that $\lim_{k\to+\infty} \|\mathcal{R}^k\|^{1/k} \leq \nu$. Then we construct, for any k, a function ϕ_k, such that:

$$\|\mathcal{R}^k(\phi_k)\|^{1/k} \geq \rho(S_k)^{1/k}$$

This construction is a generalization to the multivariate cases of Farmer et al. (2009a) and Farmer et al. (2010).

We compute

$$\mathcal{R}^k(\phi)(s^t) = \sum_{i_1,\cdots,i_k} p_{s_t i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} B_{s_t}^{-1} A_{s_t} B_{i_1}^{-1} A_{i_1} \cdots B_{i_{k-1}}^{-1} A_{i_{k-1}} \mathcal{F}_{i_1} \cdots \mathcal{F}_{i_k}(\phi)(s^t)$$

We will find an upper bound and a lower bound for $\|\mathcal{R}^k\|$, in terms of a sequence (u_k) associated to well-chosen norms on $\mathcal{M}_n(\mathbb{R})$. First, we consider the triple norm associated to the infinite norm on $\mathcal{M}_n(\mathbb{R})$ and the associated sequence u_k. For any ϕ such that $\|\phi\|_\infty = 1$, we obtain by sub-additivity of the norm,

$$\|\mathcal{R}^k(\phi)\|_\infty \leq \sum_{i_1,\cdots,i_k} p_{s_t i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} |||B_{s_t}^{-1} A_{s_t} B_{i_1}^{-1} A_{i_1} \cdots B_{i_{k-1}}^{-1} A_{i_{k-1}}||| = u_k^k$$

which leads to $\lim_{k\to+\infty} \|\mathcal{R}^k\|^{1/k} \leq \nu$.

Reciprocally, we consider on $\mathcal{M}_{r,s}(\mathbb{R})$ the norm $|\cdot|$ defined by:

$$|M| = \sum_{i,j} |m_{i,j}|, \quad \text{where} \quad M = [m_{i,j}]_{(i,j)\in\{1,\cdots,r\}\times\{1,\cdots,s\}}$$

This norm satisfies:

- $|M| \leq r|||M|||_\infty$
- If we write $M = [M_1, M_2, \cdots, M_l]$ by blocks, we notice the following useful property:

$$|M| = \sum_{i=1}^{l} |M_i|$$

Fix $s_t \in \{1,\cdots,N\}$ and let us denote by $\{w_{i_1\cdots i_{k+1}}, \forall (i_1 \cdots i_{k+1} \in \{1,\cdots,N\})\}$ a family of $n \times 1$ vectors and rewrite the following sum as a product of matrices by blocks:

$$\sum_{(i_1,\cdots,i_k)\in\{1,\cdots,N\}^k} p_{s_t i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} A_{s_t} A_{i_1} \cdots A_{i_{k-1}} w_{s_t i_1 \cdots i_{k-1}}$$

$$= \left[\begin{array}{ccc} p_{s_t 1} \cdots p_{11} [B_{s_t}^{-1} A_{s_t} \cdots B_1^{-1} A_1] & \cdots & | p_{s_t N} \cdots p_{NN} [B_{s_t}^{-1} A_{s_t} \cdots B_N^{-1} A_N] \end{array}\right] \times \begin{bmatrix} w_{s_t 1 \cdots 1} \\ \vdots \\ w_{s_t N \cdots N} \end{bmatrix}$$

Thus,

$$\sup_{\|w_{i_1\cdots i_p}\|_\infty\leq 1}\|\sum_{i_1,\cdots,i_k} p_{s_t i_1}p_{i_1 i_2}\cdots p_{i_{k-1}i_k}A_{s_t}A_{i_1}\cdots A_{i_{k-1}}w_{i_1\cdots i_p}\|_\infty$$

$$=\sup_{\|w_{i_1\cdots i_p}\|_\infty\leq 1}\|\begin{bmatrix} p_{s_t 1}\cdots p_{11}[B_{s_t}^{-1}A_{s_t}\cdots B_1^{-1}A_1] & \cdots & |p_{s_t N}\cdots p_{NN}[B_{s_t}^{-1}A_{s_t}\cdots B_N^{-1}A_N]\end{bmatrix}\times\begin{bmatrix} w_{s_t 1\cdots 1}\\ \vdots \\ w_{s_t N\cdots N}\end{bmatrix}\|_\infty$$

$$=|||\begin{bmatrix} p_{s_t 1}\cdots p_{11}[B_{s_t}^{-1}A_{s_t}\cdots B_1^{-1}A_1] & \cdots & |p_{s_t N}\cdots p_{NN}[B_{s_t}^{-1}A_{s_t}\cdots B_N^{-1}A_N]\end{bmatrix}|||_\infty$$

$$\geq\frac{1}{Nn}|\begin{bmatrix} p_{s_t 1}\cdots p_{11}[B_{s_t}^{-1}A_{s_t}\cdots B_1^{-1}A_1] & \cdots & |p_{s_t N}\cdots p_{NN}[B_{s_t}^{-1}A_{s_t}\cdots B_N^{-1}A_N]\end{bmatrix}|$$

$$\geq\frac{1}{Nn}\sum_{i_1,\cdots,i_k} p_{s_t i_1}p_{i_1 i_2}\cdots p_{i_{k-1}i_k}|A_{s_t}A_{i_1}\cdots A_{i_{k-1}}|$$

Furthermore, as the considered space is a bounded subset of finite-dimensional vectorial space, the supremum is reached and there exist N^k vectors $(w_{s_t i_1\cdots i_{k-1}})$ for $(i_1,\cdots,i_{k-1})\in\{1,\cdots,N\}^{k-1}$ such that:

$$\|\sum_{i_1,\cdots,i_k} p_{s_t i_1}p_{i_1 i_2}\cdots p_{i_{k-1}i_k}B_{s_t}^{-1}A_{s_t}B_{i_1}^{-1}A_{i_1}\cdots B_{i_{k-1}}^{-1}A_{i_{k-1}}w_{i_1\cdots i_p}\|$$
$$\geq\frac{1}{Nn}\sum_{i_1,\cdots,i_k} p_{s_t i_1}p_{i_1 i_2}\cdots p_{i_{k-1}i_k}|B_{s_t}^{-1}A_{s_t}B_{i_1}^{-1}A_{i_1}\cdots B_{i_{k-1}}^{-1}A_{i_{k-1}}|$$

We define the function ϕ_0 by:$\phi_0(s^t)=w_{s_t s_{t-1}s_{t-2}\cdots s_{t-k}}$. This function is bounded and of norm 1. Moreover, ϕ_0 satisfies:

$$\sum_{s_t}\|\mathcal{R}^k(\phi_0)(s^t)\|\geq\frac{1}{Nn}\sum_{s_t,i_1,\cdots,i_k} p_{s_t i_1}p_{i_1 i_2}\cdots p_{i_{k-1}i_k}|B_{s_t}^{-1}A_{s_t}B_{i_1}^{-1}A_{i_1}\cdots B_{i_{k-1}}^{-1}A_{i_{k-1}}|=\frac{1}{Nn}\left(\tilde{u}_k\right)^k$$

which leads to:

$$\|\mathcal{R}^k(\phi_0)\|_\infty\geq\frac{1}{N^2 n}\left(\tilde{u}_k\right)^k$$

Finally, this implies that:

$$\|\mathcal{R}^k\|^{1/k}\geq(N^2 n)^{-1/k}\tilde{u}_k$$

Taking the limit, we get that $\lim_{k\to+\infty}\|\mathcal{R}^k\|^{1/k}\geq\nu$. This ends the proof of Lemma 4.

C.3 Proof of Proposition 2

A consequence of Lemma 4 is that $1 \in \sigma(\mathcal{R})$ if and only if $\nu \geq 1$, and thus $(\mathbb{1} - \mathcal{R})$ is invertible if and only if $\nu < 1$, which proves Proposition 2.

D Proofs of Lemmas 2 and 3

Lemma 2 follows directly from Equation (13).

To prove Lemma 3, we notice that

$$u_k^k = \sum_{(i_1,\cdots,i_k,i_{k+1})\in\{1,\cdots,N\}^k} p_{i_1 i_2}\cdots p_{i_{k-1}i_k} p_{i_k i_{k+1}} \|B_{i_1}^{-1}A_{i_1}\cdots B_{i_k}^{-1}A_{i_k}\|$$

Then by considering the multiples of p ($k = np$) and by only keeping the diverging trajectory (the hypothesis of the Lemma), we can rewrite the above equation as follows:

$$u_{np}^{np} \geq [p_{i_0 i_1} p_{i_1 i_2}\cdots p_{i_{p_0} i_0} \|B_{i_0}^{-1}A_{i_0}\cdots B_{i_{p_0}}^{-1}A_{i_{p_0}}\|]^n$$

and hence,

$$u_{np} \geq [p_{i_0 i_1} p_{i_1 i_2}\cdots p_{i_{p_0} i_0} \|B_{i_0}^{-1}A_{i_0}\cdots B_{i_{p_0}}^{-1}A_{i_{p_0}}\|]^{1/p}$$

Besides,

$$[p_{i_0 i_1} p_{i_1 i_2}\cdots p_{i_{p_0} i_0} \|B_{i_0}^{-1}A_{i_0}\cdots B_{i_{p_0}}^{-1}A_{i_{p_0}}\|]^{1/p} \geq \rho(p_{i_0 i_1} p_{i_1 i_2}\cdots p_{i_{p_0} i_0} B_{i_0}^{-1}A_{i_0}\cdots B_{i_{p_0}}^{-1}A_{i_{p_0}})$$

Thus,

$$\lim_{n\to\infty} u_{np} \geq \rho(p_{i_0 i_1} p_{i_1 i_2}\cdots p_{i_{p_0} i_0} B_{i_0}^{-1}A_{i_0}\cdots B_{i_{p_0}}^{-1}A_{i_{p_0}})$$

The right-hand-side of the inequality is larger than one by hypothesis which implies that $\nu > 1$.

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