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VARIATIONAL REPRESENTATIONS FOR N -CYCLICALLY MONOTONE VECTOR FIELDS

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Given a convex bounded domain Ω in \mathbb{R}^d and an integer $N \geq 2$, we associate to any *jointly N -monotone* $(N-1)$ -tuple $(u_1, u_2, \dots, u_{N-1})$ of vector fields from Ω into \mathbb{R}^d a Hamiltonian H on $\mathbb{R}^d \times \mathbb{R}^d \times \dots \times \mathbb{R}^d$ that is concave in the first variable, jointly convex in the last $N-1$ variables, and such that

$$(u_1(x), u_2(x), \dots, u_{N-1}(x)) = \nabla_{2,\dots,N} H(x, x, \dots, x)$$

for almost all $x \in \Omega$. Moreover, H is N -antisymmetric in a sense made precise later, and also N -sub-antisymmetric in the sense that for all $X \in \Omega^N$ the sum $\sum_{i=0}^{N-1} H(\sigma^i(X)) \leq 0$ is nonpositive, σ being the permutation that shifts the coordinates of X leftward one slot and places the first coordinate last. This result can be seen as an extension of a theorem of E. Krauss, which associates to any monotone operator a concave-convex antisymmetric saddle function. We also give various variational characterizations of vector fields that are almost everywhere N -monotone, showing that they are dual to the class of measure-preserving N -involutions on Ω .

1. Introduction

Given a domain Ω in \mathbb{R}^d , recall that a single-valued map u from Ω to \mathbb{R}^d is said to be *N -cyclically monotone* if for every cycle $x_1, \dots, x_N, x_{N+1} = x_1$ of points in Ω , one has

$$(1) \quad \sum_{i=1}^N \langle u(x_i), x_i - x_{i+1} \rangle \geq 0.$$

A classical theorem of Rockafellar [Phelps 1993] states that a map u from Ω to \mathbb{R}^d

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is N -cyclically monotone for every $N \geq 2$ if and only if

$$(2) \quad u(x) \in \partial\phi(x) \quad \text{for all } x \in \Omega,$$

where $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a convex function. On the other hand, a result of E. Krauss [1985] yields that u is a monotone map, i.e., a 2-cyclically monotone map, if and only if

$$(3) \quad u(x) \in \partial_2 H(x, x) \quad \text{for all } x \in \Omega,$$

where H is a concave-convex antisymmetric Hamiltonian on $\mathbb{R}^d \times \mathbb{R}^d$, and $\partial_2 H$ is the subdifferential of H as a convex function in the second variable.

In this paper, we extend the result of Krauss to the class of N -cyclically monotone vector fields, where $N \geq 3$. We shall give a representation for a family of $N-1$ vector fields, which may or may not be individually N -cyclically monotone. Here is the needed concept.

Definition 1. Let u_1, \dots, u_{N-1} be bounded vector fields from a domain $\Omega \subset \mathbb{R}^d$ into \mathbb{R}^d . We shall say that the $(N-1)$ -tuple $(u_1, u_2, \dots, u_{N-1})$ is *jointly N -monotone* if for every cycle x_1, \dots, x_{2N-1} of points in Ω such that $x_{N+i} = x_i$ for $1 \leq i \leq N-1$, one has

$$(4) \quad \sum_{i=1}^N \sum_{\ell=1}^{N-1} \langle u_\ell(x_i), x_i - x_{\ell+i} \rangle \geq 0.$$

Examples of jointly N -monotone families of vector fields:

- It is clear that $(u, 0, 0, \dots, 0)$ is jointly N -monotone if and only if u is N -monotone.
- More generally, if each u_ℓ is N -monotone, then the family $(u_1, u_2, \dots, u_{N-1})$ is jointly N -monotone. Actually, one only needs that for $1 \leq \ell \leq N-1$, the vector field u_ℓ be (N, ℓ) -monotone in the following sense: for every cycle $x_1, \dots, x_{N+\ell}$ of points in Ω such that $x_{N+i} = x_i$ for $1 \leq i \leq \ell$, we have

$$(5) \quad \sum_{i=1}^N \langle u_\ell(x_i), x_i - x_{\ell+i} \rangle \geq 0.$$

This notion is sometimes weaker than N -monotonicity since if ℓ divides N , then it suffices for u to be N/ℓ -monotone in order to be an (N, ℓ) -monotone vector field. For example, if u_1 and u_3 are 4-monotone operators and u_2 is 2-monotone, then the triplet (u_1, u_2, u_3) is jointly 4-monotone.

- Another example is if (u_1, u_2, u_3) are vector fields such that u_2 is 2-monotone and

$$\langle u_1(x) - u_3(y), x - y \rangle \geq 0 \quad \text{for every } x, y \in \mathbb{R}^d.$$

In this case, the triplet (u_1, u_2, u_3) is jointly 4-monotone. In particular, if u_1 and u_2 are both 2-monotone, then the triplet (u_1, u_2, u_1) is jointly 4-monotone.

- More generally, it is easy to show that (u, u, \dots, u) is jointly N -monotone if and only if u is 2-cyclically monotone.

We shall always denote by σ the cyclic permutation on $\mathbb{R}^d \times \dots \times \mathbb{R}^d$ defined by

$$\sigma(x_1, x_2, \dots, x_{N-1}, x_N) = (x_2, x_3, \dots, x_N, x_1).$$

We let

$$(6) \quad \mathcal{H}_N(\Omega) = \left\{ H \in C(\Omega^N) : \sum_{i=0}^{N-1} H(\sigma^i(x_1, \dots, x_N)) = 0 \right\}$$

be the family of continuous Hamiltonians on Ω^N that are N -antisymmetric, i.e., satisfy the condition to the right of the colon in (6). We say that H is N -sub-antisymmetric on Ω if

$$(7) \quad \sum_{i=0}^{N-1} H(\sigma^i(x_1, \dots, x_N)) \leq 0 \quad \text{on } \Omega^N.$$

We shall also say that a function F of two variables is N -cyclically sub-antisymmetric on Ω if

$$(8) \quad \begin{aligned} &F(x, x) = 0 \quad \text{and} \\ &\sum_{i=1}^N F(x_i, x_{i+1}) \leq 0 \quad \text{for all cyclic families } x_1, \dots, x_N, x_{N+1} = x_1 \text{ in } \Omega. \end{aligned}$$

Note that if a function $H(x_1, \dots, x_N)$ is N -sub-antisymmetric and if it only depends on the first two variables, then the function $F(x_1, x_2) := H(x_1, x_2, \dots, x_N)$ is N -cyclically sub-antisymmetric.

We associate to any function H on Ω^N the functional given by on $\Omega \times (\mathbb{R}^d)^{N-1}$

$$(9) \quad L_H(x, p_1, \dots, p_{N-1}) = \sup \left\{ \sum_{i=1}^{N-1} \langle p_i, y_i \rangle - H(x, y_1, \dots, y_{N-1}) : y_i \in \Omega \right\}.$$

Note that if Ω is convex and if H is convex in the last $N-1$ variables, then L_H is nothing but the Legendre transform of \tilde{H} with respect to the last $N-1$ variables, where \tilde{H} is the extension of H over $(\mathbb{R}^d)^N$, defined by $\tilde{H} = H$ on Ω^N and $\tilde{H} = +\infty$ outside Ω^N . Since $H(x, \dots, x) = 0$ for any $H \in \mathcal{H}_N(\Omega)$, we have, for any such H ,

$$(10) \quad L_H(x, p_1, \dots, p_{N-1}) \geq \sum_{i=1}^{N-1} \langle x, p_i \rangle,$$

for $x \in \Omega$ and $p_1, \dots, p_{N-1} \in \mathbb{R}^d$. To formulate variational principles for such

vector fields, we shall consider the class of σ -invariant probability measures on Ω^N , which are those $\pi \in \mathcal{P}(\Omega^N)$ such that for all $h \in L^1(\Omega^N, d\pi)$, we have

$$(11) \quad \int_{\Omega^N} h(x_1, \dots, x_N) d\pi = \int_{\Omega^N} h(\sigma(x_1, \dots, x_N)) d\pi.$$

We set

$$(12) \quad \mathcal{P}_{\text{sym}}(\Omega^N) = \{\pi \in \mathcal{P}(\Omega^N) : \pi \text{ } \sigma\text{-invariant probability on } \Omega^N\}.$$

For a given probability measure μ on Ω , we also consider the class

$$(13) \quad \mathcal{P}_{\text{sym}}^\mu(\Omega^N) = \{\pi \in \mathcal{P}_{\text{sym}}(\Omega^N) : \text{proj}_1 \pi = \mu\},$$

i.e., the set of all $\pi \in \mathcal{P}_{\text{sym}}(\Omega^N)$ with a given first marginal μ , meaning that

$$(14) \quad \int_{\Omega^N} f(x_1) d\pi(x_1, \dots, x_N) = \int_{\Omega} f(x_1) d\mu(x_1) \quad \text{for every } f \in L^1(\Omega, \mu).$$

Now consider the set $\mathcal{S}(\Omega, \mu)$ of μ -measure-preserving transformations on Ω , which can be identified with a closed subset of the sphere of $L^2(\Omega, \mathbb{R}^d)$. We shall also consider the subset of $\mathcal{S}(\Omega, \mu)$ consisting of N -involutions, that is,

$$\mathcal{S}_N(\Omega, \mu) = \{S \in \mathcal{S}(\Omega, \mu) : S^N = I \text{ } \mu\text{-a.e.}\}.$$

2. Monotone vector fields and N -antisymmetric Hamiltonians

In this section, we establish the following extension of a theorem of Krauss.

Theorem 2. *Let $N \geq 2$ be an integer, and let u_1, \dots, u_{N-1} be bounded vector fields from a convex domain $\Omega \subset \mathbb{R}^d$ into \mathbb{R}^d .*

- 1) *If the $(N-1)$ -tuple (u_1, \dots, u_{N-1}) is jointly N -monotone, then there exists an N -sub-antisymmetric Hamiltonian H that is zero on the diagonal of Ω^N , concave in the first variable, convex in the other $N-1$ variables, and such that*

$$(15) \quad (u_1(x), \dots, u_{N-1}(x)) = \nabla_{2, \dots, N} H(x, x, \dots, x) \quad \text{for a.e. } x \in \Omega.$$

Moreover, H is N -antisymmetric in the sense that

$$(16) \quad H(x_1, x_2, \dots, x_N) + H_{2, \dots, N}(x_1, x_2, \dots, x_N) = 0,$$

where $H_{2, \dots, N}$ is the concavification of the function $K(x) = \sum_{i=1}^{N-1} H(\sigma^i(x))$ with respect to the last $N-1$ variables.

Furthermore, there exists a continuous N -antisymmetric Hamiltonian \bar{H} on Ω^N , such that

$$(17) \quad L_{\bar{H}}(x, u_1(x), u_2(x), \dots, u_{N-1}(x)) = \sum_{i=1}^{N-1} \langle u_i(x), x \rangle \quad \text{for all } x \in \Omega.$$

2) Conversely, if (u_1, \dots, u_{N-1}) satisfies (15) for some N -sub-antisymmetric Hamiltonian H that is zero on the diagonal of Ω^N , concave in the first variable, and convex in the other variables, then the $(N-1)$ -tuple (u_1, \dots, u_{N-1}) is jointly N -monotone on Ω .

Remark 3. In the case $N = 2$, $K(x) = H(x_2, x_1)$ is concave with respect to x_2 , hence $H_2(x_1, x_2) = H(x_2, x_1)$, and (16) becomes

$$H(x_1, x_2) + H(x_2, x_1) = 0;$$

thus H is antisymmetric, recovering well-known results [Krauss 1985; Ghossoub 2009; Ghossoub and Moameni 2013a; Millien 2011].

Lemma 4. Assume the $(N-1)$ -tuple of bounded vector fields (u_1, \dots, u_{N-1}) on Ω is jointly N -monotone. Define

$$f(x_1, \dots, x_N) := \sum_{l=1}^{N-1} \langle u_l(x_1), x_1 - x_{l+1} \rangle$$

and let \tilde{f} be the convexification of f with respect to the first variable, given by

$$(18) \quad \tilde{f}(x_1, x_2, \dots, x_N) = \inf \left\{ \sum_{k=1}^n \lambda_k f(x_1^k, x_2, \dots, x_N) : n \in \mathbb{N}, \lambda_k \geq 0, \sum_{k=1}^n \lambda_k = 1, \sum_{k=1}^n \lambda_k x_1^k = x_1 \right\}.$$

- 1) We have $f \geq \tilde{f}$ on Ω^N .
- 2) \tilde{f} is convex in the first variable and concave with respect to the other variables.
- 3) $\tilde{f}(x, x, \dots, x) = 0$ for each $x \in \Omega$.
- 4) \tilde{f} satisfies

$$(19) \quad \sum_{i=0}^{N-1} \tilde{f}(\sigma^i(x_1, \dots, x_N)) \geq 0 \quad \text{on } \Omega^N.$$

Proof. Since the $(N-1)$ -tuple (u_1, \dots, u_{N-1}) is jointly N -monotone, it is easy to see that the function

$$f(x_1, \dots, x_N) := \sum_{l=1}^{N-1} \langle u_l(x_1), x_1 - x_{l+1} \rangle$$

is linear in the last $N-1$ variables, that $f(x, x, \dots, x) = 0$, and that

$$(20) \quad \sum_{i=0}^{N-1} f(\sigma^i(x_1, \dots, x_N)) \geq 0 \quad \text{on } \Omega^N.$$

It is also clear that $f \geq \tilde{f}$, that \tilde{f} is convex with respect to the first variable x_1 ,

and that it is concave with respect to the other variables x_2, \dots, x_N , since f itself is concave (actually linear) with respect to x_2, \dots, x_N . We now show that \tilde{f} satisfies (19).

For that, we fix x_1, x_2, \dots, x_N in Ω and consider $(x_1^k)_{k=1}^n$ in Ω , and $(\lambda_k)_k$ in \mathbb{R} such that $\lambda_k \geq 0$ such that $\sum_{k=1}^n \lambda_k = 1$ and $\sum_{k=1}^n \lambda_k x_1^k = x_1$. For each k , we have

$$f(x_1^k, x_2, \dots, x_N) + f(x_2, \dots, x_N, x_1^k) + \dots + f(x_N, x_1^k, x_2, \dots, x_{N-1}) \geq 0.$$

Multiplying by λ_k , summing over k , and using that f is linear in the last $N-1$ variables, we have

$$\sum_{k=1}^n \lambda_k f(x_1^k, x_2, \dots, x_N) + f(x_2, \dots, x_N, x_1) + \dots + f(x_N, x_1, x_2, \dots, x_{N-1}) \geq 0.$$

By taking the infimum, we obtain

$$\tilde{f}(x_1, x_2, \dots, x_N) + \sum_{i=1}^{N-1} f(\sigma^i(x_1, x_2, \dots, x_N)) \geq 0.$$

Let now $n \in \mathbb{N}$, $\lambda_k \geq 0$, $x_N^k \in \Omega$ be such that $\sum_{k=1}^n \lambda_k = 1$ and $\sum_{k=1}^n \lambda_k x_2^k = x_2$. For every $1 \leq k \leq n$, we have

$$\tilde{f}(x_1, x_2^k, x_3, \dots, x_N) + f(x_2^k, x_3, \dots, x_1) + \dots + f(x_N, x_1, x_2^k, x_3, \dots, x_{N-1}) \geq 0.$$

Multiplying by λ_k , summing over k and using that \tilde{f} is convex in the first variable and f is linear in the last $N-1$ variables, we obtain

$$\begin{aligned} & \tilde{f}(x_1, x_2, x_3, \dots, x_N) + \sum_{k=1}^n \lambda_k f(x_2^k, x_3, \dots, x_1) + \dots + f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \\ & \geq \sum_{k=1}^n \lambda_k \tilde{f}(x_1, x_2^k, x_3, \dots, x_N) + \sum_{k=1}^n \lambda_k f(x_2^k, x_3, \dots, x_1) \\ & \quad + \dots + \sum_{k=1}^n \lambda_k f(x_N, x_1, x_2^k, x_3, \dots, x_{N-1}) \\ & \geq 0. \end{aligned}$$

By taking the infimum over all possible such choices, we get

$$\tilde{f}(x_1, x_2, x_3, \dots, x_N) + \tilde{f}(x_2, x_3, \dots, x_1) + \dots + f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \geq 0.$$

By repeating this procedure with x_3, \dots, x_{N-1} , we get

$$\sum_{i=0}^{N-2} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N)) + f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \geq 0.$$

Finally, since

$$f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \geq - \sum_{i=0}^{N-2} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N))$$

and since \tilde{f} is concave in the last $N-1$ variables, the function

$$x_N \rightarrow - \sum_{i=0}^{N-2} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N))$$

for fixed x_1, x_2, \dots, x_{N-1} is a convex minorant of $x_N \rightarrow f(x_N, x_1, x_2, \dots, x_{N-1})$. It follows that

$$\begin{aligned} f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) &\geq \tilde{f}(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \\ &\geq - \sum_{i=0}^{N-2} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N)), \end{aligned}$$

which yields $\sum_{i=0}^{N-1} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N)) \geq 0$. This implies that $\tilde{f}(x, x, \dots, x) \geq 0$ for $x \in \Omega$.

On the other hand, since $\tilde{f}(x, x, \dots, x) \leq f(x, x, \dots, x) = 0$, we get that $\tilde{f}(x, x, \dots, x) = 0$ for all $x \in \Omega$. □

Proof of Theorem 2. Assume the $(N-1)$ -tuple of vector fields (u_1, \dots, u_{N-1}) is jointly N -monotone on Ω , and consider the function

$$f(x_1, \dots, x_N) := \sum_{l=1}^{N-1} \langle u_l(x_1), x_1 - x_{l+1} \rangle$$

as well as its convexification with respect to the first variable $\tilde{f}(x_1, \dots, x_N)$.

By Lemma 4, the function $\psi(x_1, \dots, x_N) := -\tilde{f}(x_1, \dots, x_N)$ satisfies the following properties:

- (i) $x_1 \rightarrow \psi(x_1, \dots, x_N)$ is concave.
- (ii) $(x_2, x_3, \dots, x_N) \rightarrow \psi(x_1, \dots, x_N)$ is convex.
- (iii) $\psi(x_1, \dots, x_N) \geq -f(x_1, \dots, x_N) = \sum_{l=1}^{N-1} \langle u_l(x_1), x_{l+1} - x_1 \rangle$.
- (iv) ψ is N -sub-antisymmetric.

Now consider the family $\overline{\mathcal{H}}$ of functions $H : \Omega^N \rightarrow \mathbb{R}$ such that

- 1) $H(x_1, x_2, \dots, x_N) \geq \sum_{l=1}^{N-1} \langle u_l(x_1), x_{l+1} - x_1 \rangle$ for every N -tuple (x_1, \dots, x_N) in Ω^N ,
- 2) H is concave in the first variable,
- 3) H is jointly convex in the last $N-1$ variables,

- 4) H is N -sub-antisymmetric,
- 5) H is zero on the diagonal of Ω^N .

Note that $\overline{\mathcal{H}} \neq \emptyset$ since ψ belongs to $\overline{\mathcal{H}}$. Note that any H satisfying conditions 1 and 4 automatically satisfies 5. Indeed, by N -sub-antisymmetry, for all $\mathbf{x} = (x_1, \dots, x_N) \in \Omega^N$ we have

$$(21) \quad H(\mathbf{x}) \leq - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x})) \leq - \sum_{i=1}^{N-1} \psi(\sigma^i(\mathbf{x})).$$

This also yields that

$$(22) \quad \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_{\ell+1} - x_1 \rangle \leq H(\mathbf{x}) \leq - \sum_{i=2}^N \sum_{\ell=1}^{N-1} \langle u_\ell(x_i), x_i - x_{i+\ell} \rangle,$$

where we denote $x_{i+N} := x_i$ for $i = 1, \dots, \ell$. This yields that $H(x, x, \dots, x) = 0$ for any $x \in \Omega$.

It is also easy to see that every directed family $(H_i)_i$ in $\overline{\mathcal{H}}$ has a supremum $H_\infty \in \overline{\mathcal{H}}$, meaning that $\overline{\mathcal{H}}$ is a Zorn family, and therefore has a maximal element H .

Now consider the function

$$\overline{H}(\mathbf{x}) = \frac{1}{N} \left((N-1)H(\mathbf{x}) - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x})) \right).$$

- (i) \overline{H} is N -antisymmetric, since $\overline{H}(\mathbf{x}) = \frac{1}{N} \sum_{i=1}^{N-1} [H(\mathbf{x}) - H(\sigma^i(\mathbf{x}))]$, and each summand is N -antisymmetric.
- (ii) $\overline{H} \geq H$ on Ω^N , since $N[\overline{H}(\mathbf{x}) - H(\mathbf{x})] = - \sum_{i=0}^{N-1} H(\sigma^i(\mathbf{x})) \geq 0$ (because H itself is N -sub-antisymmetric).

The maximality of H would have implied that $H = \overline{H}$ is N -antisymmetric if only \overline{H} was jointly convex in the last $N-1$ variables, but since this is not necessarily the case, we consider for $\mathbf{x} = (x_1, x_2, \dots, x_N)$ the function

$$K(x_1, x_2, \dots, x_N) = K(\mathbf{x}) := - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x})),$$

which is already concave in the first variable x_1 . Its convexification in the last $N-1$ variables, that is,

$$K^{2, \dots, N}(\mathbf{x}) = \inf \left\{ \sum_{i=1}^n \lambda_i K(x_1, x_2^i, \dots, x_N^i) : \lambda_i \geq 0, \sum_{i=1}^n \lambda_i (x_2^i, \dots, x_N^i, 1) = (x_2, \dots, x_N, 1) \right\},$$

is still concave in the first variable, but is now convex in the last $N-1$ variables. Moreover,

$$(23) \quad H \leq K^{2,\dots,N} \leq K = - \sum_{i=1}^{N-1} H \circ \sigma^i.$$

Indeed, $K^{2,\dots,N} \leq K$ from the definition of $K^{2,\dots,N}$, while $H \leq K^{2,\dots,N}$ because $H \leq K$ and H is already convex in the last $N-1$ variables. It follows that

$$H \leq \frac{(N-1)H + K^{2,\dots,N}}{N} \leq \frac{(N-1)H + K}{N} = \frac{1}{N} \left((N-1)H - \sum_{i=1}^{N-1} H \circ \sigma^i \right) = \bar{H}.$$

The function $H' = ((N-1)H + K^{2,\dots,N})/N$ belongs to the family $\bar{\mathcal{H}}$ and therefore $H = H'$ by the maximality of H .

This finally yields that H is N -sub-antisymmetric, that $H(x, \dots, x) = 0$ for all $x \in \Omega$ and that

$$H(x) + H_{2,\dots,N}(x) = 0 \quad \text{for every } x \in \Omega^N,$$

where $H_{2,\dots,N} = -K^{2,\dots,N}$, which for a fixed x_1 is nothing but the concavification of $(x_2, \dots, x_N) \rightarrow \sum_{i=1}^{N-1} H(\sigma^i(x_1, x_2, \dots, x_N))$.

Note now that since for any x_1, \dots, x_N in Ω

$$(24) \quad H(x_1, x_2, \dots, x_N) \geq \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_{\ell+1} - x_1 \rangle,$$

and

$$(25) \quad H(x_1, x_1, \dots, x_1) = 0,$$

we have

$$(26) \quad H(x_1, x_2, \dots, x_N) - H(x_1, \dots, x_1) \geq \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_{\ell+1} - x_1 \rangle.$$

Since H is convex in the last $N-1$ variables, this means that for all $x \in \Omega$, we have

$$(27) \quad (u_1(x), u_2(x), \dots, u_{N-1}(x)) \in \partial_{2,\dots,N} H(x, x, \dots, x),$$

as claimed in (15). This also yields

$$L_H(x, u_1(x), \dots, u_{N-1}(x)) + H(x, x, \dots, x) = \sum_{\ell=1}^{N-1} \langle u_\ell(x), x \rangle \quad \text{for all } x \in \Omega.$$

In other words, $L_H(x, u_1(x), \dots, u_{N-1}(x)) = \sum_{\ell=1}^{N-1} \langle u_\ell(x), x \rangle$ for all $x \in \Omega$. As above, consider

$$\bar{H}(x) = \frac{1}{N} \left((N-1)H(x) - \sum_{i=1}^{N-1} H(\sigma^i(x)) \right).$$

We have $\bar{H} \in \bar{\mathcal{H}}_N(\Omega)$ and $\bar{H} \geq H$, and therefore $L_{\bar{H}} \leq L_H$. On the other hand, for all $x \in \Omega$ we have

$$\begin{aligned} L_{\bar{H}}(x, u_1(x), \dots, u_{N-1}(x)) &= L_{\bar{H}}(x, u_1(x), \dots, u_{N-1}(x)) + \bar{H}(x, x, \dots, x) \\ &\geq \sum_{\ell=1}^{N-1} \langle u_\ell(x), x \rangle. \end{aligned}$$

To prove (17), we use the appendix in [Ghoussoub and Moameni 2013b] to deduce that for $i = 2, \dots, N$, the gradients $\nabla_i H(x, x, \dots, x)$ actually exist for a.e. x in Ω .

The converse is straightforward since if (27) holds, then (26) does, and since we also have (25), then the property that (u_1, \dots, u_{N-1}) is jointly N -monotone follows from (24) and the sub-antisymmetry of H . □

In the case of a single N -monotone vector field, we can obviously apply the above theorem to the $(N-1)$ -tuple $(u, 0, \dots, 0)$, which is then N -monotone, to find an N -sub-antisymmetric Hamiltonian H , which is concave in the first variable and convex in the last $N-1$ variables such that

$$(28) \quad (-u(x), u(x), 0, \dots, 0) = \nabla H(x, x, \dots, x) \quad \text{for a.e. } x \in \Omega.$$

However, in this case we can restrict ourselves to N -cyclically sub-antisymmetric functions of two variables and establish the following extension of the theorem of Krauss.

Theorem 5. *If u is N -cyclically monotone on Ω , then there exists a concave-convex function of two variables F that is N -cyclically sub-antisymmetric and zero on the diagonal, such that*

$$(29) \quad (-u(x), u(x)) \in \partial F(x, x) \quad \text{for all } x \in \Omega,$$

where ∂H is the subdifferential of H as a concave-convex function [Rockafellar 1970]. Moreover,

$$(30) \quad u(x) = \nabla_2 F(x, x) \quad \text{for a.e. } x \in \Omega.$$

Proof. Let $f(x, y) = \langle u(x), x - y \rangle$ and let $f^1(x, y)$ be its convexification in x for fixed y , that is,

$$(31) \quad f^1(x, y) = \inf \left\{ \sum_{k=1}^n \lambda_k f(x_k, y) : \lambda_k \geq 0, \sum_{k=1}^n \lambda_k = 1, \sum_{k=1}^n \lambda_k x_k = x \right\}.$$

Since $f(x, x) = 0$, f is linear in y , and $\sum_{i=1}^N f(x_i, x_{i+1}) \geq 0$ for any cyclic family

$x_1, \dots, x_N, x_{N+1} = x_1$ in Ω , it is easy to show that $f \geq f^1$ on Ω , f^1 is convex in the first variable and concave with respect to the second, $f^1(x, x) = 0$ for each $x \in \Omega$, and that f^1 is N -cyclically supersymmetric in the sense that for any cyclic family $x_1, \dots, x_N, x_{N+1} = x_1$ in Ω , we have $\sum_{i=1}^N f^1(x_i, x_{i+1}) \geq 0$.

Now consider $F(x, y) = -f^1(x, y)$ and note that $x \rightarrow F(x, y)$ is concave, $y \rightarrow F(x, y)$ is convex, $F(x, y) \geq -f(x, y) = \langle u(x), y - x \rangle$ and F is N -cyclically sub-antisymmetric. By the antisymmetry, we have

$$(32) \quad \langle u(x_1), x_2 - x_1 \rangle \leq F(x_1, x_2) \leq \langle u(x_2), x_2 - x_1 \rangle,$$

which yields that $(-u(x), u(x)) \in \partial F(x, x)$ for all $x \in \Omega$.

Since F is antisymmetric and concave-convex, the possibly multivalued map $x \rightarrow \partial_2 F(x, x)$ is monotone on Ω , and therefore single-valued and differentiable almost everywhere [Phelps 1993]. This completes the proof. \square

Remark 6. We cannot expect to have a function F such that $\sum_{i=1}^N F(x_i, x_{i+1}) = 0$ for all cyclic families $x_1, \dots, x_N, x_{N+1} = x_1$ in Ω . Actually, we believe that the only function satisfying such an N -antisymmetry for $N \geq 3$ must be of the form $F(x, y) = f(x) - f(y)$. This is why one needs to consider functions of N variables in order to get N -antisymmetry. In other words, the function defined by

$$(33) \quad H(x_1, x_2, \dots, x_N) := \frac{1}{N} \left((N - 1)F(x_1, x_2) - \sum_{i=2}^{N-1} F(x_i, x_{i+1}) \right)$$

is N -antisymmetric in the sense of (6) and $H(x_1, x_2, \dots, x_N) \geq F(x_1, x_2)$ for all (x_1, x_2, \dots, x_N) in Ω^N .

3. Variational characterization of monotone vector fields

In order to simplify the exposition, we shall always assume in the sequel that $d\mu$ is Lebesgue measure dx normalized to be a probability on Ω . We shall also assume that Ω is convex and that its boundary has measure zero.

Theorem 7. *Let $u_1, \dots, u_{N-1} : \Omega \rightarrow \mathbb{R}^d$ be bounded measurable vector fields. The following properties are then equivalent:*

- 1) *The $(N-1)$ -tuple (u_1, \dots, u_{N-1}) is jointly N -monotone a.e., that is, there exists a measure-zero set Ω_0 such that (u_1, \dots, u_{N-1}) is jointly N -monotone on $\Omega \setminus \Omega_0$.*
- 2) *The infimum of the Monge–Kantorovich problem*

$$(34) \quad \inf \left\{ \int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_1 - x_{\ell+1} \rangle d\pi(x_1, x_2, \dots, x_N) : \pi \in \mathcal{P}_{\text{sym}}^\mu(\Omega^N) \right\}$$

is equal to zero, and is therefore attained by the push-forward of μ by the map $x \rightarrow (x, x, \dots, x)$.

3) (u_1, \dots, u_{N-1}) is in the polar of $\mathcal{P}_N(\Omega, \mu)$ in the following sense:

$$(35) \quad \inf \left\{ \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), x - S^{\ell}x \rangle d\mu : S \in \mathcal{P}_N(\Omega, \mu) \right\} = 0.$$

4) The following holds:

$$(36) \quad \inf \left\{ \int_{\Omega} \sum_{\ell=1}^{N-1} |u_{\ell}(x) - S^{\ell}x|^2 d\mu : S \in \mathcal{P}_N(\Omega, \mu) \right\} = \sum_{\ell=1}^{N-1} \int_{\Omega} |u_{\ell}(x) - x|^2 d\mu.$$

5) There exists an N -sub-antisymmetric Hamiltonian H which is concave in the first variable, convex in the last $N-1$ variables, and vanishing on the diagonal such that

$$(37) \quad (u_1(x), \dots, u_{N-1}(x)) = \nabla_{2,\dots,N} H(x, x, \dots, x) \quad \text{for a.e. } x \in \Omega.$$

Moreover, H is N -symmetric in the sense of (16).

6) The following duality holds:

$$\begin{aligned} \inf \left\{ \int_{\Omega} L_H(x, u_1(x), \dots, u_{N-1}(x)) d\mu : H \in \mathcal{H}_N(\Omega) \right\} \\ = \sup \left\{ \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), S^{\ell}x \rangle d\mu : S \in \mathcal{P}_N(\Omega, \mu) \right\} \end{aligned}$$

and the latter is attained at the identity map.

We start with the following lemma, which identifies those probabilities in $\mathcal{P}_{\text{sym}}^{\mu}(\Omega^N)$ that are carried by graphs of functions from Ω to Ω^N .

Lemma 8. *Let $S : \Omega \rightarrow \Omega$ be a μ -measurable map. The following properties are equivalent:*

- 1) *The image of μ by the map $x \rightarrow (x, Sx, \dots, S^{N-1}x)$ belongs to $\mathcal{P}_{\text{sym}}^{\mu}(\Omega^N)$.*
- 2) *S is μ -measure-preserving and $S^N(x) = x$ μ -a.e.*
- 3) *For any bounded Borel measurable N -antisymmetric H on Ω^N , we have $\int_{\Omega} H(x, Sx, \dots, S^{N-1}x) d\mu = 0$.*

Proof. Clearly 1) implies 3), since $\int_{\Omega^N} H(x) d\pi(x) = 0$ for any N -antisymmetric Hamiltonian H and any $\pi \in \mathcal{P}_{\text{sym}}^{\mu}(\Omega^N)$.

That 2) implies 1) is also straightforward since if π is the push-forward of μ by a map of the form $x \rightarrow (x, Sx, \dots, S^{N-1}x)$, where S is a μ -measure-preserving S

with $S^N x = x$ μ -a.e. on Ω , then for all $h \in L^1(\Omega^N, d\pi)$, we have

$$\begin{aligned} \int_{\Omega^N} h(x_1, \dots, x_N) d\pi &= \int_{\Omega} h(x, Sx, \dots, S^{N-1}x) d\mu(x) \\ &= \int_{\Omega} h(Sx, S^2x, \dots, S^{N-1}x, S^Nx) d\mu(x) \\ &= \int_{\Omega} h(Sx, S^2x, \dots, S^{N-1}x, x) d\mu(x) \\ &= \int_{\Omega^N} h(\sigma(x_1, \dots, x_N)) d\pi. \end{aligned}$$

We now prove that 2) and 3) are equivalent. Assuming first that S is μ -measure-preserving such that $S^N = I$ μ -a.e., then for every Borel bounded N -antisymmetric H , we have

$$\begin{aligned} \int_{\Omega} H(x, Sx, S^2x, \dots, S^{N-1}x) d\mu &= \int_{\Omega} H(Sx, S^2x, \dots, S^{N-1}x, x) d\mu \\ &= \dots = \int_{\Omega} H(S^{N-1}x, x, Sx, \dots, S^{N-2}x) d\mu. \end{aligned}$$

Since H is N -antisymmetric, we can see that

$$\begin{aligned} H(x, Sx, \dots, S^{N-1}x) + H(Sx, S^2x, \dots, S^{N-1}x, x) \\ + \dots + H(S^{N-1}x, x, Sx, \dots, S^{N-2}x) = 0. \end{aligned}$$

It follows that $N \int_{\Omega} H(x, Sx, S^2x, \dots, S^{N-1}x) d\mu = 0$.

For the reverse implication, assume $\int_{\Omega} H(x, Sx, S^2x, \dots, S^{N-1}x) d\mu = 0$ for every N -antisymmetric Hamiltonian H . By testing this identity with the Hamiltonians

$$H(x_1, x_2, \dots, x_N) = f(x_1) - f(x_i),$$

where f is any continuous function on Ω , one gets that S is μ -measure-preserving. Now take the Hamiltonian

$$H(x_1, x_2, \dots, x_N) = |x_1 - Sx_N| - |Sx_1 - x_2| - |x_2 - Sx_1| + |Sx_2 - x_3|.$$

Note that $H \in \mathcal{H}_N(\Omega)$ since it is of the form

$$H(x_1, \dots, x_N) = f(x_1, x_2, x_N) - f(x_2, x_3, x_1).$$

Now test the above identity with such an H to obtain

$$0 = \int_{\Omega} H(x, Sx, S^2x, \dots, S^{N-1}x) d\mu = \int_{\Omega} |x - SS^{N-1}x| d\mu.$$

It follows that $S^N = I$ μ -a.e. on ω , and we are done. □

Proof of Theorem 7. To show that 1) implies 2), it suffices to notice that if π is a σ -invariant probability measure on Ω^N such that $\text{proj}_1\pi = \mu$, then

$$\begin{aligned} \int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_1 - x_{\ell+1} \rangle d\pi(x_1, \dots, x_N) \\ &= \frac{1}{N} \sum_{i=1}^N \int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_\ell(x_i), x_i - x_{i+\ell} \rangle d\pi(x_1, \dots, x_N) \\ &= \frac{1}{N} \int_{\Omega^N} \left(\sum_{i=1}^N \sum_{\ell=1}^{N-1} \langle u_\ell(x_i), x_i - x_{i+\ell} \rangle \right) d\pi(x_1, \dots, x_N) \\ &\geq 0, \end{aligned}$$

since (u_1, \dots, u_{N-1}) is jointly N -monotone. On the other hand, if π is the σ -invariant measure obtained by taking the image of $\mu := dx$ by $x \rightarrow (x, \dots, x)$, then

$$\int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_1 - x_{\ell+1} \rangle d\pi(x_1, \dots, x_N) = 0.$$

To show that 2) implies 3), let S be a μ -measure-preserving transformation on Ω such that $S^N = I$ μ -a.e. on Ω . Then the image π_S of μ by the map

$$x \rightarrow (x, Sx, S^2x, \dots, S^{N-1}x)$$

is σ -invariant, hence

$$\int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_1 - x_{\ell+1} \rangle d\pi_S(x_1, \dots, x_N) = \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_\ell(x), x - S^\ell x \rangle d\mu \geq 0.$$

By taking $S = I$, we get that the infimum is necessarily zero.

The equivalence of 3) and 4) follows immediately from developing the square.

We now show that 3) implies 1). Take N points x_1, x_2, \dots, x_N in Ω , and let $R > 0$ be such that $B(x_i, R) \subset \Omega$. Consider the transformation

$$S_R(x) = \begin{cases} x - x_1 + x_2 & \text{for } x \in B(x_1, R), \\ x - x_2 + x_3 & \text{for } x \in B(x_2, R), \\ \vdots & \\ x - x_N + x_1 & \text{for } x \in B(x_N, R), \\ x & \text{otherwise.} \end{cases}$$

It is easy to see that S_R is a measure-preserving transformation and that $S_R^N = \text{Id}$.

We then have

$$0 \leq \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), x - S_R^{\ell} x \rangle d\mu \leq \sum_{i=1}^N \int_{B(x_i, R)} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), x_i - x_{\ell+i} \rangle d\mu.$$

Letting $R \rightarrow 0$, we get from Lebesgue’s density theorem that

$$\frac{1}{|B(x_i, R)|} \int_{B(x_i, R)} \langle u_{\ell}(x), x_i - x_{\ell+i} \rangle d\mu \rightarrow \langle u_{\ell}(x_i), x_i - x_{\ell+i} \rangle,$$

from which follows that (u_1, \dots, u_{N-1}) are jointly N -monotone a.e. on Ω . The fact that 1) is equivalent to 5) follows immediately from Theorem 2.

To prove that 5) implies 6), note that for all $p_i \in \mathbb{R}^d$, $x \in \Omega$, $y_i \in \Omega$, $i = 1, \dots, N - 1$,

$$L_H(x, p_1, \dots, p_{N-1}) + H(x, y_1, \dots, y_{N-1}) \geq \sum_{i=1}^{N-1} \langle p_i, y_i \rangle,$$

which yields that for any $S \in \mathcal{S}_N(\Omega, \mu)$,

$$\begin{aligned} \int_{\Omega} [L_H(x, u_1(x), \dots, u_{N-1}(x)) d\mu + H(x, Sx, \dots, S^{N-1}x)] d\mu \\ \geq \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), S^{\ell}x \rangle d\mu. \end{aligned}$$

If $H \in \mathcal{H}_N(\Omega)$ and $S \in \mathcal{S}_N(\Omega, \mu)$, we then have $\int_{\Omega} H(x, Sx, \dots, S^{N-1}x) d\mu = 0$, and therefore

$$\int_{\Omega} L_H(x, u_1(x), \dots, u_{N-1}(x)) d\mu \geq \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), S^{\ell}x \rangle d\mu.$$

If now H is the N -sub-antisymmetric Hamiltonian obtained by 5), which is concave in the first variable and convex in the last $N - 1$ variables, then

$$L_H(x, u_1(x), \dots, u_{N-1}(x)) + H(x, x, \dots, x) = \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), x \rangle \quad \text{for all } x \in \Omega \setminus \Omega_0,$$

and therefore $\int_{\Omega} L_H(x, u_1(x), \dots, u_{N-1}(x)) d\mu = \sum_{\ell=1}^{N-1} \int_{\Omega} \langle u_{\ell}(x), x \rangle d\mu$.

Now consider

$$\bar{H}(x) = \frac{1}{N} \left((N - 1)H(x) - \sum_{i=1}^{N-1} H(\sigma^i(x)) \right).$$

As before, we have $\bar{H} \in \mathcal{H}_N(\Omega)$ and $\bar{H} \geq H$. Since $L_{\bar{H}} \leq L_H$, we have

$$\int_{\Omega} L_{\bar{H}}(x, u_1(x), \dots, u_{N-1}(x)) \, d\mu = \sum_{\ell=1}^{N-1} \int_{\Omega} \langle u_{\ell}(x), x \rangle \, d\mu$$

and 6) is proved.

Finally, note that 6) readily implies 3), which means that (u_1, \dots, u_{N-1}) is then jointly N -monotone. \square

We now consider again the case of a single N -cyclically monotone vector field.

Corollary 9. *Let $u : \Omega \rightarrow \mathbb{R}^d$ be a bounded measurable vector field. The following properties are then equivalent:*

- 1) *The vector field u is N -cyclically monotone a.e., that is, there exists a measure-zero set Ω_0 such that u is N -cyclically monotone on $\Omega \setminus \Omega_0$.*
- 2) *The infimum of the Monge–Kantorovich problem*

$$(38) \quad \inf \left\{ \int_{\Omega^N} \langle u(x_1), x_1 - x_2 \rangle \, d\pi(x) : \pi \in \mathcal{P}_{\text{sym}}^{\mu}(\Omega^N) \right\}$$

is equal to zero, and is therefore attained by the push-forward of μ by the map $x \rightarrow (x, x, \dots, x)$.

- 3) *The vector field u is in the polar of $\mathcal{S}_N(\Omega, \mu)$, that is,*

$$(39) \quad \inf \left\{ \int_{\Omega} \langle u(x), x - Sx \rangle \, d\mu : S \in \mathcal{S}_N(\Omega, \mu) \right\} = 0.$$

- 4) *The projection of u on $\mathcal{S}_N(\Omega, \mu)$ is the identity map, that is,*

$$(40) \quad \inf \left\{ \int_{\Omega} |u(x) - Sx|^2 \, d\mu : S \in \mathcal{S}_N(\Omega, \mu) \right\} = \int_{\Omega} |u(x) - x|^2 \, d\mu.$$

- 5) *There exists an N -cyclically sub-antisymmetric function H of two variables, which is concave in the first variable, convex in the second variable, vanishing on the diagonal and such that*

$$(41) \quad u(x) = \nabla_2 H(x, x) \quad \text{for a.e. } x \in \Omega.$$

- 6) *The following duality holds:*

$$\begin{aligned} \inf \left\{ \int_{\Omega} L_H(x, u(x), 0, \dots, 0) \, d\mu : H \in \mathcal{H}_N(\Omega) \right\} \\ = \sup \left\{ \int_{\Omega} \langle u(x), Sx \rangle \, d\mu : S \in \mathcal{S}_N(\Omega, \mu) \right\} \end{aligned}$$

and the latter is attained at the identity map.

Proof. This is an immediate application of Theorem 7 applied to the $(N-1)$ -tuple vector fields $(u, 0, \dots, 0)$, which is clearly jointly N -monotone on $\Omega \setminus \Omega_0$, whenever u is N -monotone on $\Omega \setminus \Omega_0$. \square

Remark 10. The sets of μ -measure-preserving N -involutions $(\mathcal{S}_N(\Omega, \mu))_N$ do not form a nested family, that is, $\mathcal{S}_N(\Omega, \mu)$ is not necessarily included in $\mathcal{S}_M(\Omega, \mu)$, whenever $N \leq M$, unless of course M is a multiple of N . On the other hand, the above theorem shows that their polar sets, i.e.,

$$\mathcal{S}_N(\Omega, \mu)^0 = \left\{ u \in L^2(\Omega, \mathbb{R}^d) : \int_{\Omega} \langle u(x), x - Sx \rangle d\mu \geq 0 \text{ for all } S \in \mathcal{S}_N(\Omega, \mu) \right\},$$

which coincide with the N -cyclically monotone maps, satisfy

$$\mathcal{S}_{N+1}(\Omega, \mu)^0 \subset \mathcal{S}_N(\Omega, \mu)^0,$$

for every $N \geq 1$. This can also be seen directly. Indeed, it is clear that a 2-involution is a 4-involution but not necessarily a 3-involution. On the other hand, assume that u is a 3-cyclically monotone operator. Then for any transformation $S : \Omega \rightarrow \Omega$, we have

$$\int_{\Omega} \langle u(x), x - Sx \rangle d\mu + \int_{\Omega} \langle u(Sx), Sx - S^2x \rangle d\mu + \int_{\Omega} \langle u(S^2x), S^2x - x \rangle d\mu \geq 0.$$

Now if S is measure-preserving, we have

$$\int_{\Omega} \langle u(x), x - Sx \rangle d\mu + \int_{\Omega} \langle u(x), x - Sx \rangle d\mu + \int_{\Omega} \langle u(S^2x), S^2x - x \rangle d\mu \geq 0,$$

and if $S^2 = I$, then $\int_{\Omega} \langle u(x), x - Sx \rangle d\mu \geq 0$, which means that $u \in \mathcal{S}_2(\Omega, \mu)^0$. Similarly, one can show that any $(N+1)$ -cyclically monotone operator belongs to $\mathcal{S}_N(\Omega, \mu)^0$. In other words, $\mathcal{S}_{N+1}(\Omega, \mu)^0 \subset \mathcal{S}_N(\Omega, \mu)^0$ for all $N \geq 2$. Note that $\mathcal{S}_1(\Omega, \mu)^0 = \{I\}^0 = L^2(\Omega, \mathbb{R}^d)$, while

$$\begin{aligned} \mathcal{S}(\Omega, \mu)^0 &= \bigcap_N \mathcal{S}_N(\Omega, \mu)^0 \\ &= \{u \in L^2(\Omega, \mathbb{R}^d), u = \nabla \phi \text{ for some convex function } \phi \text{ in } W^{1,2}(\mathbb{R}^d)\}, \end{aligned}$$

in view of classical results of Rockafellar [1970] and Brenier [1991].

Remark 11. In [Ghoussoub and Moameni 2013b], the preceding result is extended to give a similar decomposition for any family of bounded measurable vector fields u_1, u_2, \dots, u_{N-1} on Ω . It is shown there that there exists a measure-preserving N -involution S on Ω and an N -antisymmetric Hamiltonian H on Ω^N such that for $i = 1, \dots, N-1$, we have

$$u_i(x) = \nabla_{i+1} H(x, Sx, S^2x, \dots, S^{N-1}x) \quad \text{for a.e. } x \in \Omega.$$

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
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