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► **To cite this version:**

Alfred Galichon, Nassif Ghoussoub. Variational representations for N-cyclically monotone vector fields. Pacific Journal of Mathematics, Mathematical Sciences Publishers, 2014, 269 (2), pp.323 - 340. 10.2140/pjm.2014.269.323 . hal-03460569

HAL Id: hal-03460569

<https://hal-sciencespo.archives-ouvertes.fr/hal-03460569>

Submitted on 1 Dec 2021

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*Pacific
Journal of
Mathematics*

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Volume 269 No. 2

June 2014

VARIATIONAL REPRESENTATIONS FOR N -CYCLICALLY MONOTONE VECTOR FIELDS

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Given a convex bounded domain Ω in \mathbb{R}^d and an integer $N \geq 2$, we associate to any *jointly N -monotone* $(N-1)$ -tuple $(u_1, u_2, \dots, u_{N-1})$ of vector fields from Ω into \mathbb{R}^d a Hamiltonian H on $\mathbb{R}^d \times \mathbb{R}^d \times \dots \times \mathbb{R}^d$ that is concave in the first variable, jointly convex in the last $N-1$ variables, and such that

$$(u_1(x), u_2(x), \dots, u_{N-1}(x)) = \nabla_{2,\dots,N} H(x, x, \dots, x)$$

for almost all $x \in \Omega$. Moreover, H is N -antisymmetric in a sense made precise later, and also N -sub-antisymmetric in the sense that for all $X \in \Omega^N$ the sum $\sum_{i=0}^{N-1} H(\sigma^i(X)) \leq 0$ is nonpositive, σ being the permutation that shifts the coordinates of X leftward one slot and places the first coordinate last. This result can be seen as an extension of a theorem of E. Krauss, which associates to any monotone operator a concave-convex antisymmetric saddle function. We also give various variational characterizations of vector fields that are almost everywhere N -monotone, showing that they are dual to the class of measure-preserving N -involutions on Ω .

1. Introduction

Given a domain Ω in \mathbb{R}^d , recall that a single-valued map u from Ω to \mathbb{R}^d is said to be *N -cyclically monotone* if for every cycle $x_1, \dots, x_N, x_{N+1} = x_1$ of points in Ω , one has

$$(1) \quad \sum_{i=1}^N \langle u(x_i), x_i - x_{i+1} \rangle \geq 0.$$

A classical theorem of Rockafellar [Phelps 1993] states that a map u from Ω to \mathbb{R}^d

Galichon's research has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007–2013)/ ERC grant agreement n° 313699. Support from FiME, Laboratoire de Finance des Marchés de l'Énergie (<http://www.fime-lab.org>) is gratefully acknowledged. Ghoussoub is partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

MSC2010: 49J40.

Keywords: N -cyclically monotone vector fields, Krauss theorem.

is N -cyclically monotone for every $N \geq 2$ if and only if

$$(2) \quad u(x) \in \partial\phi(x) \quad \text{for all } x \in \Omega,$$

where $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a convex function. On the other hand, a result of E. Krauss [1985] yields that u is a monotone map, i.e., a 2-cyclically monotone map, if and only if

$$(3) \quad u(x) \in \partial_2 H(x, x) \quad \text{for all } x \in \Omega,$$

where H is a concave-convex antisymmetric Hamiltonian on $\mathbb{R}^d \times \mathbb{R}^d$, and $\partial_2 H$ is the subdifferential of H as a convex function in the second variable.

In this paper, we extend the result of Krauss to the class of N -cyclically monotone vector fields, where $N \geq 3$. We shall give a representation for a family of $N-1$ vector fields, which may or may not be individually N -cyclically monotone. Here is the needed concept.

Definition 1. Let u_1, \dots, u_{N-1} be bounded vector fields from a domain $\Omega \subset \mathbb{R}^d$ into \mathbb{R}^d . We shall say that the $(N-1)$ -tuple $(u_1, u_2, \dots, u_{N-1})$ is *jointly N -monotone* if for every cycle x_1, \dots, x_{2N-1} of points in Ω such that $x_{N+i} = x_i$ for $1 \leq i \leq N-1$, one has

$$(4) \quad \sum_{i=1}^N \sum_{\ell=1}^{N-1} \langle u_\ell(x_i), x_i - x_{\ell+i} \rangle \geq 0.$$

Examples of jointly N -monotone families of vector fields:

- It is clear that $(u, 0, 0, \dots, 0)$ is jointly N -monotone if and only if u is N -monotone.
- More generally, if each u_ℓ is N -monotone, then the family $(u_1, u_2, \dots, u_{N-1})$ is jointly N -monotone. Actually, one only needs that for $1 \leq \ell \leq N-1$, the vector field u_ℓ be (N, ℓ) -monotone in the following sense: for every cycle $x_1, \dots, x_{N+\ell}$ of points in Ω such that $x_{N+i} = x_i$ for $1 \leq i \leq \ell$, we have

$$(5) \quad \sum_{i=1}^N \langle u_\ell(x_i), x_i - x_{\ell+i} \rangle \geq 0.$$

This notion is sometimes weaker than N -monotonicity since if ℓ divides N , then it suffices for u to be N/ℓ -monotone in order to be an (N, ℓ) -monotone vector field. For example, if u_1 and u_3 are 4-monotone operators and u_2 is 2-monotone, then the triplet (u_1, u_2, u_3) is jointly 4-monotone.

- Another example is if (u_1, u_2, u_3) are vector fields such that u_2 is 2-monotone and

$$\langle u_1(x) - u_3(y), x - y \rangle \geq 0 \quad \text{for every } x, y \in \mathbb{R}^d.$$

In this case, the triplet (u_1, u_2, u_3) is jointly 4-monotone. In particular, if u_1 and u_2 are both 2-monotone, then the triplet (u_1, u_2, u_1) is jointly 4-monotone.

- More generally, it is easy to show that (u, u, \dots, u) is jointly N -monotone if and only if u is 2-cyclically monotone.

We shall always denote by σ the cyclic permutation on $\mathbb{R}^d \times \dots \times \mathbb{R}^d$ defined by

$$\sigma(x_1, x_2, \dots, x_{N-1}, x_N) = (x_2, x_3, \dots, x_N, x_1).$$

We let

$$(6) \quad \mathcal{H}_N(\Omega) = \left\{ H \in C(\Omega^N) : \sum_{i=0}^{N-1} H(\sigma^i(x_1, \dots, x_N)) = 0 \right\}$$

be the family of continuous Hamiltonians on Ω^N that are N -antisymmetric, i.e., satisfy the condition to the right of the colon in (6). We say that H is N -sub-antisymmetric on Ω if

$$(7) \quad \sum_{i=0}^{N-1} H(\sigma^i(x_1, \dots, x_N)) \leq 0 \quad \text{on } \Omega^N.$$

We shall also say that a function F of two variables is N -cyclically sub-antisymmetric on Ω if

$$(8) \quad \begin{aligned} &F(x, x) = 0 \quad \text{and} \\ &\sum_{i=1}^N F(x_i, x_{i+1}) \leq 0 \quad \text{for all cyclic families } x_1, \dots, x_N, x_{N+1} = x_1 \text{ in } \Omega. \end{aligned}$$

Note that if a function $H(x_1, \dots, x_N)$ is N -sub-antisymmetric and if it only depends on the first two variables, then the function $F(x_1, x_2) := H(x_1, x_2, \dots, x_N)$ is N -cyclically sub-antisymmetric.

We associate to any function H on Ω^N the functional given by on $\Omega \times (\mathbb{R}^d)^{N-1}$

$$(9) \quad L_H(x, p_1, \dots, p_{N-1}) = \sup \left\{ \sum_{i=1}^{N-1} \langle p_i, y_i \rangle - H(x, y_1, \dots, y_{N-1}) : y_i \in \Omega \right\}.$$

Note that if Ω is convex and if H is convex in the last $N-1$ variables, then L_H is nothing but the Legendre transform of \tilde{H} with respect to the last $N-1$ variables, where \tilde{H} is the extension of H over $(\mathbb{R}^d)^N$, defined by $\tilde{H} = H$ on Ω^N and $\tilde{H} = +\infty$ outside Ω^N . Since $H(x, \dots, x) = 0$ for any $H \in \mathcal{H}_N(\Omega)$, we have, for any such H ,

$$(10) \quad L_H(x, p_1, \dots, p_{N-1}) \geq \sum_{i=1}^{N-1} \langle x, p_i \rangle,$$

for $x \in \Omega$ and $p_1, \dots, p_{N-1} \in \mathbb{R}^d$. To formulate variational principles for such

vector fields, we shall consider the class of σ -invariant probability measures on Ω^N , which are those $\pi \in \mathcal{P}(\Omega^N)$ such that for all $h \in L^1(\Omega^N, d\pi)$, we have

$$(11) \quad \int_{\Omega^N} h(x_1, \dots, x_N) d\pi = \int_{\Omega^N} h(\sigma(x_1, \dots, x_N)) d\pi.$$

We set

$$(12) \quad \mathcal{P}_{\text{sym}}(\Omega^N) = \{\pi \in \mathcal{P}(\Omega^N) : \pi \text{ } \sigma\text{-invariant probability on } \Omega^N\}.$$

For a given probability measure μ on Ω , we also consider the class

$$(13) \quad \mathcal{P}_{\text{sym}}^\mu(\Omega^N) = \{\pi \in \mathcal{P}_{\text{sym}}(\Omega^N) : \text{proj}_1 \pi = \mu\},$$

i.e., the set of all $\pi \in \mathcal{P}_{\text{sym}}(\Omega^N)$ with a given first marginal μ , meaning that

$$(14) \quad \int_{\Omega^N} f(x_1) d\pi(x_1, \dots, x_N) = \int_{\Omega} f(x_1) d\mu(x_1) \quad \text{for every } f \in L^1(\Omega, \mu).$$

Now consider the set $\mathcal{S}(\Omega, \mu)$ of μ -measure-preserving transformations on Ω , which can be identified with a closed subset of the sphere of $L^2(\Omega, \mathbb{R}^d)$. We shall also consider the subset of $\mathcal{S}(\Omega, \mu)$ consisting of N -involutions, that is,

$$\mathcal{S}_N(\Omega, \mu) = \{S \in \mathcal{S}(\Omega, \mu) : S^N = I \text{ } \mu\text{-a.e.}\}.$$

2. Monotone vector fields and N -antisymmetric Hamiltonians

In this section, we establish the following extension of a theorem of Krauss.

Theorem 2. *Let $N \geq 2$ be an integer, and let u_1, \dots, u_{N-1} be bounded vector fields from a convex domain $\Omega \subset \mathbb{R}^d$ into \mathbb{R}^d .*

- 1) *If the $(N-1)$ -tuple (u_1, \dots, u_{N-1}) is jointly N -monotone, then there exists an N -sub-antisymmetric Hamiltonian H that is zero on the diagonal of Ω^N , concave in the first variable, convex in the other $N-1$ variables, and such that*

$$(15) \quad (u_1(x), \dots, u_{N-1}(x)) = \nabla_{2, \dots, N} H(x, x, \dots, x) \quad \text{for a.e. } x \in \Omega.$$

Moreover, H is N -antisymmetric in the sense that

$$(16) \quad H(x_1, x_2, \dots, x_N) + H_{2, \dots, N}(x_1, x_2, \dots, x_N) = 0,$$

where $H_{2, \dots, N}$ is the concavification of the function $K(x) = \sum_{i=1}^{N-1} H(\sigma^i(x))$ with respect to the last $N-1$ variables.

Furthermore, there exists a continuous N -antisymmetric Hamiltonian \bar{H} on Ω^N , such that

$$(17) \quad L_{\bar{H}}(x, u_1(x), u_2(x), \dots, u_{N-1}(x)) = \sum_{i=1}^{N-1} \langle u_i(x), x \rangle \quad \text{for all } x \in \Omega.$$

2) Conversely, if (u_1, \dots, u_{N-1}) satisfies (15) for some N -sub-antisymmetric Hamiltonian H that is zero on the diagonal of Ω^N , concave in the first variable, and convex in the other variables, then the $(N-1)$ -tuple (u_1, \dots, u_{N-1}) is jointly N -monotone on Ω .

Remark 3. In the case $N = 2$, $K(x) = H(x_2, x_1)$ is concave with respect to x_2 , hence $H_2(x_1, x_2) = H(x_2, x_1)$, and (16) becomes

$$H(x_1, x_2) + H(x_2, x_1) = 0;$$

thus H is antisymmetric, recovering well-known results [Krauss 1985; Ghossoub 2009; Ghossoub and Moameni 2013a; Millien 2011].

Lemma 4. Assume the $(N-1)$ -tuple of bounded vector fields (u_1, \dots, u_{N-1}) on Ω is jointly N -monotone. Define

$$f(x_1, \dots, x_N) := \sum_{l=1}^{N-1} \langle u_l(x_1), x_1 - x_{l+1} \rangle$$

and let \tilde{f} be the convexification of f with respect to the first variable, given by

$$(18) \quad \tilde{f}(x_1, x_2, \dots, x_N) = \inf \left\{ \sum_{k=1}^n \lambda_k f(x_1^k, x_2, \dots, x_N) : n \in \mathbb{N}, \lambda_k \geq 0, \sum_{k=1}^n \lambda_k = 1, \sum_{k=1}^n \lambda_k x_1^k = x_1 \right\}.$$

- 1) We have $f \geq \tilde{f}$ on Ω^N .
- 2) \tilde{f} is convex in the first variable and concave with respect to the other variables.
- 3) $\tilde{f}(x, x, \dots, x) = 0$ for each $x \in \Omega$.
- 4) \tilde{f} satisfies

$$(19) \quad \sum_{i=0}^{N-1} \tilde{f}(\sigma^i(x_1, \dots, x_N)) \geq 0 \quad \text{on } \Omega^N.$$

Proof. Since the $(N-1)$ -tuple (u_1, \dots, u_{N-1}) is jointly N -monotone, it is easy to see that the function

$$f(x_1, \dots, x_N) := \sum_{l=1}^{N-1} \langle u_l(x_1), x_1 - x_{l+1} \rangle$$

is linear in the last $N-1$ variables, that $f(x, x, \dots, x) = 0$, and that

$$(20) \quad \sum_{i=0}^{N-1} f(\sigma^i(x_1, \dots, x_N)) \geq 0 \quad \text{on } \Omega^N.$$

It is also clear that $f \geq \tilde{f}$, that \tilde{f} is convex with respect to the first variable x_1 ,

and that it is concave with respect to the other variables x_2, \dots, x_N , since f itself is concave (actually linear) with respect to x_2, \dots, x_N . We now show that \tilde{f} satisfies (19).

For that, we fix x_1, x_2, \dots, x_N in Ω and consider $(x_1^k)_{k=1}^n$ in Ω , and $(\lambda_k)_k$ in \mathbb{R} such that $\lambda_k \geq 0$ such that $\sum_{k=1}^n \lambda_k = 1$ and $\sum_{k=1}^n \lambda_k x_1^k = x_1$. For each k , we have

$$f(x_1^k, x_2, \dots, x_N) + f(x_2, \dots, x_N, x_1^k) + \dots + f(x_N, x_1^k, x_2, \dots, x_{N-1}) \geq 0.$$

Multiplying by λ_k , summing over k , and using that f is linear in the last $N-1$ variables, we have

$$\sum_{k=1}^n \lambda_k f(x_1^k, x_2, \dots, x_N) + f(x_2, \dots, x_N, x_1) + \dots + f(x_N, x_1, x_2, \dots, x_{N-1}) \geq 0.$$

By taking the infimum, we obtain

$$\tilde{f}(x_1, x_2, \dots, x_N) + \sum_{i=1}^{N-1} f(\sigma^i(x_1, x_2, \dots, x_N)) \geq 0.$$

Let now $n \in \mathbb{N}$, $\lambda_k \geq 0$, $x_N^k \in \Omega$ be such that $\sum_{k=1}^n \lambda_k = 1$ and $\sum_{k=1}^n \lambda_k x_2^k = x_2$. For every $1 \leq k \leq n$, we have

$$\tilde{f}(x_1, x_2^k, x_3, \dots, x_N) + f(x_2^k, x_3, \dots, x_1) + \dots + f(x_N, x_1, x_2^k, x_3, \dots, x_{N-1}) \geq 0.$$

Multiplying by λ_k , summing over k and using that \tilde{f} is convex in the first variable and f is linear in the last $N-1$ variables, we obtain

$$\begin{aligned} & \tilde{f}(x_1, x_2, x_3, \dots, x_N) + \sum_{k=1}^n \lambda_k f(x_2^k, x_3, \dots, x_1) + \dots + f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \\ & \geq \sum_{k=1}^n \lambda_k \tilde{f}(x_1, x_2^k, x_3, \dots, x_N) + \sum_{k=1}^n \lambda_k f(x_2^k, x_3, \dots, x_1) \\ & \quad + \dots + \sum_{k=1}^n \lambda_k f(x_N, x_1, x_2^k, x_3, \dots, x_{N-1}) \\ & \geq 0. \end{aligned}$$

By taking the infimum over all possible such choices, we get

$$\tilde{f}(x_1, x_2, x_3, \dots, x_N) + \tilde{f}(x_2, x_3, \dots, x_1) + \dots + f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \geq 0.$$

By repeating this procedure with x_3, \dots, x_{N-1} , we get

$$\sum_{i=0}^{N-2} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N)) + f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \geq 0.$$

Finally, since

$$f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \geq - \sum_{i=0}^{N-2} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N))$$

and since \tilde{f} is concave in the last $N-1$ variables, the function

$$x_N \rightarrow - \sum_{i=0}^{N-2} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N))$$

for fixed x_1, x_2, \dots, x_{N-1} is a convex minorant of $x_N \rightarrow f(x_N, x_1, x_2, \dots, x_{N-1})$. It follows that

$$\begin{aligned} f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) &\geq \tilde{f}(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \\ &\geq - \sum_{i=0}^{N-2} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N)), \end{aligned}$$

which yields $\sum_{i=0}^{N-1} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N)) \geq 0$. This implies that $\tilde{f}(x, x, \dots, x) \geq 0$ for $x \in \Omega$.

On the other hand, since $\tilde{f}(x, x, \dots, x) \leq f(x, x, \dots, x) = 0$, we get that $\tilde{f}(x, x, \dots, x) = 0$ for all $x \in \Omega$. □

Proof of Theorem 2. Assume the $(N-1)$ -tuple of vector fields (u_1, \dots, u_{N-1}) is jointly N -monotone on Ω , and consider the function

$$f(x_1, \dots, x_N) := \sum_{l=1}^{N-1} \langle u_l(x_1), x_1 - x_{l+1} \rangle$$

as well as its convexification with respect to the first variable $\tilde{f}(x_1, \dots, x_N)$.

By Lemma 4, the function $\psi(x_1, \dots, x_N) := -\tilde{f}(x_1, \dots, x_N)$ satisfies the following properties:

- (i) $x_1 \rightarrow \psi(x_1, \dots, x_N)$ is concave.
- (ii) $(x_2, x_3, \dots, x_N) \rightarrow \psi(x_1, \dots, x_N)$ is convex.
- (iii) $\psi(x_1, \dots, x_N) \geq -f(x_1, \dots, x_N) = \sum_{l=1}^{N-1} \langle u_l(x_1), x_{l+1} - x_1 \rangle$.
- (iv) ψ is N -sub-antisymmetric.

Now consider the family $\overline{\mathcal{H}}$ of functions $H : \Omega^N \rightarrow \mathbb{R}$ such that

- 1) $H(x_1, x_2, \dots, x_N) \geq \sum_{l=1}^{N-1} \langle u_l(x_1), x_{l+1} - x_1 \rangle$ for every N -tuple (x_1, \dots, x_N) in Ω^N ,
- 2) H is concave in the first variable,
- 3) H is jointly convex in the last $N-1$ variables,

- 4) H is N -sub-antisymmetric,
- 5) H is zero on the diagonal of Ω^N .

Note that $\overline{\mathcal{H}} \neq \emptyset$ since ψ belongs to $\overline{\mathcal{H}}$. Note that any H satisfying conditions 1 and 4 automatically satisfies 5. Indeed, by N -sub-antisymmetry, for all $\mathbf{x} = (x_1, \dots, x_N) \in \Omega^N$ we have

$$(21) \quad H(\mathbf{x}) \leq - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x})) \leq - \sum_{i=1}^{N-1} \psi(\sigma^i(\mathbf{x})).$$

This also yields that

$$(22) \quad \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_{\ell+1} - x_1 \rangle \leq H(\mathbf{x}) \leq - \sum_{i=2}^N \sum_{\ell=1}^{N-1} \langle u_\ell(x_i), x_i - x_{i+\ell} \rangle,$$

where we denote $x_{i+N} := x_i$ for $i = 1, \dots, \ell$. This yields that $H(x, x, \dots, x) = 0$ for any $x \in \Omega$.

It is also easy to see that every directed family $(H_i)_i$ in $\overline{\mathcal{H}}$ has a supremum $H_\infty \in \overline{\mathcal{H}}$, meaning that $\overline{\mathcal{H}}$ is a Zorn family, and therefore has a maximal element H .

Now consider the function

$$\overline{H}(\mathbf{x}) = \frac{1}{N} \left((N-1)H(\mathbf{x}) - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x})) \right).$$

- (i) \overline{H} is N -antisymmetric, since $\overline{H}(\mathbf{x}) = \frac{1}{N} \sum_{i=1}^{N-1} [H(\mathbf{x}) - H(\sigma^i(\mathbf{x}))]$, and each summand is N -antisymmetric.
- (ii) $\overline{H} \geq H$ on Ω^N , since $N[\overline{H}(\mathbf{x}) - H(\mathbf{x})] = - \sum_{i=0}^{N-1} H(\sigma^i(\mathbf{x})) \geq 0$ (because H itself is N -sub-antisymmetric).

The maximality of H would have implied that $H = \overline{H}$ is N -antisymmetric if only \overline{H} was jointly convex in the last $N-1$ variables, but since this is not necessarily the case, we consider for $\mathbf{x} = (x_1, x_2, \dots, x_N)$ the function

$$K(x_1, x_2, \dots, x_N) = K(\mathbf{x}) := - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x})),$$

which is already concave in the first variable x_1 . Its convexification in the last $N-1$ variables, that is,

$$K^{2, \dots, N}(\mathbf{x}) = \inf \left\{ \sum_{i=1}^n \lambda_i K(x_1, x_2^i, \dots, x_N^i) : \lambda_i \geq 0, \sum_{i=1}^n \lambda_i (x_2^i, \dots, x_N^i, 1) = (x_2, \dots, x_N, 1) \right\},$$

is still concave in the first variable, but is now convex in the last $N-1$ variables. Moreover,

$$(23) \quad H \leq K^{2,\dots,N} \leq K = - \sum_{i=1}^{N-1} H \circ \sigma^i.$$

Indeed, $K^{2,\dots,N} \leq K$ from the definition of $K^{2,\dots,N}$, while $H \leq K^{2,\dots,N}$ because $H \leq K$ and H is already convex in the last $N-1$ variables. It follows that

$$H \leq \frac{(N-1)H + K^{2,\dots,N}}{N} \leq \frac{(N-1)H + K}{N} = \frac{1}{N} \left((N-1)H - \sum_{i=1}^{N-1} H \circ \sigma^i \right) = \bar{H}.$$

The function $H' = ((N-1)H + K^{2,\dots,N})/N$ belongs to the family $\bar{\mathcal{H}}$ and therefore $H = H'$ by the maximality of H .

This finally yields that H is N -sub-antisymmetric, that $H(x, \dots, x) = 0$ for all $x \in \Omega$ and that

$$H(x) + H_{2,\dots,N}(x) = 0 \quad \text{for every } x \in \Omega^N,$$

where $H_{2,\dots,N} = -K^{2,\dots,N}$, which for a fixed x_1 is nothing but the concavification of $(x_2, \dots, x_N) \rightarrow \sum_{i=1}^{N-1} H(\sigma^i(x_1, x_2, \dots, x_N))$.

Note now that since for any x_1, \dots, x_N in Ω

$$(24) \quad H(x_1, x_2, \dots, x_N) \geq \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_{\ell+1} - x_1 \rangle,$$

and

$$(25) \quad H(x_1, x_1, \dots, x_1) = 0,$$

we have

$$(26) \quad H(x_1, x_2, \dots, x_N) - H(x_1, \dots, x_1) \geq \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_{\ell+1} - x_1 \rangle.$$

Since H is convex in the last $N-1$ variables, this means that for all $x \in \Omega$, we have

$$(27) \quad (u_1(x), u_2(x), \dots, u_{N-1}(x)) \in \partial_{2,\dots,N} H(x, x, \dots, x),$$

as claimed in (15). This also yields

$$L_H(x, u_1(x), \dots, u_{N-1}(x)) + H(x, x, \dots, x) = \sum_{\ell=1}^{N-1} \langle u_\ell(x), x \rangle \quad \text{for all } x \in \Omega.$$

In other words, $L_H(x, u_1(x), \dots, u_{N-1}(x)) = \sum_{\ell=1}^{N-1} \langle u_\ell(x), x \rangle$ for all $x \in \Omega$. As above, consider

$$\bar{H}(x) = \frac{1}{N} \left((N-1)H(x) - \sum_{i=1}^{N-1} H(\sigma^i(x)) \right).$$

We have $\bar{H} \in \bar{\mathcal{H}}_N(\Omega)$ and $\bar{H} \geq H$, and therefore $L_{\bar{H}} \leq L_H$. On the other hand, for all $x \in \Omega$ we have

$$\begin{aligned} L_{\bar{H}}(x, u_1(x), \dots, u_{N-1}(x)) &= L_{\bar{H}}(x, u_1(x), \dots, u_{N-1}(x)) + \bar{H}(x, x, \dots, x) \\ &\geq \sum_{\ell=1}^{N-1} \langle u_\ell(x), x \rangle. \end{aligned}$$

To prove (17), we use the appendix in [Ghoussoub and Moameni 2013b] to deduce that for $i = 2, \dots, N$, the gradients $\nabla_i H(x, x, \dots, x)$ actually exist for a.e. x in Ω .

The converse is straightforward since if (27) holds, then (26) does, and since we also have (25), then the property that (u_1, \dots, u_{N-1}) is jointly N -monotone follows from (24) and the sub-antisymmetry of H . \square

In the case of a single N -monotone vector field, we can obviously apply the above theorem to the $(N-1)$ -tuple $(u, 0, \dots, 0)$, which is then N -monotone, to find an N -sub-antisymmetric Hamiltonian H , which is concave in the first variable and convex in the last $N-1$ variables such that

$$(28) \quad (-u(x), u(x), 0, \dots, 0) = \nabla H(x, x, \dots, x) \quad \text{for a.e. } x \in \Omega.$$

However, in this case we can restrict ourselves to N -cyclically sub-antisymmetric functions of two variables and establish the following extension of the theorem of Krauss.

Theorem 5. *If u is N -cyclically monotone on Ω , then there exists a concave-convex function of two variables F that is N -cyclically sub-antisymmetric and zero on the diagonal, such that*

$$(29) \quad (-u(x), u(x)) \in \partial F(x, x) \quad \text{for all } x \in \Omega,$$

where ∂H is the subdifferential of H as a concave-convex function [Rockafellar 1970]. Moreover,

$$(30) \quad u(x) = \nabla_2 F(x, x) \quad \text{for a.e. } x \in \Omega.$$

Proof. Let $f(x, y) = \langle u(x), x - y \rangle$ and let $f^1(x, y)$ be its convexification in x for fixed y , that is,

$$(31) \quad f^1(x, y) = \inf \left\{ \sum_{k=1}^n \lambda_k f(x_k, y) : \lambda_k \geq 0, \sum_{k=1}^n \lambda_k = 1, \sum_{k=1}^n \lambda_k x_k = x \right\}.$$

Since $f(x, x) = 0$, f is linear in y , and $\sum_{i=1}^N f(x_i, x_{i+1}) \geq 0$ for any cyclic family

$x_1, \dots, x_N, x_{N+1} = x_1$ in Ω , it is easy to show that $f \geq f^1$ on Ω , f^1 is convex in the first variable and concave with respect to the second, $f^1(x, x) = 0$ for each $x \in \Omega$, and that f^1 is N -cyclically supersymmetric in the sense that for any cyclic family $x_1, \dots, x_N, x_{N+1} = x_1$ in Ω , we have $\sum_{i=1}^N f^1(x_i, x_{i+1}) \geq 0$.

Now consider $F(x, y) = -f^1(x, y)$ and note that $x \rightarrow F(x, y)$ is concave, $y \rightarrow F(x, y)$ is convex, $F(x, y) \geq -f(x, y) = \langle u(x), y - x \rangle$ and F is N -cyclically sub-antisymmetric. By the antisymmetry, we have

$$(32) \quad \langle u(x_1), x_2 - x_1 \rangle \leq F(x_1, x_2) \leq \langle u(x_2), x_2 - x_1 \rangle,$$

which yields that $(-u(x), u(x)) \in \partial F(x, x)$ for all $x \in \Omega$.

Since F is antisymmetric and concave-convex, the possibly multivalued map $x \rightarrow \partial_2 F(x, x)$ is monotone on Ω , and therefore single-valued and differentiable almost everywhere [Phelps 1993]. This completes the proof. \square

Remark 6. We cannot expect to have a function F such that $\sum_{i=1}^N F(x_i, x_{i+1}) = 0$ for all cyclic families $x_1, \dots, x_N, x_{N+1} = x_1$ in Ω . Actually, we believe that the only function satisfying such an N -antisymmetry for $N \geq 3$ must be of the form $F(x, y) = f(x) - f(y)$. This is why one needs to consider functions of N variables in order to get N -antisymmetry. In other words, the function defined by

$$(33) \quad H(x_1, x_2, \dots, x_N) := \frac{1}{N} \left((N - 1)F(x_1, x_2) - \sum_{i=2}^{N-1} F(x_i, x_{i+1}) \right)$$

is N -antisymmetric in the sense of (6) and $H(x_1, x_2, \dots, x_N) \geq F(x_1, x_2)$ for all (x_1, x_2, \dots, x_N) in Ω^N .

3. Variational characterization of monotone vector fields

In order to simplify the exposition, we shall always assume in the sequel that $d\mu$ is Lebesgue measure dx normalized to be a probability on Ω . We shall also assume that Ω is convex and that its boundary has measure zero.

Theorem 7. Let $u_1, \dots, u_{N-1} : \Omega \rightarrow \mathbb{R}^d$ be bounded measurable vector fields. The following properties are then equivalent:

- 1) The $(N-1)$ -tuple (u_1, \dots, u_{N-1}) is jointly N -monotone a.e., that is, there exists a measure-zero set Ω_0 such that (u_1, \dots, u_{N-1}) is jointly N -monotone on $\Omega \setminus \Omega_0$.
- 2) The infimum of the Monge–Kantorovich problem

$$(34) \quad \inf \left\{ \int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_1 - x_{\ell+1} \rangle d\pi(x_1, x_2, \dots, x_N) : \pi \in \mathcal{P}_{\text{sym}}^\mu(\Omega^N) \right\}$$

is equal to zero, and is therefore attained by the push-forward of μ by the map $x \rightarrow (x, x, \dots, x)$.

3) (u_1, \dots, u_{N-1}) is in the polar of $\mathcal{P}_N(\Omega, \mu)$ in the following sense:

$$(35) \quad \inf \left\{ \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), x - S^{\ell}x \rangle d\mu : S \in \mathcal{P}_N(\Omega, \mu) \right\} = 0.$$

4) The following holds:

$$(36) \quad \inf \left\{ \int_{\Omega} \sum_{\ell=1}^{N-1} |u_{\ell}(x) - S^{\ell}x|^2 d\mu : S \in \mathcal{P}_N(\Omega, \mu) \right\} = \sum_{\ell=1}^{N-1} \int_{\Omega} |u_{\ell}(x) - x|^2 d\mu.$$

5) There exists an N -sub-antisymmetric Hamiltonian H which is concave in the first variable, convex in the last $N-1$ variables, and vanishing on the diagonal such that

$$(37) \quad (u_1(x), \dots, u_{N-1}(x)) = \nabla_{2,\dots,N} H(x, x, \dots, x) \quad \text{for a.e. } x \in \Omega.$$

Moreover, H is N -symmetric in the sense of (16).

6) The following duality holds:

$$\begin{aligned} \inf \left\{ \int_{\Omega} L_H(x, u_1(x), \dots, u_{N-1}(x)) d\mu : H \in \mathcal{H}_N(\Omega) \right\} \\ = \sup \left\{ \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), S^{\ell}x \rangle d\mu : S \in \mathcal{P}_N(\Omega, \mu) \right\} \end{aligned}$$

and the latter is attained at the identity map.

We start with the following lemma, which identifies those probabilities in $\mathcal{P}_{\text{sym}}^{\mu}(\Omega^N)$ that are carried by graphs of functions from Ω to Ω^N .

Lemma 8. *Let $S : \Omega \rightarrow \Omega$ be a μ -measurable map. The following properties are equivalent:*

- 1) *The image of μ by the map $x \rightarrow (x, Sx, \dots, S^{N-1}x)$ belongs to $\mathcal{P}_{\text{sym}}^{\mu}(\Omega^N)$.*
- 2) *S is μ -measure-preserving and $S^N(x) = x$ μ -a.e.*
- 3) *For any bounded Borel measurable N -antisymmetric H on Ω^N , we have $\int_{\Omega} H(x, Sx, \dots, S^{N-1}x) d\mu = 0$.*

Proof. Clearly 1) implies 3), since $\int_{\Omega^N} H(x) d\pi(x) = 0$ for any N -antisymmetric Hamiltonian H and any $\pi \in \mathcal{P}_{\text{sym}}^{\mu}(\Omega^N)$.

That 2) implies 1) is also straightforward since if π is the push-forward of μ by a map of the form $x \rightarrow (x, Sx, \dots, S^{N-1}x)$, where S is a μ -measure-preserving S

with $S^N x = x$ μ -a.e. on Ω , then for all $h \in L^1(\Omega^N, d\pi)$, we have

$$\begin{aligned} \int_{\Omega^N} h(x_1, \dots, x_N) d\pi &= \int_{\Omega} h(x, Sx, \dots, S^{N-1}x) d\mu(x) \\ &= \int_{\Omega} h(Sx, S^2x, \dots, S^{N-1}x, S^Nx) d\mu(x) \\ &= \int_{\Omega} h(Sx, S^2x, \dots, S^{N-1}x, x) d\mu(x) \\ &= \int_{\Omega^N} h(\sigma(x_1, \dots, x_N)) d\pi. \end{aligned}$$

We now prove that 2) and 3) are equivalent. Assuming first that S is μ -measure-preserving such that $S^N = I$ μ -a.e., then for every Borel bounded N -antisymmetric H , we have

$$\begin{aligned} \int_{\Omega} H(x, Sx, S^2x, \dots, S^{N-1}x) d\mu &= \int_{\Omega} H(Sx, S^2x, \dots, S^{N-1}x, x) d\mu \\ &= \dots = \int_{\Omega} H(S^{N-1}x, x, Sx, \dots, S^{N-2}x) d\mu. \end{aligned}$$

Since H is N -antisymmetric, we can see that

$$\begin{aligned} H(x, Sx, \dots, S^{N-1}x) + H(Sx, S^2x, \dots, S^{N-1}x, x) \\ + \dots + H(S^{N-1}x, x, Sx, \dots, S^{N-2}x) = 0. \end{aligned}$$

It follows that $N \int_{\Omega} H(x, Sx, S^2x, \dots, S^{N-1}x) d\mu = 0$.

For the reverse implication, assume $\int_{\Omega} H(x, Sx, S^2x, \dots, S^{N-1}x) d\mu = 0$ for every N -antisymmetric Hamiltonian H . By testing this identity with the Hamiltonians

$$H(x_1, x_2, \dots, x_N) = f(x_1) - f(x_i),$$

where f is any continuous function on Ω , one gets that S is μ -measure-preserving. Now take the Hamiltonian

$$H(x_1, x_2, \dots, x_N) = |x_1 - Sx_N| - |Sx_1 - x_2| - |x_2 - Sx_1| + |Sx_2 - x_3|.$$

Note that $H \in \mathcal{H}_N(\Omega)$ since it is of the form

$$H(x_1, \dots, x_N) = f(x_1, x_2, x_N) - f(x_2, x_3, x_1).$$

Now test the above identity with such an H to obtain

$$0 = \int_{\Omega} H(x, Sx, S^2x, \dots, S^{N-1}x) d\mu = \int_{\Omega} |x - SS^{N-1}x| d\mu.$$

It follows that $S^N = I$ μ -a.e. on ω , and we are done. □

Proof of Theorem 7. To show that 1) implies 2), it suffices to notice that if π is a σ -invariant probability measure on Ω^N such that $\text{proj}_1\pi = \mu$, then

$$\begin{aligned} \int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_1 - x_{\ell+1} \rangle d\pi(x_1, \dots, x_N) \\ &= \frac{1}{N} \sum_{i=1}^N \int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_\ell(x_i), x_i - x_{i+\ell} \rangle d\pi(x_1, \dots, x_N) \\ &= \frac{1}{N} \int_{\Omega^N} \left(\sum_{i=1}^N \sum_{\ell=1}^{N-1} \langle u_\ell(x_i), x_i - x_{i+\ell} \rangle \right) d\pi(x_1, \dots, x_N) \\ &\geq 0, \end{aligned}$$

since (u_1, \dots, u_{N-1}) is jointly N -monotone. On the other hand, if π is the σ -invariant measure obtained by taking the image of $\mu := dx$ by $x \rightarrow (x, \dots, x)$, then

$$\int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_1 - x_{\ell+1} \rangle d\pi(x_1, \dots, x_N) = 0.$$

To show that 2) implies 3), let S be a μ -measure-preserving transformation on Ω such that $S^N = I$ μ -a.e. on Ω . Then the image π_S of μ by the map

$$x \rightarrow (x, Sx, S^2x, \dots, S^{N-1}x)$$

is σ -invariant, hence

$$\int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_1 - x_{\ell+1} \rangle d\pi_S(x_1, \dots, x_N) = \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_\ell(x), x - S^\ell x \rangle d\mu \geq 0.$$

By taking $S = I$, we get that the infimum is necessarily zero.

The equivalence of 3) and 4) follows immediately from developing the square.

We now show that 3) implies 1). Take N points x_1, x_2, \dots, x_N in Ω , and let $R > 0$ be such that $B(x_i, R) \subset \Omega$. Consider the transformation

$$S_R(x) = \begin{cases} x - x_1 + x_2 & \text{for } x \in B(x_1, R), \\ x - x_2 + x_3 & \text{for } x \in B(x_2, R), \\ \vdots & \\ x - x_N + x_1 & \text{for } x \in B(x_N, R), \\ x & \text{otherwise.} \end{cases}$$

It is easy to see that S_R is a measure-preserving transformation and that $S_R^N = \text{Id}$.

We then have

$$0 \leq \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), x - S_R^{\ell} x \rangle d\mu \leq \sum_{i=1}^N \int_{B(x_i, R)} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), x_i - x_{\ell+i} \rangle d\mu.$$

Letting $R \rightarrow 0$, we get from Lebesgue’s density theorem that

$$\frac{1}{|B(x_i, R)|} \int_{B(x_i, R)} \langle u_{\ell}(x), x_i - x_{\ell+i} \rangle d\mu \rightarrow \langle u_{\ell}(x_i), x_i - x_{\ell+i} \rangle,$$

from which follows that (u_1, \dots, u_{N-1}) are jointly N -monotone a.e. on Ω . The fact that 1) is equivalent to 5) follows immediately from Theorem 2.

To prove that 5) implies 6), note that for all $p_i \in \mathbb{R}^d$, $x \in \Omega$, $y_i \in \Omega$, $i = 1, \dots, N - 1$,

$$L_H(x, p_1, \dots, p_{N-1}) + H(x, y_1, \dots, y_{N-1}) \geq \sum_{i=1}^{N-1} \langle p_i, y_i \rangle,$$

which yields that for any $S \in \mathcal{S}_N(\Omega, \mu)$,

$$\begin{aligned} \int_{\Omega} [L_H(x, u_1(x), \dots, u_{N-1}(x)) d\mu + H(x, Sx, \dots, S^{N-1}x)] d\mu \\ \geq \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), S^{\ell}x \rangle d\mu. \end{aligned}$$

If $H \in \mathcal{H}_N(\Omega)$ and $S \in \mathcal{S}_N(\Omega, \mu)$, we then have $\int_{\Omega} H(x, Sx, \dots, S^{N-1}x) d\mu = 0$, and therefore

$$\int_{\Omega} L_H(x, u_1(x), \dots, u_{N-1}(x)) d\mu \geq \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), S^{\ell}x \rangle d\mu.$$

If now H is the N -sub-antisymmetric Hamiltonian obtained by 5), which is concave in the first variable and convex in the last $N - 1$ variables, then

$$L_H(x, u_1(x), \dots, u_{N-1}(x)) + H(x, x, \dots, x) = \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), x \rangle \quad \text{for all } x \in \Omega \setminus \Omega_0,$$

and therefore $\int_{\Omega} L_H(x, u_1(x), \dots, u_{N-1}(x)) d\mu = \sum_{\ell=1}^{N-1} \int_{\Omega} \langle u_{\ell}(x), x \rangle d\mu$.

Now consider

$$\bar{H}(x) = \frac{1}{N} \left((N - 1)H(x) - \sum_{i=1}^{N-1} H(\sigma^i(x)) \right).$$

As before, we have $\bar{H} \in \mathcal{H}_N(\Omega)$ and $\bar{H} \geq H$. Since $L_{\bar{H}} \leq L_H$, we have

$$\int_{\Omega} L_{\bar{H}}(x, u_1(x), \dots, u_{N-1}(x)) \, d\mu = \sum_{\ell=1}^{N-1} \int_{\Omega} \langle u_{\ell}(x), x \rangle \, d\mu$$

and 6) is proved.

Finally, note that 6) readily implies 3), which means that (u_1, \dots, u_{N-1}) is then jointly N -monotone. \square

We now consider again the case of a single N -cyclically monotone vector field.

Corollary 9. *Let $u : \Omega \rightarrow \mathbb{R}^d$ be a bounded measurable vector field. The following properties are then equivalent:*

- 1) *The vector field u is N -cyclically monotone a.e., that is, there exists a measure-zero set Ω_0 such that u is N -cyclically monotone on $\Omega \setminus \Omega_0$.*
- 2) *The infimum of the Monge–Kantorovich problem*

$$(38) \quad \inf \left\{ \int_{\Omega^N} \langle u(x_1), x_1 - x_2 \rangle \, d\pi(x) : \pi \in \mathcal{P}_{\text{sym}}^{\mu}(\Omega^N) \right\}$$

is equal to zero, and is therefore attained by the push-forward of μ by the map $x \rightarrow (x, x, \dots, x)$.

- 3) *The vector field u is in the polar of $\mathcal{S}_N(\Omega, \mu)$, that is,*

$$(39) \quad \inf \left\{ \int_{\Omega} \langle u(x), x - Sx \rangle \, d\mu : S \in \mathcal{S}_N(\Omega, \mu) \right\} = 0.$$

- 4) *The projection of u on $\mathcal{S}_N(\Omega, \mu)$ is the identity map, that is,*

$$(40) \quad \inf \left\{ \int_{\Omega} |u(x) - Sx|^2 \, d\mu : S \in \mathcal{S}_N(\Omega, \mu) \right\} = \int_{\Omega} |u(x) - x|^2 \, d\mu.$$

- 5) *There exists an N -cyclically sub-antisymmetric function H of two variables, which is concave in the first variable, convex in the second variable, vanishing on the diagonal and such that*

$$(41) \quad u(x) = \nabla_2 H(x, x) \quad \text{for a.e. } x \in \Omega.$$

- 6) *The following duality holds:*

$$\begin{aligned} \inf \left\{ \int_{\Omega} L_H(x, u(x), 0, \dots, 0) \, d\mu : H \in \mathcal{H}_N(\Omega) \right\} \\ = \sup \left\{ \int_{\Omega} \langle u(x), Sx \rangle \, d\mu : S \in \mathcal{S}_N(\Omega, \mu) \right\} \end{aligned}$$

and the latter is attained at the identity map.

Proof. This is an immediate application of Theorem 7 applied to the $(N-1)$ -tuple vector fields $(u, 0, \dots, 0)$, which is clearly jointly N -monotone on $\Omega \setminus \Omega_0$, whenever u is N -monotone on $\Omega \setminus \Omega_0$. \square

Remark 10. The sets of μ -measure-preserving N -involutions $(\mathcal{S}_N(\Omega, \mu))_N$ do not form a nested family, that is, $\mathcal{S}_N(\Omega, \mu)$ is not necessarily included in $\mathcal{S}_M(\Omega, \mu)$, whenever $N \leq M$, unless of course M is a multiple of N . On the other hand, the above theorem shows that their polar sets, i.e.,

$$\mathcal{S}_N(\Omega, \mu)^0 = \left\{ u \in L^2(\Omega, \mathbb{R}^d) : \int_{\Omega} \langle u(x), x - Sx \rangle d\mu \geq 0 \text{ for all } S \in \mathcal{S}_N(\Omega, \mu) \right\},$$

which coincide with the N -cyclically monotone maps, satisfy

$$\mathcal{S}_{N+1}(\Omega, \mu)^0 \subset \mathcal{S}_N(\Omega, \mu)^0,$$

for every $N \geq 1$. This can also be seen directly. Indeed, it is clear that a 2-involution is a 4-involution but not necessarily a 3-involution. On the other hand, assume that u is a 3-cyclically monotone operator. Then for any transformation $S : \Omega \rightarrow \Omega$, we have

$$\int_{\Omega} \langle u(x), x - Sx \rangle d\mu + \int_{\Omega} \langle u(Sx), Sx - S^2x \rangle d\mu + \int_{\Omega} \langle u(S^2x), S^2x - x \rangle d\mu \geq 0.$$

Now if S is measure-preserving, we have

$$\int_{\Omega} \langle u(x), x - Sx \rangle d\mu + \int_{\Omega} \langle u(x), x - Sx \rangle d\mu + \int_{\Omega} \langle u(S^2x), S^2x - x \rangle d\mu \geq 0,$$

and if $S^2 = I$, then $\int_{\Omega} \langle u(x), x - Sx \rangle d\mu \geq 0$, which means that $u \in \mathcal{S}_2(\Omega, \mu)^0$. Similarly, one can show that any $(N+1)$ -cyclically monotone operator belongs to $\mathcal{S}_N(\Omega, \mu)^0$. In other words, $\mathcal{S}_{N+1}(\Omega, \mu)^0 \subset \mathcal{S}_N(\Omega, \mu)^0$ for all $N \geq 2$. Note that $\mathcal{S}_1(\Omega, \mu)^0 = \{I\}^0 = L^2(\Omega, \mathbb{R}^d)$, while

$$\begin{aligned} \mathcal{S}(\Omega, \mu)^0 &= \bigcap_N \mathcal{S}_N(\Omega, \mu)^0 \\ &= \{u \in L^2(\Omega, \mathbb{R}^d), u = \nabla \phi \text{ for some convex function } \phi \text{ in } W^{1,2}(\mathbb{R}^d)\}, \end{aligned}$$

in view of classical results of Rockafellar [1970] and Brenier [1991].

Remark 11. In [Ghoussoub and Moameni 2013b], the preceding result is extended to give a similar decomposition for any family of bounded measurable vector fields u_1, u_2, \dots, u_{N-1} on Ω . It is shown there that there exists a measure-preserving N -involution S on Ω and an N -antisymmetric Hamiltonian H on Ω^N such that for $i = 1, \dots, N-1$, we have

$$u_i(x) = \nabla_{i+1} H(x, Sx, S^2x, \dots, S^{N-1}x) \quad \text{for a.e. } x \in \Omega.$$

Acknowledgement

We are grateful to the anonymous referee for a careful reading of this paper, which led to several improvements.

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Received January 7, 2013. Revised October 2, 2013.

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
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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

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PACIFIC JOURNAL OF MATHEMATICS

Volume 269 No. 2 June 2014

Totaro's question for simply connected groups of low rank	257
JODI BLACK and RAMAN PARIMALA	
Uniform hyperbolicity of the curve graphs	269
BRIAN H. BOWDITCH	
Constant Gaussian curvature surfaces in the 3-sphere via loop groups	281
DAVID BRANDER, JUN-ICHI INOBUCHI and SHIMPEI KOBAYASHI	
On embeddings into compactly generated groups	305
PIERRE-EMMANUEL CAPRACE and YVES CORNULIER	
Variational representations for N -cyclically monotone vector fields	323
ALFRED GALICHON and NASSIF GHOUSOUB	
Restricted successive minima	341
MARTIN HENK and CARSTEN THIEL	
Radial solutions of non-Archimedean pseudodifferential equations	355
ANATOLY N. KOCHUBEI	
A Jantzen sum formula for restricted Verma modules over affine Kac–Moody algebras at the critical level	371
JOHANNES KÜBEL	
Notes on the extension of the mean curvature flow	385
YAN LENG, ENTAO ZHAO and HAORAN ZHAO	
Hypersurfaces with prescribed angle function	393
HENRIQUE F. DE LIMA, ERALDO A. LIMA JR. and ULISSES L. PARENTE	
Existence of nonparametric solutions for a capillary problem in warped products	407
JORGE H. LIRA and GABRIELA A. WANDERLEY	
A counterexample to the simple loop conjecture for $\mathrm{PSL}(2, \mathbb{R})$	425
KATHRYN MANN	
Twisted Alexander polynomials of 2-bridge knots for parabolic representations	433
TAKAYUKI MORIFUJI and ANH T. TRAN	
Schwarzian differential equations associated to Shimura curves of genus zero	453
FANG-TING TU	
Polynomial invariants of Weyl groups for Kac–Moody groups	491
ZHAO XU-AN and JIN CHUNHUA	



0030-8730(201406)269:2;1-Z