Profile-score adjustments for incidental-parameter problems

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PROFILE-SCORE ADJUSTMENTS FOR INCIDENTAL-PARAMETER PROBLEMS

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We propose a scheme of iterative adjustments to the profile score to deal with incidental-parameter bias in models for stratified data with few observations on a large number of strata. The first-order adjustment is based on a calculation of the profile-score bias and evaluation of this bias at maximum-likelihood estimates of the incidental parameters. If the bias does not depend on the incidental parameters, the first-order adjusted profile score is fully recentered, solving the incidental-parameter problem. Otherwise, it is approximately recentered, alleviating the incidental-parameter problem. In the latter case, the adjustment can be iterated to give higher-order adjustments, possibly until convergence. The adjustments are generally applicable (e.g., not requiring parameter orthogonality) and lead to estimates that generally improve on maximum likelihood. We examine a range of nonlinear models with covariates. In many of them, we obtain an adjusted profile score that is exactly unbiased. In the others, we obtain approximate bias adjustments that yield much improved estimates, relative to maximum likelihood, even when there are only two observations per stratum.

Keywords: *adjusted profile score, bias reduction, incidental parameters.*

INTRODUCTION

Consider inference about a finite-dimensional parameter ψ based on m observations from each of n independent strata in the presence of incidental parameters λ_i $(i = 1, 2, \ldots, n)$, one for each stratum. It is well known that maximum likelihood does not, in general, yield consistent point estimates of ψ as the number of strata, n, increases while their size, m, is kept fixed; see Neyman and Scott (1948). Only in some cases is it possible to separate inference about ψ from inference about the λ_i by means of a conditional or marginal likelihood; see Andersen (1970) and Lancaster (2000) for examples. Estimation of ψ may, alternatively, be based on an integrated likelihood, as discussed in Kalbfleisch and Sprott (1970), but the choice of prior density on λ_i may be hard to justify.

An alternative route is to work with either a modified likelihood (Barndorff-Nielsen 1983) or an approximate conditional likelihood (Cox and Reid 1987). In a rectangular-array embedding (Li et al. 2003), Sartori (2003) showed that the modified likelihood generally leads to superior inference. Lancaster (2002) found similar improvements for approximate conditional likelihoods. Arellano and Bonhomme (2009) extended these to situations where an information-orthogonalizing reparameterization is not possible. Hahn and Newey (2004) developed bias corrections with similar gains.

In this paper, we study estimation of ψ based on adjustments to the profile score, as in McCullagh and Tibshirani (1990). The basic (or first-order) adjustment is to calculate the bias of the profile score, evaluate it at maximum-likelihood estimates of the λ_i, and subsequently recenter the profile score. When the bias is free of incidental parameters, the so-adjusted profile score is fully recentered and leads to consistent estimation of

ψ under Neyman-Scott asymptotics. We find that this is the case in a number of relevant models. When the bias is not free of incidental parameters, the first-order adjusted profile score leads to point estimates whose bias is $O(m^{-2})$, as opposed to the standard $O(m^{-1})$. We show that the adjustment can be iterated, possibly until convergence, generating higher-order adjustments. Assuming sufficient regularity, at each iteration the order of the bias is reduced, and the fully iterated bias adjustment may yield consistent estimates under Neyman-Scott asymptotics. We study the adjustments analytically and numerically in a range of nonlinear models. Invariably, the profile score adjustments are found to be very effective, either leading to consistent estimates under Neyman-Scott asymptotics or, else, to estimates with much smaller bias than maximum likelihood, even for $m = 2$. In the examples examined, we also find that, when a conditional or marginal likelihood exists, its score function coincides with the fully iterated adjusted profile score.

Focusing on the profile score has several advantages. First, the calculation of its bias will reveal whether the presence of incidental parameters, in fact, leads to an incidental-parameter problem. In contrast, verifying whether ψ and λ_i are information orthogonal can be a cumbersome task, not in the least because orthogonality may hold in one parameterization but not in another. Second, if the profile-score bias is zero or free of incidental parameters, or if the fully iterated adjusted profile score has zero bias, the adjusted profile-score equation is unbiased. This property is not shared by approaches based on the modified likelihood which, in general, do not fully recenter the profile score and, in cases where there is no incidental-parameter problem, can induce bias rather than eliminate it. Third, the adjustment to the profile score does not require calculating sample-space derivatives. The computational burden involved in setting up modified profile likelihoods is one major reason for the development of approximations to it, such as those of Cox and Reid (1987) and Severini (1998), for example. Fourth, profile-score adjustments can always be computed, analytically or numerically. They do not require the existence of a sufficient statistic or an information-orthogonal parameterization. It is well known that both do not exist, in general (Severini 2000). Fifth, the iterative procedure leads to higher-order improvements relative to the other general approaches available. The fully iterated adjustment yields estimators whose bias, in principle, shrinks exponentially fast in m. This is of particular importance given the difficulty with which the other methods can be modified to yield higher-order improvements.

1. ADJUSTING THE PROFILE SCORE

1.1. Bias of the profile score

Suppose we are given a rectangular-array data set $\{y_{ij}; i = 1, \dots, n; j = 1, \dots, m\}$ with n strata and m observations for each stratum. The observations y_{ij} are sampled from a probability density (or mass) function $f(y_{ij}; \psi, \lambda_i)$, where ψ and λ_i are finite-dimensional parameters. The density may depend on covariates but this will be suppressed in the notation. The parameter of interest is ψ, with $\lambda = (\lambda_1, \dots, \lambda_n)$ being treated as a nuisance parameter. For simplicity of the exposition, we shall assume that the observations y_{ij} are independent across i and j, although some examples with dependent data will also be discussed.

The profile log-likelihood and score functions for ψ, and their ith contributions, are

$$l(\psi) = \sum_{i=1}^{n} l_i(\psi), \qquad l_i(\psi) = \sum_{j=1}^{m} \log f(y_{ij}; \psi, \widehat{\lambda}_i(\psi)),$$
$$s(\psi) = \sum_{i=1}^{n} s_i(\psi), \qquad s_i(\psi) = \sum_{j=1}^{m} \nabla_\psi \log f(y_{ij}; \psi, \widehat{\lambda}_i(\psi)),$$

where $\widehat{\lambda}_i(\psi) = \arg\max_{\lambda_i} \sum_{j=1}^{m} \log f(y_{ij}; \psi, \lambda_i)$ is the maximum likelihood estimator of λ_i for fixed ψ. Let

$\widehat{\psi} = \arg\max_{\psi} l(\psi)$ be the maximum likelihood estimator of ψ and assume sufficient regularity to ensure that $\widehat{\psi}$ satisfies $s(\widehat{\psi}) = 0$. Neyman and Scott (1948) showed that $\widehat{\psi}$ is not, in general, a consistent estimator of the true value of ψ as $n \to \infty$ while m remains fixed. This is the incidental-parameter problem; it has been documented in numerous examples in the literature.

When $\widehat{\psi}$ is inconsistent, the inconsistency is due to a bias in the profile score function. One may view $s(\psi)$ as an approximation to the infeasible profile score function

$$s^{\mathrm{in}}(\psi) = \sum_{i=1}^{n} s_i^{\mathrm{in}}(\psi), \qquad s_i^{\mathrm{in}}(\psi) = \sum_{j=1}^{m} \nabla_{\psi} \log f(y_{ij}; \psi, \lambda_i(\psi)),$$

where $\lambda_i(\psi) = \arg\max_{\lambda_i^*} E_{\psi,\lambda_i} \sum_{j=1}^{m} \log f(y_{ij}; \psi, \lambda_i^*)$ and $E_{\psi,\lambda_i}(\cdot)$ denotes the expectation under the density $f(\cdot; \psi, \lambda_i)$. So $s(\psi)$ differs from $s^{\mathrm{in}}(\psi)$ in that $s(\psi)$ uses $\widehat{\lambda}(\psi) = (\widehat{\lambda}_1(\psi), \ldots, \widehat{\lambda}_n(\psi))$ whereas $s^{\mathrm{in}}(\psi)$ uses the infeasible $\lambda(\psi) = (\lambda_1(\psi), \ldots, \lambda_n(\psi))$, a difference that often introduces a bias. While $s^{\mathrm{in}}(\psi)$ is unbiased, i.e., $E_{\psi,\lambda} s^{\mathrm{in}}(\psi) = 0$ (see, e.g., Pace and Salvan 2006), it is often the case that

$$E_{\psi,\lambda} s(\psi) \neq 0,$$

causing $\widehat{\psi}$ to be inconsistent for fixed m and the limit distribution of $\widehat{\psi}$ to be incorrectly centered unless $m/n \to \infty$; see, e.g., Portnoy (1988) and Li et al. (2003). Typically, when the profile score is biased, its bias is of order $O(n)$ and the inconsistency of $\widehat{\psi}$, as $n \to \infty$ with fixed m, is of order $O(m^{-1})$.

1.2. Iterated bias adjustment

Our approach is to bias-adjust $s(\psi)$ and, therefore, requires calculating $E_{\psi,\lambda} s(\psi)$, analytically or numerically, for given ψ and λ. Three cases arise:

(a) $E_{\psi,\lambda} s(\psi) = 0$;
(b) $E_{\psi,\lambda} s(\psi) \neq 0$ but $E_{\psi,\lambda} s(\psi)$ is free of λ;
(c) $E_{\psi,\lambda} s(\psi) \neq 0$ and $E_{\psi,\lambda} s(\psi)$ depends on λ.

In Case (a), $s(\psi) = 0$ is an unbiased estimating equation and $\widehat{\psi}$ is consistent as $n \to \infty$ for fixed m. The interesting point here is that a simple calculation, that of $E_{\psi,\lambda} s(\psi)$, will reveal so. In the examples, we show that the profile score is unbiased in the Poisson and the exponential regression model.

In Case (b), $\widehat{\psi}$ is inconsistent for fixed m, but an unbiased estimating equation is readily obtained. The adjusted profile score

$$s_{\mathrm{a}}(\psi) = s(\psi) - E_{\psi,\lambda} s(\psi)$$

is unbiased by construction and feasible because it does not depend on λ. Neyman and Scott (1948) already noted that, when $E_{\psi,\lambda} s(\psi)$ is free of λ, a fixed-m consistent estimator can be obtained by centering the profile score. We will discuss several models where this is the case.

In Case (c), consider the first-order adjusted profile score

$$s_{\mathrm{a}}^{(1)}(\psi) = s(\psi) - E_{\psi,\widehat{\lambda}(\psi)} s(\psi).$$

McCullagh and Tibshirani (1990) suggested this approximate centering of the profile score in the generic context where nuisance parameters are profiled out of the likelihood. Under regularity conditions, the first-order adjusted profile score reduces the large-m asymptotic bias of the profile score by a factor $O(m^{-1})$. The profile score bias is $E_{\psi,\lambda} s(\psi) = E_{\psi,\lambda(\psi)} s(\psi) = \sum_{i=1}^{n} E_{\psi,\lambda_i(\psi)} s_i(\psi)$, which we are approximating by $E_{\psi,\widehat{\lambda}(\psi)} s(\psi) =$

$\sum_{i=1}^{n} E_{\psi,\widehat{\lambda}_i(\psi)} s_i(\psi)$. Thus, in each $E_{\psi,\lambda_i(\psi)} s_i(\psi)$ the infeasible $\lambda_i(\psi)$ is approximated by $\widehat{\lambda}_i(\psi)$. This introduces a relative bias of order $O(m^{-1})$ as $m \to \infty$ (see, e.g., DiCiccio et al. 1996), i.e., $E_{\psi,\lambda_i} E_{\psi,\widehat{\lambda}_i(\psi)} s_i(\psi) = \big(1 + O(m^{-1})\big) E_{\psi,\lambda_i} s_i(\psi)$ and, on summing over i, $E_{\psi,\lambda} E_{\psi,\widehat{\lambda}(\psi)} s(\psi) = \big(1 + O(m^{-1})\big) E_{\psi,\lambda} s(\psi)$. Therefore, relative to profile score, the first-order adjusted profile score reduces the bias from $E_{\psi,\lambda} s(\psi) = O(n)$ to $E_{\psi,\lambda} s_{\mathrm{a}}^{(1)}(\psi) = O(n/m)$. The adjustment removes the first-order asymptotic bias from $s(\psi)$; see also Sartori (2003).

In Case (c), the adjustment can be iterated, each iteration giving a further asymptotic improvement. The bias $E_{\psi,\lambda} s_{\mathrm{a}}^{(1)}(\psi)$ can be approximated by $E_{\psi,\widehat{\lambda}(\psi)} s_{\mathrm{a}}^{(1)}(\psi)$, again with relative bias $O(m^{-1})$, leading to the second-order adjusted profile score

$$\begin{aligned} s_{\mathrm{a}}^{(2)}(\psi) &= s_{\mathrm{a}}^{(1)}(\psi) - E_{\psi,\widehat{\lambda}(\psi)} s_{\mathrm{a}}^{(1)}(\psi) \\ &= s(\psi) - 2E_{\psi,\widehat{\lambda}(\psi)} s(\psi) + E_{\psi,\widehat{\lambda}(\psi)} \left(E_{\psi,\widehat{\lambda}(\psi,\widehat{\lambda}(\psi))} s(\psi) \right), \end{aligned}$$

with bias $E_{\psi,\lambda} s_{\mathrm{a}}^{(2)}(\psi) = O(n/m^2)$. Here, $\widehat{\lambda}(\psi, \widehat{\lambda}(\psi))$ is the maximum likelihood estimator of λ, for fixed ψ, based on a data set $\{y_{ij}^*; i = 1, \ldots, n; j = 1, \ldots, m\}$ where y_{ij}^* is sampled from $f(\cdot; \psi, \widehat{\lambda}_i(\psi))$. The structure of the iterated adjustments is now apparent. Defining the p-fold iteration of $E_{\psi,\widehat{\lambda}(\psi)}(\cdot)$ by the recursion

$$\begin{aligned} E_{\psi,\widehat{\lambda}(\psi)}^{(0)}(\cdot) &= (\cdot), \\ E_{\psi,\widehat{\lambda}(\psi)}^{(p)}(\cdot) &= E_{\psi,\widehat{\lambda}(\psi)} \left(E_{\psi,\widehat{\lambda}(\psi,\widehat{\lambda}(\psi))}^{(p-1)}(\cdot) \right), \qquad p = 1, 2, \ldots, \end{aligned}$$

the kth order adjusted profile score is

$$\begin{aligned} s_{\mathrm{a}}^{(k)}(\psi) &= s_{\mathrm{a}}^{(k-1)}(\psi) - E_{\psi,\widehat{\lambda}(\psi)} s_{\mathrm{a}}^{(k-1)}(\psi) \\ &= s(\psi) - \sum_{p=1}^{k} \binom{k}{p} (-1)^{p-1} E_{\psi,\widehat{\lambda}(\psi)}^{(p)} s(\psi), \end{aligned}$$

with bias $O(n/m^k)$, given sufficient regularity. The associated estimator, $\widehat{\psi}_{\mathrm{a}}^{(k)}$, is defined as the solution to $s_{\mathrm{a}}^{(k)}(\psi) = 0$. The adjustment may be iterated until convergence. Upon existence of the limits, we define the (fully iterated) adjusted profile score as $s_{\mathrm{a}}(\psi) = \lim_{k\to\infty} s_{\mathrm{a}}^{(k)}(\psi)$ and the adjusted score estimator as $\widehat{\psi}_{\mathrm{a}} = \lim_{k\to\infty} \widehat{\psi}_{\mathrm{a}}^{(k)}$.

1.3. Further discussion of Case (c)

Our intuitive motivation for iterating the adjustment, possibly until convergence, uses large-m arguments. Nevertheless, the fully iterated adjustment may have good properties even when m is very small. In particular, it may occur that $\widehat{\psi}_{\mathrm{a}}$ is fixed-m consistent whereas $\widehat{\psi}$ and $\widehat{\psi}_{\mathrm{a}}^{(k)}$ (for any finite k) are not. Also, in cases where $\widehat{\psi}_{\mathrm{a}}$ is not fixed-m consistent, it may still reduce the asymptotic bias of $\widehat{\psi}$. Such situations arise when ψ is not point identified for a given m. Obviously, when point identification fails, $s_{\mathrm{a}}(\psi)$ cannot deliver an unbiased estimating equation, i.e., it must be the case that $E_{\psi,\lambda} s_{\mathrm{a}}(\psi)$ is generically nonzero or that $s_{\mathrm{a}}(\psi)$ vanishes in the neighborhood of the true value of ψ. Yet, in either case, $\widehat{\psi}_{\mathrm{a}}$ may still be well defined (i.e., the required limit exists) and have better properties than $\widehat{\psi}$. We will discuss examples to illustrate these points.

Implementing the adjustment and solving $s_{\mathrm{a}}(\psi) = 0$ or $s_{\mathrm{a}}^{(k)}(\psi) = 0$ for some chosen k requires evaluating $E_{\psi,\lambda} s(\psi)$ for given ψ and λ. Often $E_{\psi,\lambda} s(\psi)$ is not available in closed form, but it can be approximated by simulation. For the sake of bias adjustment, any number of simulations suffices, even a single one. For reasons of accuracy, however, the number of simulations should not be too small, unless perhaps in models

where evaluating $s(\psi)$ is computationally costly. For the higher-order adjustments, which require evaluating the terms $E^{(p)}_{\psi,\widehat{\lambda}(\psi)} s(\psi)$, we suggest to use a small number of simulations in the inner expectations and a larger number in the outermost expectation, and to keep the basic stream of random numbers constant for all values of ψ.

The classification (a)–(c) helps to discern if there is an incidental-parameter problem for ψ and, if so, how it may be solved or mitigated. When ψ is multidimensional, there may be an incidental-parameter problem only for a subvector of ψ. Let ψ and $s(\psi)$ be conformably partitioned as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad s(\psi) = \begin{pmatrix} s_{\psi_1}(\psi) \\ s_{\psi_2}(\psi) \end{pmatrix}.$$

Then, if for any ψ_2',

$$E_{\psi,\lambda} s_{\psi_1}(\psi') = 0 \qquad \text{for } \psi' = \begin{pmatrix} \psi_1 \\ \psi_2' \end{pmatrix},$$
$$E_{\psi,\lambda} s_{\psi_2}(\psi) \neq 0,$$

there is an incidental-parameter problem for ψ_2 but it does not carry over to ψ_1. Note that $E_{\psi,\lambda} s_{\psi_1}(\psi) = 0$ is necessary but not sufficient to prevent the incidental-parameter problem for ψ_2 to carry over to ψ_1.

2. EXAMPLES

Many of our examples are conditional models of a variable y_{ij} given a vector of covariates x_{ij}. Accordingly, expectations are taken conditionally on the covariates.

2.1. Case (a) models

Poisson regression. Consider Poisson random variables y_{ij} with mean $\mu_{ij} = \lambda_i \exp(x_{ij}^\top \psi)$. The probability mass function is $f(y_{ij}; \psi, \lambda_i) = \mu_{ij}^{y_{ij}} \exp(-\mu_{ij})/y_{ij}!$. For fixed ψ, the maximum likelihood estimator of λ_i is $\widehat{\lambda}_i(\psi) = \sum_j y_{ij} / \sum_j \exp(x_{ij}^\top \psi)$. The profile log-likelihood and score are

$$l(\psi) = \sum_{i,j} y_{ij} \left(-\log \sum_j \exp(x_{ij}^\top \psi) + x_{ij}^\top \psi \right),$$
$$s(\psi) = \sum_{i,j} y_{ij} \left(-\frac{\sum_j \exp(x_{ij}^\top \psi) x_{ij}}{\sum_j \exp(x_{ij}^\top \psi)} + x_{ij} \right).$$

(We omit additive constants from $l(\psi)$ in all examples.) Taking expectations gives $E_{\psi,\lambda} s(\psi) = 0$. There is no incidental-parameter problem in this model and, accordingly, the adjustment leaves the profile score unaltered.

Blundell et al. (1999) gave a closely related derivation. Further, Lancaster (2002) showed that ψ and λ are likelihood orthogonal after an interest-respecting reparametrization (i.e., the unprofiled likelihood is separable), which is another way of showing that maximum likelihood is consistent. The modifications of Barndorff-Nielsen (1983) and Cox and Reid (1987) to the profile likelihood also leave it unchanged. The profile likelihood is equal to the conditional likelihood given $\sum_j y_{ij}$, $i = 1, \ldots, n$, which is a sufficient statistic for λ. Hence, from Hahn (1997) it follows that maximum likelihood attains the semiparametric efficiency bound.

Exponential regression. Let y_{ij} be exponentially distributed with scale $\mu_{ij} = \lambda_i \exp(x_{ij}^\top \psi)$, i.e., $f(y_{ij}; \psi, \lambda_i) = \mu_{ij}^{-1} \exp(-y_{ij}/\mu_{ij})$. Then $\widehat{\lambda}_i(\psi) = m^{-1} \sum_j y_{ij} \exp(-x_{ij}^\top \psi)$ and the profile log-likelihood and

score are

$$l(\psi) = \sum_{i,j} \left(-\log \sum_j y_{it} \exp(-x_{ij}^\top \psi) - x_{ij}^\top \psi \right),$$
$$s(\psi) = \sum_{i,j} \left(\frac{\sum_j y_{ij} \exp(-x_{ij}^\top \psi) x_{ij}}{\sum_j y_{ij} \exp(-x_{ij}^\top \psi)} - x_{ij} \right).$$

Now write

$$\frac{\sum_j y_{ij} \exp(-x_{ij}^\top \psi) x_{ij}}{\sum_j y_{ij} \exp(-x_{ij}^\top \psi)} = \frac{\sum_j z_{ij} x_{ij}}{\sum_j z_{ij}},$$

$z_{ij} = y_{ij}/\mu_{ij}$ being independent unit-exponential random variables. Because the z_{ij} are identically distributed,

$$E\left(\frac{\sum_j z_{ij} x_{ij}}{\sum_j z_{ij}} \right) = \sum_j E\left(\frac{z_{ij}}{\sum_j z_{ij}} \right) x_{ij} = \frac{1}{m} \sum_j x_{ij}.$$

Hence $E_{\psi,\lambda} s(\psi) = 0$. There is no incidental-parameter problem. Again, the profile likelihood coincides with the modified profile likelihood of Barndorff-Nielsen (1983). It is also identical to the marginal likelihood of the ratios y_{ij}/y_{i1}, which is free of λ.

2.2. Case (b) models

Many normal means. This is the classic Neyman and Scott (1948) example of the incidental-parameter problem. The goal is to infer the variance ψ from independent observations $y_{ij} \sim \mathcal{N}(\lambda_i, \psi)$. The profile score is

$$s(\psi) = -\frac{nm}{2\psi} + \frac{1}{2\psi^2} \sum_{i,j} (y_{ij} - \overline{y}_i)^2,$$

with bias $E_{\psi,\lambda} s(\psi) = -n(2\psi)^{-1}$, free of λ, and $\widehat{\psi} = (nm)^{-1} \sum_{i,j} (y_{ij} - \overline{y}_i)^2$ converges to $(1 - m^{-1})\psi$. The solution of the adjusted profile score equation $s_{\mathrm{a}}(\psi) = 0$ is $\widehat{\psi}/(1 - m^{-1})$, which is consistent for fixed m. Numerous other approaches lead to the same estimator.

A regression version of this model has $y_{ij} \sim \mathcal{N}(\lambda_i + x_{ij}^\top \psi_1, \psi_2)$, with ψ consisting of ψ_1 and ψ_2. Here, the bias of the profile score for ψ_1 is zero and for ψ_2 it is $-n(2\psi_2)^{-1}$, as in the no-covariate case. Again, solving the adjusted profile score equation yields the standard solution: (i) the maximum likelihood estimator of ψ_1 (which is least-squares applied to the pooled group-wise demeaned data) is unaltered; (ii) a one-degree-of-freedom correction is applied to the maximum likelihood estimator of ψ_2.

Autoregression. Since Nickell (1981), autoregressive models have become another classic instance of the incidental-parameter problem. Suppose that

$$y_{ij} = \lambda_i + \psi_1 y_{ij-1} + \varepsilon_{ij}, \qquad \varepsilon_{ij} \sim \mathcal{N}(0, \psi_2),$$

where we observe y_{ij} for $j = 0, 1, \ldots, m$ and leave the initial observations, y_{i0}, unrestricted (i.e., we condition on them). The profile score is

$$s(\psi) = \begin{pmatrix} -\psi_2^{-1} \sum_i (y_i - \psi_1 y_{i-})^\top M y_{i-} \\ -(2\psi_2)^{-1} (nm - \psi_2^{-1} \sum_i (y_i - \psi_1 y_{i-})^\top M (y_i - \psi_1 y_{i-})) \end{pmatrix},$$

where $y_i = (y_{i1}, y_{i2}, \ldots, y_{im})^\top$, $y_{i-} = (y_{i0}, y_{i1}, \ldots, y_{im-1})^\top$, $M = I - m^{-1}\iota\iota^\top$, I is the $m \times m$ identity matrix, and ι is an m-vector of ones. Using backward substitution, it follows that

$$E_{\psi,\lambda} s(\psi) = \begin{pmatrix} -n\,(m-1)^{-1} \sum_{j=1}^{m-1} (m-j)\,\psi_1^{j-1} \\ -n(2\psi_2)^{-1} \end{pmatrix},$$

which is free of λ.

Lancaster (2002) showed that λ and ψ can be orthogonalized and derived a Cox and Reid (1987) conditional profile log-likelihood, whose score function coincides with the adjusted profile score. When the model is extended to include covariates or more than one autoregressive term, the bias of the profile score still admits a closed form and remains free of incidental parameters. In contrast, an orthogonalizing reparameterization of λ and ψ no longer exists; see Dhaene and Jochmans (2015).

Weibull regression. Suppose that y_{ij} is Weibull distributed with survival function $\exp(-(y_{ij}/\mu_{ij})^{\psi_2})$, where $\mu_{ij} = \lambda_i \exp(x_{ij}^\top \psi_1)$. Then $\widehat{\lambda}_i(\psi) = (m^{-1} \sum_j w_{ij}(\psi))^{1/\psi_2}$, where $w_{ij}(\psi) = (y_{ij} \exp(-x_{ij}^\top \psi_1))^{\psi_2}$. The profile log-likelihood and score are

$$l(\psi) = \sum_{i,j} \left(\log \psi_2 + \psi_2 \log y_{ij} - \log \sum_j w_{ij}(\psi) - \psi_2 x_{ij}^\top \psi_1 \right),$$
$$s(\psi) = \sum_{i,j} \begin{pmatrix} \psi_2 \sum_j w_{ij}(\psi) x_{ij} / \sum_j w_{ij}(\psi) - \psi_2 x_{ij} \\ \psi_2^{-1} + \log y_{ij} - \psi_2^{-1} \sum_j w_{ij}(\psi) \log w_{ij}(\psi) / \sum_j w_{ij}(\psi) - x_{ij}^\top \psi_1 \end{pmatrix}.$$

A calculation summarized in the Appendix gives the bias of the profile score as

$$E_{\psi,\lambda} s(\psi) = \begin{pmatrix} 0 \\ n\psi_2^{-1} \end{pmatrix},$$

which is free of λ. The solutions of $s(\psi) = 0$ and $s_{\mathrm{a}}(\psi) = 0$ differ for both ψ_1 and ψ_2. However, there is an incidental-parameter problem only for ψ_2 because, as shown in the Appendix, the first component of $s(\psi')$, with $\psi' = (\psi_1, \psi_2')$, has zero expectation for any ψ_2'.

Several other approaches lead to the same result. Lancaster (2000) showed that λ and ψ can be orthogonalized. Integrating the reparameterized λ from the likelihood using a uniform prior gives a Cox and Reid (1987) conditional profile likelihood. Chamberlain (1985) suggested to use the marginal likelihood of the ratios y_{ij}/y_{i1}, which is free of λ. The conditional profile log-likelihood and the marginal log-likelihood are identical, and their score functions are identical to the adjusted profile score function.

Gamma regression. Here, y_{ij} is gamma distributed with scale $\mu_{ij} = \lambda_i \exp(x_{ij}^\top \psi_1)$ and shape parameter ψ_2. The density is $f(y_{ij}; \psi, \lambda_i) = y_{ij}^{\psi_2 - 1} \mu_{ij}^{-\psi_2} \exp(-y_{ij}/\mu_{ij})/\Gamma(\psi_2)$ and the maximum likelihood estimator of λ_i for fixed ψ is $\widehat{\lambda}_i(\psi) = \psi_2^{-1} m^{-1} \sum_j y_{ij} \exp(-x_{ij}^\top \psi_1)$. The profile log-likelihood and score are

$$l(\psi) = \sum_{i,j} \left(-\log \Gamma(\psi_2) + \psi_2 \log(m\psi_2 y_{ij}) - \psi_2 - \psi_2 \log \sum_j y_{ij} \exp(-x_{ij}^\top \psi_1) - \psi_2 x_{ij}^\top \psi_1 \right),$$
$$s(\psi) = \sum_{i,j} \begin{pmatrix} \psi_2 \sum_j y_{ij} \exp(-x_{ij}^\top \psi_1) x_{ij} / \sum_j y_{ij} \exp(-x_{ij}^\top \psi_1) - \psi_2 x_{ij}) \\ -\mathrm{psi}(\psi_2) + \log(m\psi_2 y_{ij}) - \log \sum_j y_{ij} \exp(-x_{ij}^\top \psi_1) - x_{ij}^\top \psi_1 \end{pmatrix},$$

where $\mathrm{psi}(z) = \nabla_z \log \Gamma(z)$, the digamma function. A calculation, given in the Appendix, yields

$$E_{\psi,\lambda} s(\psi) = \begin{pmatrix} 0 \\ nm(\log(m\psi_2) - \mathrm{psi}(m\psi_2)) \end{pmatrix},$$

which is free of λ. In this model, again, there is an incidental-parameter problem only for ψ_2. The solutions of $s(\psi)=0$ and $s_{\mathrm{a}}(\psi)=0$ coincide for ψ_1 but differ for ψ_2.

The adjusted profile score function is equal to the score function of the marginal log-likelihood of the ratios y_{ij}/y_{i1}, which is free of λ (Chamberlain 1985).

Inverse Gaussian regression. Suppose y_{ij} has the inverse Gaussian distribution with mean $\mu_{ij}=\lambda_i\exp(x_{ij}^\top\psi_1)$ and variance μ_{ij}^3/ψ_2. The density is $f(y_{ij};\psi,\lambda_i)=\sqrt{\psi_2/(2\pi y_{ij}^3)}\exp(-\psi_2(2y_{ij})^{-1}(y_{ij}\mu_{ij}^{-1}-1)^2)$. For fixed ψ, the maximum likelihood estimator of λ_i is $\widehat{\lambda}_i(\psi)=\sum_j y_{ij}\exp(-2x_{ij}^\top\psi_1)/\sum_j\exp(-x_{ij}^\top\psi_1)$. The profile log-likelihood and score are

$$l(\psi)=\sum_{i,j}\left(2^{-1}\log\psi_2-\psi_2(2y_{ij})^{-1}(y_{ij}\widehat{\mu}_{ij}^{-1}-1)^2\right),$$
$$s(\psi)=\sum_{i,j}\begin{pmatrix}\psi_2(y_{ij}\widehat{\mu}_{ij}^{-2}-\widehat{\mu}_{ij}^{-1})x_{ij}\\ (2\psi_2)^{-1}-(2y_{ij})^{-1}(y_{ij}\widehat{\mu}_{ij}^{-1}-1)^2\end{pmatrix},$$

where $\widehat{\mu}_{ij}=\widehat{\lambda}_i(\psi)\exp(x_{ij}^\top\psi_1)$. The profile score bias is

$$E_{\psi,\lambda}s(\psi)=\begin{pmatrix}0\\ n(2\psi_2)^{-1}\end{pmatrix},$$

as shown in the Appendix. Here, again, there is an incidental-parameter problem only for ψ_2, and the solutions of $s(\psi)=0$ and $s_{\mathrm{a}}(\psi)=0$ coincide for ψ_1 but differ for ψ_2.

2.3. Case (c) models

Binary matched pairs. Consider n pairs $y_i=(y_{i1},y_{i2})$ of independent binary variables with success probabilities $\Pr(y_{i1}=1)=(1+e^{-\lambda_i})^{-1}$ and $\Pr(y_{i2}=1)=(1+e^{-\lambda_i-\psi})^{-1}$; cf. Cox (1958). The parameter ψ is the log odds ratio and it is well known that $\operatorname{plim}_{n\to\infty}\widehat{\psi}=2\psi$. The classic solution to this incidental-parameter problem is to use the conditional likehood given $y_{i1}+y_{i2}$, $i=1,\ldots,n$ (Rasch 1961, Andersen 1970). The conditional maximum likelihood estimator is consistent and semi-parametrically efficient (Hahn 1997). Here, we show that the adjusted profile score coincides with the score of the conditional log-likelihood.

For pairs of the form $y_i=(0,0)$ or $y_i=(1,1)$, the maximum likelihood estimator of λ_i for any fixed ψ is $\widehat{\lambda}_i(\psi)=-\infty$ and $\widehat{\lambda}_i(\psi)=+\infty$, respectively, so $l_i(\psi)=s_i(\psi)=0$ for such pairs. For pairs of the form $y_i=(0,1)$ or $y_i=(1,0)$, $\widehat{\lambda}_i(\psi)=-\psi/2$. The profile log-likehood and score are

$$l(\psi)=-2n_{01}\log(1+e^{-\psi/2})-2n_{10}\log(1+e^{\psi/2}),$$
$$s(\psi)=\frac{n_{01}}{1+e^{\psi/2}}-\frac{n_{10}}{1+e^{-\psi/2}},$$

where n_{01} an n_{10} are the number of $(0,1)$ and $(1,0)$ pairs, respectively, with expected values

$$E_{\psi,\lambda}n_{01}=\sum_i(1+e^{\lambda_i})^{-1}(1+e^{-\lambda_i-\psi})^{-1},$$
$$E_{\psi,\lambda}n_{10}=\sum_i(1+e^{-\lambda_i})^{-1}(1+e^{\lambda_i+\psi})^{-1}=e^{-\psi}E_{\psi,\lambda}n_{01}.$$

Defining $a_\psi=(1-e^{-\psi/2})(1+e^{\psi/2})^{-1}$, the bias of the profile score is

$$E_{\psi,\lambda}s(\psi)=a_\psi E_{\psi,\lambda}n_{01},$$

which depends on λ via $E_{\psi,\lambda}n_{01}$. Now consider the sequence $s_{\mathrm{a}}^{(k)}(\psi)$ of finite-order adjusted profile scores.

We have

$$E_{\psi,\widehat{\lambda}(\psi)} n_{01} = (n_{01}+n_{10})(1+e^{-\psi/2})^{-2},$$
$$E_{\psi,\lambda}\left(E_{\psi,\widehat{\lambda}(\psi)} n_{01}\right) = b_\psi E_{\psi,\lambda} n_{01},$$

where $b_\psi = (1+e^{-\psi})(1+e^{-\psi/2})^{-2}$. Hence, for $k = 1, 2, \ldots$,

$$E^{(k)}_{\psi,\widehat{\lambda}(\psi)} s(\psi) = a_\psi b_\psi^{k-1} E_{\psi,\widehat{\lambda}(\psi)} n_{01},$$
$$E_{\psi,\lambda}\left(E^{(k)}_{\psi,\widehat{\lambda}(\psi)} s(\psi)\right) = a_\psi b_\psi^{k} E_{\psi,\lambda} n_{01},$$

and we obtain

$$\begin{aligned} s_{\mathrm{a}}^{(k)}(\psi) &= s(\psi) - \sum_{p=1}^{k} \binom{k}{p} (-1)^{p-1}\, a_\psi b_\psi^{p-1} E_{\psi,\widehat{\lambda}(\psi)} n_{01} \\ &= s(\psi) - (1-(1-b_\psi)^k) a_\psi b_\psi^{-1} E_{\psi,\widehat{\lambda}(\psi)} n_{01}, \end{aligned}$$

with bias

$$E_{\psi,\lambda} s_{\mathrm{a}}^{(k)} = a_\psi (1-b_\psi)^k E_{\psi,\lambda} n_{01}.$$

For all k, the bias has the same sign as ψ and, since $0 < b_\psi < 1$, decays monotonically to zero at a geometric rate in k. Letting $k \to \infty$, we find

$$s_{\mathrm{a}}(\psi) = \frac{n_{01}}{1+e^{\psi}} - \frac{n_{10}}{1+e^{-\psi}} = s_{\mathrm{c}}(\psi),$$

where $s_{\mathrm{c}}(\psi)$ is the score function of the conditional log-likelihood. Thus, fully iterating the bias adjustment of the profile score leads to the conditional likelihood. Accordingly, $\widehat{\psi}_{\mathrm{a}} = \log(n_{01}/n_{10})$, the conditional maximum likelihood estimator.

It may be remarked that, in this model, the fully iterated adjustment can also be obtained without iterating, essentially as in Case (b). First rescale $s(\psi)$ as

$$q(\psi) = \frac{s(\psi)}{n_{01}+n_{10}} = \frac{n_{01}}{n_{01}+n_{10}} - \frac{1}{1+e^{-\psi/2}},$$

assuming $n_{01}+n_{10} > 0$. As $n \to \infty$, $q(\psi)$ converges in probability to

$$q_\infty(\psi) = \frac{1}{1+e^{-\psi}} - \frac{1}{1+e^{-\psi/2}}$$

for any sequence $\lambda_1, \lambda_2, \ldots$ for which $E_{\psi,\lambda} n_{01}/n$ converges. Since $q_\infty(\psi)$ is free of λ, $q_{\mathrm{a}}(\psi) = q(\psi) - q_\infty(\psi)$ is a bias-adjusted version of $q(\psi)$. This yields $q_{\mathrm{a}}(\psi) = s_{\mathrm{c}}(\psi)/(n_{01}+n_{10})$, again leading to the conditional maximum likelihood estimator.

Now consider the following generalization. Suppose $\Pr(y_{i1}=1) = G(\lambda_i)$ and $\Pr(y_{i2}=1) = G(\lambda_i+\psi)$, where G is a distribution function with a density g that is symmetric about zero, unimodal, continuous, and non-zero everywhere. A conditional likelihood that is free of λ exists only when G is logistic, as above. The profile score and the adjusted profile score are

$$s(\psi) = (n_{01} - Q(\psi/2) n_{10})\, c_\psi,$$
$$s_{\mathrm{a}}(\psi) = \left(n_{01} - Q(\psi/2)^2 n_{10}\right) d_\psi,$$

where $Q(z) = G(z)/G(-z)$, $c_\psi = g(\psi/2)/G(\psi/2)$, and $d_\psi = g(\psi/2)Q(-\psi/2)/(G(\psi/2)^2 + G(-\psi/2)^2)$; the derivation is given in the Appendix. It follows that $s_{\mathrm{a}}(\psi)$ is unbiased if and only if

$$\frac{E_{\psi,\lambda} n_{01}}{E_{\psi,\lambda} n_{10}} = Q(\psi/2)^2,$$

regardless of λ. Therefore, unbiasedness requires that

$$\frac{E_{\psi,\lambda} n_{01}}{E_{\psi,\lambda} n_{10}} = \frac{\sum_i G(-\lambda_i) G(\lambda_i + \psi)}{\sum_i G(\lambda_i) G(-\lambda_i - \psi)}$$

be free of λ. Setting all λ_i equal gives the requirement

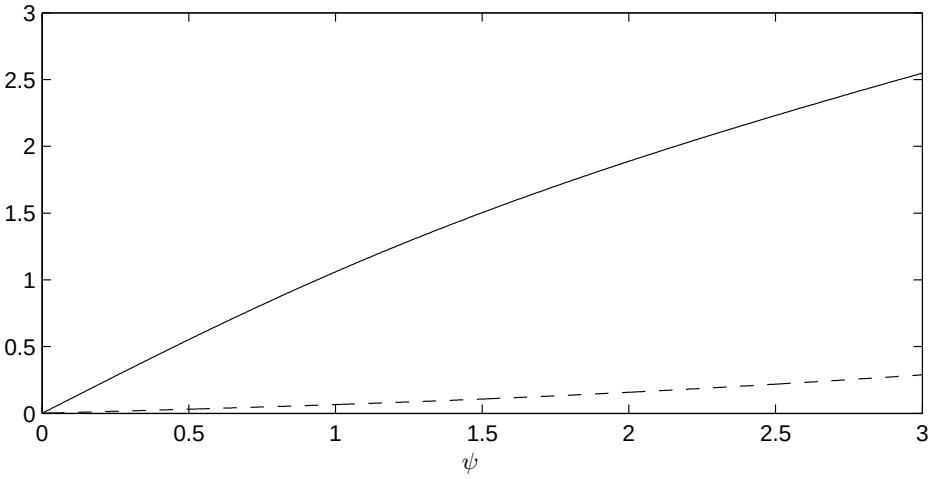
$$\frac{Q(\lambda_i + \psi)}{Q(\lambda_i)} = h(\psi)$$

for some function h. Setting $\lambda_i = 0$ gives $h(\psi) = Q(\psi)$ and hence

$$Q(\lambda_i + \psi) = Q(\lambda_i) Q(\psi),$$

whose solution is of the form $Q(z) = e^{\gamma z}$ and, therefore, $G(z) = (1 + e^{-\gamma z})^{-1}$ is logistic. When G is logistic, $s_{\mathrm{a}}(\psi)$ is unbiased, as shown earlier. When G is not logistic, $E_{\psi,\lambda} n_{01} / E_{\psi,\lambda} n_{10}$ is not free of λ and, therefore, ψ is not point identified everywhere and an unbiased estimating equation does not exist; see also Chamberlain (1980, 2010).

Figure 1. Asymptotic biases in the probit model, $m = 2$



Asymptotic bias of $\widehat{\psi}$ (solid) and $\widehat{\psi}_{\mathrm{a}}$ (dashed) when $\lambda_i \sim N(0,1)$ and G is standard normal.

When G is not logistic, the asymptotic bias of $\widehat{\psi}$ and $\widehat{\psi}_{\mathrm{a}}$ can be signed. Suppose $\psi > 0$ (the case $\psi < 0$ follows by symmetry). Given $Q^{-1}(z) = G^{-1}(z/(1+z))$, we have

$$\widehat{\psi} = 2G^{-1}\left(\frac{n_{01}/n_{10}}{1 + n_{01}/n_{10}}\right),$$
$$\widehat{\psi}_{\mathrm{a}} = 2G^{-1}\left(\frac{\sqrt{n_{01}/n_{10}}}{1 + \sqrt{n_{01}/n_{10}}}\right).$$

Assume that $E_{\psi,\lambda} n_{01}/n$ converges, so that the probability limits of $\widehat{\psi}$ and $\widehat{\psi}_{\mathrm{a}}$ exist. Now, since $Q(\lambda_i + \psi)/Q(\lambda_i) \geq Q(\psi/2)/Q(-\psi/2)$, with equality if and only if $\lambda_i = -\psi/2$,

$$\frac{E_{\psi,\lambda} n_{01}}{E_{\psi,\lambda} n_{10}} \geq Q(\psi/2)^2.$$

Therefore, $\operatorname{plim}_{n\to\infty} \widehat{\psi} > \operatorname{plim}_{n\to\infty} \widehat{\psi}_{\mathrm{a}} \geq \psi$, with equality if and only if $\lambda_i = -\psi/2$ for almost all i. Hence, although $\widehat{\psi}_{\mathrm{a}}$ is generally inconsistent, it improves on maximum likelihood uniformly across the parameter

space. Figure 1 illustrates the improvement for the case where $\lambda_1, \lambda_2, \ldots$ are drawn independently from $N(0,1)$ and G is the standard normal distribution function. The curves are the asymptotic biases of $\widehat{\psi}$ (solid) and $\widehat{\psi}_{\mathrm{a}}$ (dashed). The asymptotic bias of $\widehat{\psi}_{\mathrm{a}}$ is small when most of the $|\lambda_i + \psi/2|$ are only moderately large, but is unbounded since $Q(\lambda_i + \psi)/Q(\lambda_i) \to \infty$ as $\lambda_i \to \pm\infty$.

Binary autoregressive pairs. Consider n independent pairs $y_i = (y_{i1}, y_{i2})$ of binary variables with success probabilities

$$\Pr(y_{i1} = 1) = (1 + e^{-\lambda_i})^{-1},$$
$$\Pr(y_{i2} = 1 | y_{i1}) = (1 + e^{-\lambda_i - \psi y_{i1}})^{-1}.$$

This is an autoregressive logit model with $y_{i0} = 0$ for all i. Point identification of ψ in this setting requires at least triplets of observations (Cox 1958; Honoré and Tamer 2006). Here, we examine the profile score adjustment when only pairs of data are available.

Pairs of the form $y_i = (0,0)$ or $y_i = (1,1)$ give $\widehat{\lambda}_i(\psi) = -\infty$ and $\widehat{\lambda}_i(\psi) = +\infty$, respectively, with $l_i(\psi) = s_i(\psi) = 0$ for such pairs. The pairs $y_i = (0,1)$ give $\widehat{\lambda}_i(\psi) = 0$, $l_i(\psi) = -\log 4$, and $s_i(\psi) = 0$. Only the pairs $y_i = (1,0)$, for which $\widehat{\lambda}_i(\psi) = -\psi/2$, contribute the the profile likelihood. Let n_{01} and n_{10} be the number of $(0,1)$ and $(1,0)$ pairs. The profile log-likehood and score are

$$l(\psi) = -2n_{10} \log(1 + e^{\psi/2}),$$
$$s(\psi) = -n_{10}(1 + e^{-\psi/2})^{-1},$$

and the maximum likelihood estimator of ψ (assuming $n_{10} > 0$) is $-\infty$. The expectations of n_{01} and n_{10} are

$$E_{\psi,\lambda} n_{01} = \sum_i (1 + e^{\lambda_i})^{-1}(1 + e^{-\lambda_i})^{-1},$$
$$E_{\psi,\lambda} n_{10} = \sum_i (1 + e^{-\lambda_i})^{-1}(1 + e^{\lambda_i + \psi})^{-1}.$$

Evaluating these at $\lambda_i = \widehat{\lambda}_i(\psi)$, we obtain

$$E_{\psi,\widehat{\lambda}(\psi)} n_{01} = 4^{-1} n_{01} + (1 + e^{\psi/2})^{-1}(1 + e^{-\psi/2})^{-1} n_{10},$$
$$E_{\psi,\widehat{\lambda}(\psi)} n_{10} = 2^{-1}(1 + e^{\psi})^{-1} n_{01} + (1 + e^{\psi/2})^{-2} n_{10},$$

from which it follows that

$$E^{(k)}_{\psi,\widehat{\lambda}(\psi)} \begin{pmatrix} n_{01} \\ n_{10} \end{pmatrix} = B_\psi^k \begin{pmatrix} n_{01} \\ n_{10} \end{pmatrix}, \qquad k = 1, 2, \ldots,$$

where

$$B_\psi = \begin{pmatrix} 4^{-1} & (1 + e^{\psi/2})^{-1}(1 + e^{-\psi/2})^{-1} \\ 2^{-1}(1 + e^{\psi})^{-1} & (1 + e^{\psi/2})^{-2} \end{pmatrix}.$$

Hence, writing the profile score as

$$s(\psi) = a_\psi \begin{pmatrix} n_{01} \\ n_{10} \end{pmatrix}, \qquad a_\psi = \begin{pmatrix} 0 & -(1 + e^{-\psi/2})^{-1} \end{pmatrix},$$

the kth order adjusted profile score follows as

$$s_{\mathrm{a}}^{(k)}(\psi) = a_\psi (I - B_\psi)^k \begin{pmatrix} n_{01} \\ n_{10} \end{pmatrix}.$$

The eigenvalues of $I - B_\psi$ are less then one in absolute value, so $s_{\mathrm{a}}(\psi) = 0$ for every ψ. Unlike the case discussed in the previous example, the lack of point identification in this case results in $s_{\mathrm{a}}(\psi)$ being uninformative about ψ. However, the equation $s_{\mathrm{a}}^{(k)}(\psi) = 0$ has a unique solution, $\widehat{\psi}^{(k)}$, for every value of the ratio

n_{01}/n_{10}, and this solution converges as $k \to \infty$. The limit solution, derived in the Appendix, is

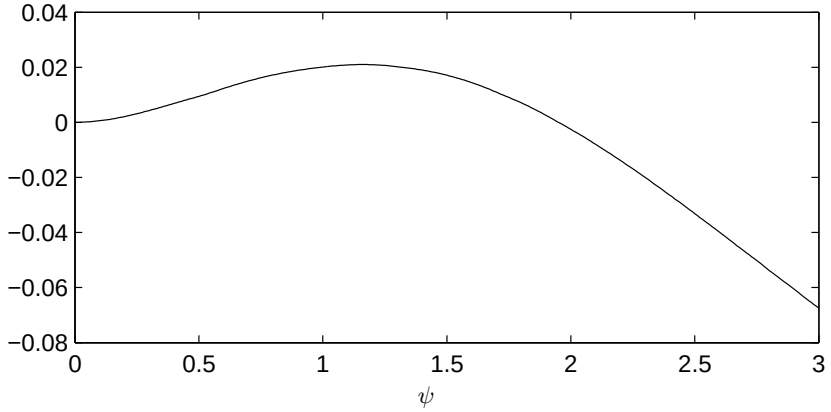
$$\widehat{\psi}_{\mathrm{a}} = g^{-1}(n_{01}/n_{10})$$

where

$$\begin{aligned} g(\psi) &= u_\psi + \sqrt{u_\psi^2 + v_\psi}, \\ u_\psi &= (1+e^\psi)(4^{-1} - (1+e^{\psi/2})^{-2}), \\ v_\psi &= 2(1+e^\psi)(1+e^{\psi/2})^{-1}(1+e^{-\psi/2})^{-1}. \end{aligned}$$

While inconsistent, $\widehat{\psi}_{\mathrm{a}}$ improves rather drastically on maximum likelihood. Figure 2 shows its asymptotic bias for the case where $\lambda_1, \lambda_2, \ldots$ are drawn independently from $N(0,1)$. The bias is uniformly small over the range of values of ψ considered.

Figure 2. Asymptotic bias in the autoregressive logit model, $m = 2$



Asymptotic bias of $\widehat{\psi}_{\mathrm{a}}$ when $\lambda_i \sim N(0,1)$.

Negative binomial regression. Let y_{ij} be a negative binomial random variable with mean $\mu_{ij} = \lambda_i \exp(x_{ij}^\top \psi_1)$ and variance $\mu_{ij} + \mu_{ij}^2/\psi_2$. The parameter $\psi_2^{-1} \geq 0$ is an overdispersion parameter, with $\psi_2 \to \infty$ yielding Poisson regression. The probability mass function is

$$f(y_{ij};\psi,\lambda_i) = \frac{\Gamma(\psi_2 + y_{ij})}{\Gamma(\psi_2)\,\Gamma(y_{ij}+1)} \left(\frac{\mu_{ij}}{\mu_{ij}+\psi_2}\right)^{y_{ij}} \left(\frac{\psi_2}{\mu_{ij}+\psi_2}\right)^{\psi_2}.$$

$\widehat{\lambda}_i(\psi)$ satisfies the equation $\sum_j g_{ij}(\psi, \widehat{\lambda}_i(\psi)) = 0$, where

$$g_{ij}(\psi,\lambda_i) = \frac{y_{ij} - \lambda_i \exp(x_{ij}^\top \psi_1)}{\psi_2 + \lambda_i \exp(x_{ij}^\top \psi_1)}, \qquad \lambda_i \geq 0.$$

This equation is equivalent to a mth order polynomial equation and has a unique nonnegative root. The profile score is

$$s(\psi) = \sum_{i,j} \begin{pmatrix} \psi_2 g_{ij}(\psi, \widehat{\lambda}_i(\psi)) x_{ij}, \\ \mathrm{psi}(\psi_2 + y_{ij}) - \mathrm{psi}(\psi_2) + \log\psi_2 - \log(\psi_2 + \widehat{\lambda}_i(\psi)\exp(x_{ij}^\top\psi_1)) \end{pmatrix},$$

with bias

$$E_{\psi,\lambda} s(\psi) = \sum_i \sum_{y_{i1}=0}^{\infty} \cdots \sum_{y_{im}=0}^{\infty} s_i(\psi) \prod_j f(y_{ij};\psi,\lambda_i).$$

For small m the bias can be computed directly. Both components of $E_{\psi,\lambda}s(\psi)$ are non-zero and depend on ψ, λ, and the covariate values.

Table 1 gives the result of a numerical computation, with the infinite sums in $E_{\psi,\lambda}s(\psi)$ truncated at 400, for the case $m = 2$, $\lambda_i = \psi_1 = \psi_2 = 1$, and $(x_{i1}, x_{i2}) = (0, \log 2)$. The overdispersion in this case is large, the means of y_{i1} and y_{i2} being 1 and 2, and the variances 2 and 6, respectively. We computed the finite-order adjusted profile score bias $E_{\psi,\lambda}s_{\mathrm{a}}^{(k)}(\psi)$ and $\mathrm{plim}_{n\to\infty}\widehat{\psi}^{(k)} = \arg\mathrm{solve}_{\psi^*}\{E_{\psi,\lambda}s_{\mathrm{a}}^{(k)}(\psi^*) = 0\}$ for $k = 0, \ldots, 5$. The row $k = 0$ shows that the profile score bias is very small for ψ_1 and large for ψ_2. Accordingly, the probability limits of the maximum likelihood estimates $\widehat{\psi}_1$ and $\widehat{\psi}_2$ are very close to ψ_1 and very different from ψ_2, respectively. This is in line with the simulation results of Allison and Waterman (2002), who found in various designs with $m = 2$ that there is hardly indication of incidental parameter bias for ψ_1, while the maximum likelihood estimator of ψ_2 was often infinity. The computation here suggests that the adjusted profile score is unbiased, and the adjusted score estimator consistent.

Table 1. Asymptotic biases in the negative binomial regression, $m = 2$

	asymptotic bias of $s_{\mathrm{a}}^{(k)}(\psi)$		asymptotic bias of $\widehat{\psi}_{\mathrm{a}}^{(k)}$	
k	ψ_1	ψ_2	ψ_1	ψ_2
0 (MLE)	.00424	.26554	−.0101	52.614
1	.00109	.05083	−.0123	.510
2	.00029	.01224	−.0031	.122
3	.00005	.00342	−.0012	.035
4	−.00003	.00097	−.0008	.010
5	−.00005	.00020	−.0007	.002

$\lambda_i = \psi_1 = \psi_2 = 1, (x_{i1}, x_{i2}) = (0, \log 2)$

3. SIMULATIONS

Table 2 presents the results of a small simulation exercise for the nonlinear models considered above where the adjusted profile score was obtained in closed form. For simulations in the linear autoregression, see Dhaene and Jochmans (2015). In all designs, we set $n = 500$ and $m = 2$. The regression models were run with a single regressor, generated as $x_{ij} \sim N(0, 1)$, and $\lambda_i \sim U(.5, 1.5)$. For the binary pairs, we generated data with $\lambda_i \sim N(0, 1)$. We set ψ as indicated in the table. The table reports the mean and standard deviation, estimated from 10,000 Monte Carlo replications, of the maximum likelihood estimator, $\widehat{\psi}$, and the adjusted score estimator, $\widehat{\psi}_{\mathrm{a}}$. The results derived above are confirmed. Whenever there is incidental-parameter bias, the adjusted profile score eliminates it, except in the probit binary-pair and the logit autoregressive-pair cases, where ψ is not point identified and a small bias remains, in line with the theory.

CONCLUSION

In models featuring incidental parameters, the profile score is often biased, which leads to inconsistent maximum-likelihood estimates under Neyman-Scott asymptotics. It is natural, then, to seek to remove this bias, as proposed by Neyman and Scott (1948) and McCullagh and Tibshirani (1990). In this paper, we propose to iterate the bias adjustment of Neyman and Scott (1948). The intuition for iterating the adjustment lies in asymptotic (i.e., large m) arguments. As the iteration proceeds, the bias of the adjusted

Table 2. Simulations, $n = 500, m = 2$ (10,000 replications)

		mean		std	
	ψ	$\widehat{\psi}$	$\widehat{\psi}_{\mathrm{a}}$	$\widehat{\psi}$	$\widehat{\psi}_{\mathrm{a}}$
Poisson regression	1	1.002	1.002	.051	.051
exponential regression	1	1.000	1.000	.055	.055
Weibull regression	1	1.000	1.000	.037	.037
	1.5	2.531	1.504	.095	.056
gamma regression	1	.999	.999	.042	.042
	1.5	2.759	1.508	.163	.084
inverse Gaussian regression	1	.999	.999	.051	.051
	1.5	3.018	1.509	.192	.096
binary matched pairs (logit)	1	2.017	1.009	.310	.155
binary matched pairs (probit)	1	2.070	1.072	.225	.124
binary autoregressive pairs (logit)	1	—	1.016	—	.246

regressions: $x_{ij} \sim N(0,1)$, $\lambda_i \sim U(.5, 1.5)$; binary pairs: $\lambda_i \sim N(0,1)$

profile score becomes less and less dependent on the incidental parameters, all the more so as m grows. It is not guaranteed that, with finite (and possibly very small) m, the full iteration, provided that the limit exists, delivers a consistent estimate under Neyman-Scott asymptotics. We find, however, that this is often the case, either because the profile-score bias is free of incidental parameters, so that no iteration is needed, or because the fully iterated adjusted profile score exists, is unbiased, and not identically zero.

APPENDIX

Weibull regression. Write the profile score as

$$s(\psi) = \sum_{i,j} \begin{pmatrix} \psi_2 \sum_j z_{ij} x_{ij} / \sum_j z_{ij} - \psi_2 x_{ij} \\ \psi_2^{-1} + \psi_2^{-1} \log z_{ij} - \psi_2^{-1} \sum_j z_{ij} \log z_{ij} / \sum_j z_{ij} \end{pmatrix}$$

with $z_{ij} = w_{ij}(\psi)\lambda_i^{-\psi_2} = (y_{it}/\mu_{it})^{\psi_2}$ being independent unit-exponential random variables. The first component of $s(\psi)$ has zero expectation, as in the exponential regression case. For the second component, write $\sum_j z_{ij} = z_{ij} + A$, where $A = \sum_{j' \neq j} z_{ij'}$ is independent of z_{ij} and is Erlang distributed with shape parameter $m-1$ and scale parameter 1. The density of A is $g_A(a) = a^{m-2}\exp(-a)/(m-2)!$, so

$$E\left(\frac{z_{ij}\log z_{ij}}{\sum_j z_{ij}}\right) = \int_0^\infty \int_0^\infty \frac{z \log z}{z+a} \exp(-z) \frac{a^{m-2}\exp(-a)}{(m-2)!}\, dz\, da = \frac{m-1-m\gamma}{m^2},$$

where γ is Euler's gamma. Setting $m = 1$ gives $E \log z_{ij} = -\gamma$. Hence the second component of $s(\psi)$ has expectation $n\psi_2^{-1}$. Now let $\psi' = (\psi_1, \psi_2')$, with arbitrary $\psi_2' > 0$. The first component of $s(\psi')$ is

$$s_{\psi_1}(\psi') = \sum_{i,j} \psi_2' \left(\frac{\sum_j z_{ij}' x_{ij}}{\sum_j z_{ij}'} - x_{ij} \right)$$

with $z_{ij}' = z_{ij}^{\psi_2'/\psi_2}$ and z_{ij} as above. It follows that $E_{\psi,\lambda} s_1(\psi') = 0$.

Gamma regression. Write the profile score as

$$s(\psi) = \sum_{i,j} \begin{pmatrix} \psi_2 \sum_j z_{ij} x_{ij} / \sum_j z_{ij} - \psi_2 x_{ij} \\ -\mathrm{psi}(\psi_2) + \log(m\psi_2) + \log z_{ij} - \log \sum_j z_{ij} \end{pmatrix},$$

$z_{ij} = y_{ij}/\mu_{ij}$ being independent gamma distributed random variables with shape parameter ψ_2 and scale 1. By the same argument as in the exponential regression model, the first component of $s(\psi)$ has zero expectation. Given that $E \log z_{ij} = \text{psi}(\psi_2)$ and that $\sum_j z_{ij}$ is gamma distributed with shape parameter $m\psi_2$ and scale 1, the second component of $s(\psi)$ has expectation $nm(\log(m\psi_2) - \text{psi}(m\psi_2))$. This expectation is $O(n)$ uniformly in m because $m(\log(m\psi_2) - \text{psi}(m\psi_2)) = (2\psi_2)^{-1} + O(m^{-1})$ as $m \to \infty$.

Inverse Gaussian regression. Write $\widehat{\mu}_{ij}$ as

$$\widehat{\mu}_{ij} = \mu_{ij}\frac{\widehat{\lambda}_i(\psi)}{\lambda_i} = \mu_{ij}\frac{\sum_j y_{ij}\lambda_i^{-2}\exp(-2x_{ij}^\top\psi_1)}{\sum_j \lambda_i^{-1}\exp(-x_{ij}^\top\psi_1)} = \mu_{ij}\frac{\sum_j y_{ij}\mu_{ij}^{-2}}{\sum_j \mu_{ij}^{-1}} = c^{-1}\mu_{ij}\sum_j z_{ij}$$

where $z_{ij} = y_{ij}\mu_{ij}^{-2}$ and $c = \sum_j \mu_{ij}^{-1}$. Denote the distribution of y_{ij} as $\mathcal{IG}(\mu_{ij}, \psi_2)$. Then, by properties of the inverse Gaussian distribution derived by Tweedie (1957),

$$z_{ij} \sim \mathcal{IG}(\mu_{ij}^{-1}, \mu_{ij}^{-2}\psi_2), \qquad \sum\nolimits_j z_{ij} \sim \mathcal{IG}\left(c, c^2\psi_2\right), \qquad \widehat{\mu}_{ij} \sim \mathcal{IG}\left(\mu_{ij}, \mu_{ij}c\psi_2\right),$$

and

$$Ey_{ij} = \mu_{ij}, \qquad Ey_{ij}^{-1} = \mu_{ij}^{-1} + \psi_2^{-1}, \qquad E\widehat{\mu}_{ij}^{-1} = \mu_{ij}^{-1} + \mu_{ij}^{-1}c^{-1}\psi_2^{-1}.$$

Hence

$$E\sum\nolimits_{i,j}(y_{ij}^{-1} - \widehat{\mu}_{ij}^{-1}) = n(m-1)\psi_2^{-1}. \tag{A.1}$$

We now calculate the expectation of z_{ij}/S^2, where $S = \sum_j z_{ij}$. The joint moment generating function of z_{ij} and S is

$$\begin{aligned}\text{MGF}_{z_{ij},S}(t_1, t_2) &= E\exp(t_1 z_{ij} + t_2 S)\\ &= \exp\left(\psi_2 c - \mu_{ij}^{-1}\sqrt{\psi_2(\psi_2 - 2t_1 - 2t_2)} - (c - \mu_{ij}^{-1})\sqrt{\psi_2(\psi_2 - 2t_2)}.\right)\end{aligned}$$

Following Cressie et al. (1981), we obtain

$$E\frac{z_{ij}}{S^2} = \int_0^\infty t_2 \lim_{t_1 \to 0}\frac{\partial}{\partial t_1}\text{MGF}_{z_{ij},S}(t_1, -t_2)dt_2 = \mu_{ij}^{-1}\frac{1 + c\psi_2}{c^3\psi_2}.$$

Hence, from $y_{ij}\widehat{\mu}_{ij}^{-2} = c^2 z_{ij} S^{-2}$, we have

$$E(y_{ij}\widehat{\mu}_{ij}^{-2} - \widehat{\mu}_{ij}^{-1}) = 0. \tag{A.2}$$

On writing the second component of $s(\psi)$ as

$$mn(2\psi_2)^{-1} - \sum_{i,j} 2^{-1}(y_{ij}\widehat{\mu}_{ij}^{-2} - \widehat{\mu}_{ij}^{-1} + y_{ij}^{-1} - \widehat{\mu}_{ij}^{-1}),$$

$E_{\psi,\lambda}s(\psi)$ follows from (A.1) and (A.2).

Binary matched pairs. Since g is symmetric about zero and unimodal, $\widehat{\lambda}_i(\psi) = -\psi/2$. The profile log-likelihood and score are

$$\begin{aligned}l(\psi) &= 2n_{01}\log G(\psi/2) + 2n_{10}\log G(-\psi/2),\\ s(\psi) &= n_{01}\frac{g(\psi/2)}{G(\psi/2)} - n_{10}\frac{g(\psi/2)}{G(-\psi/2)}\\ &= (n_{01} - Q(\psi/2)n_{10})\, c_\psi.\end{aligned}$$

Given

$$E_{\psi,\lambda} n_{01} = \sum_i G(-\lambda_i) G(\lambda_i + \psi),$$
$$E_{\psi,\lambda} n_{10} = \sum_i G(\lambda_i) G(-\lambda_i - \psi),$$

we have

$$E_{\psi,\widehat{\lambda}(\psi)} n_{01} = (n_{01} + n_{10})\alpha_{01}, \qquad \alpha_{01} = \alpha_{01}(\psi) = G(\psi/2)^2,$$
$$E_{\psi,\widehat{\lambda}(\psi)} n_{10} = (n_{01} + n_{10})\alpha_{10}, \qquad \alpha_{10} = \alpha_{10}(\psi) = G(-\psi/2)^2,$$

and, for general k,

$$\begin{aligned} E^{(k)}_{\psi,\widehat{\lambda}(\psi)} n_{01} &= \alpha_{01} E^{(k-1)}_{\psi,\widehat{\lambda}(\psi)} (n_{01} + n_{10}) = \alpha_{01}(\alpha_{01} + \alpha_{10})^{k-1}(n_{01} + n_{10}) \\ &= (\alpha_{01} + \alpha_{10})^{k-1} E_{\psi,\widehat{\lambda}(\psi)} n_{01}, \\ E^{(k)}_{\psi,\widehat{\lambda}(\psi)} n_{10} &= (\alpha_{01} + \alpha_{10})^{k-1} E_{\psi,\widehat{\lambda}(\psi)} n_{10}. \end{aligned}$$

Therefore, with $s(\psi)$ witten as

$$s(\psi) = \beta_{01} n_{01} + \beta_{10} n_{10}, \qquad \beta_{01} = \beta_{01}(\psi) = \frac{g(\psi/2)}{G(\psi/2)}, \qquad \beta_{10} = \beta_{10}(\psi) = \frac{g(\psi/2)}{G(-\psi/2)},$$

we have

$$\begin{aligned} E^{(k)}_{\psi,\widehat{\lambda}(\psi)} s(\psi) &= (\alpha_{01} + \alpha_{10})^{k-1}(\beta_{01} E_{\psi,\widehat{\lambda}(\psi)} n_{01} - \beta_{10} E_{\psi,\widehat{\lambda}(\psi)} n_{10}) \\ &= (\alpha_{01} + \alpha_{10})^{k-1}(\alpha_{01}\beta_{01} - \alpha_{10}\beta_{10})(n_{01} + n_{10}) \end{aligned}$$

and

$$\begin{aligned} s^{(k)}_{\mathrm{a}}(\psi) &= s(\psi) - \sum_{p=1}^{k} \binom{k}{p} (-1)^{p-1} (\alpha_{01} + \alpha_{10})^{p-1}(\alpha_{01}\beta_{01} - \alpha_{10}\beta_{10})(n_{01} + n_{10}) \\ &= s(\psi) - \left(1 - (1 - \alpha_{01} - \alpha_{10})^k\right) \left(\frac{\alpha_{01}\beta_{01} - \alpha_{10}\beta_{10}}{\alpha_{01} + \alpha_{10}}\right) (n_{01} + n_{10}). \end{aligned}$$

Given that $0 < \alpha_{01} + \alpha_{10} < 1$, we obtain

$$\begin{aligned} s_{\mathrm{a}}(\psi) &= s(\psi) - \left(\frac{\alpha_{01}\beta_{01} - \alpha_{10}\beta_{10}}{\alpha_{01} + \alpha_{10}}\right) (n_{01} + n_{10}) \\ &= \left(n_{01} - Q(\psi/2)^2 n_{10}\right) d_\psi. \end{aligned}$$

Binary autoregressive pairs. Write B_ψ as

$$B_\psi = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a = 4^{-1}$ and b, c, d are functions of ψ, and decompose $I - B_\psi$ as $P\Delta P^{-1}$, where

$$\Delta = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}, \qquad \delta_1 = 1 - (a + d - D)/2, \qquad \delta_2 = 1 - (a + d + D)/2,$$
$$P = \begin{pmatrix} (a - d - D)/(2c) & (a - d + D)/(2c) \\ 1 & 1 \end{pmatrix},$$
$$P^{-1} = \begin{pmatrix} -c/D & (a - d + D)/(2D) \\ c/D & -(a - d - D)/(2D) \end{pmatrix},$$

and $D = \sqrt{(a-d)^2 + 4bc}$. Note that $1 > \delta_1 > \delta_2 > 0$. For given k, $\widehat{\psi}^{(k)}_{\mathrm{a}}$ solves

$$a_\psi (I - B_\psi)^k \begin{pmatrix} n_{01} \\ n_{10} \end{pmatrix} = 0,$$

where the first element of a_ψ is zero. Given $(I - B_\psi)^k = P\Delta^k P^{-1}$, this equation is equivalent to

$$\delta_1^k \, (g - n_{01}/n_{10}) + \delta_2^k \, (h + n_{01}/n_{10}) = 0$$

where $g = (a - d + D)/(2c)$ and $h = -(a - d - D)/(2c)$. As $k \to \infty$, the first term dominates. Hence, the limiting solution solves $g = n_{01}/n_{10}$, that is, $g(\psi) = n_{01}/n_{10}$, on writing g as a function of ψ.

REFERENCES

Allison, P. D. and R. P. Waterman (2002). Fixed-effects negative binomial models. *Sociological Methodology 32*, 247–266.

Andersen, E. B. (1970). Asymptotic properties of conditional maximum-likelihood estimators. *Journal of the Royal Statistical Society, Series B 32*, 283–301.

Arellano, M. and S. Bonhomme (2009). Robust priors in nonlinear panel data models. *Econometrica 77*, 489–536.

Barndorff-Nielsen, O. (1983). On a formula for the distribution of the maximum likelihood estimator. *Biometrika 70*, 343–365.

Blundell, R., R. Griffith, and F. Windmeijer (1999). Individual effects and dynamics in count data models. The Institute for Fiscal Studies, Working Paper No. W99/3.

Chamberlain, G. (1980). Analysis of covariance with qualitative data. *Review of Economic Studies 47*, 225–238.

Chamberlain, G. (1985). Heterogeneity, omitted variable bias, and duration dependence. In J. J. Heckman and B. Singer (Eds.), *Longitudinal Analysis of Labor Market Data*. Cambridge University Press.

Chamberlain, G. (2010). Binary response models for panel data: Identification and information. *Econometrica 78*, 159–168.

Cox, D. (1958). Two further applications of a model for binary regression. *Biometrika 45*, 562–565.

Cox, D. R. and N. Reid (1987). Parameter orthogonality and approximate conditional inference. *Journal of the Royal Statistical Society, Series B 49*, 1–39.

Cressie, N., A. S. Davis, J. L. Folks, and G. E. Policello II (1981). The moment-generating function and negative integer moments. *The American Statistician 35*, 148–150.

Dhaene, G. and K. Jochmans (2015). Likelihood inference in an autoregression with fixed effects. Unpublished manuscript.

DiCiccio, T. J., M. A. Martin, S. E. Stern, and A. Young (1996). Information bias and adjusted profile likelihoods. *Journal of the Royal Statistical Society, Series B 58*, 189–203.

Hahn, J. (1997). A note on the efficient semiparametric estimation of some exponential panel models. *Econometric Theory 13*, 583–588.

Hahn, J. and W. K. Newey (2004). Jackknife and analytical bias reduction for nonlinear panel models. *Econometrica 72*, 1295–1319.

Honoré, B. E. and E. Tamer (2006). Bounds on parameters in panel dynamic discrete choice models. *Econometrica 74*, 611–629.

Kalbfleisch, J. D. and D. A. Sprott (1970). Application of likelihood methods to models involving large numbers of parameters. *Journal of the Royal Statistical Society, Series B 32*, 175–208.

Lancaster, T. (2000). The incidental parameter problem since 1948. *Journal of Econometrics 95*, 391–413.

Lancaster, T. (2002). Orthogonal parameters and panel data. *Review of Economic Studies 69*, 647–666.

Li, H., B. Lindsay, and R. Waterman (2003). Efficiency of projected score methods in rectangular array asymptotics. *Journal of the Royal Statistical Society, Series B 65*, 191–208.

McCullagh, P. and R. Tibshirani (1990). A simple method for the adjustment of profile likelihoods. *Journal of the Royal Statistical Society, Series B 52*, 325–344.

Neyman, J. and E. L. Scott (1948). Consistent estimates based on partially consistent observations. *Econometrica 16*, 1–32.

Nickell, S. (1981). Biases in dynamic models with fixed effects. *Econometrica 49*, 1417–1426.

Pace, L. and A. Salvan (2006). Adjustments of profile likelihood from a new perspective. *Journal of Statistical Planning and Inference 136*, 3554–3564.

Portnoy, S. (1988). Asymptotic behavior of likelihood methods for exponential families when the number of parameters tends to infinity. *Annals of Statistics 16*, 356–366.

Rasch, G. (1961). On general laws and the meaning of measurement in psychology. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Vol. 4*, pp. 321–333. University of California Press.

Sartori, N. (2003). Modified profile likelihood in models with stratum nuisance parameters. *Biometrika 90*, 533–549.

Severini, T. A. (1998). An approximation to the modified profile likelihood function. *Biometrika 85*, 403–411.

Severini, T. A. (2000). *Likelihood Methods in Statistics*. Oxford Statistical Science Series. Oxford University Press.

Tweedie, M. (1957). Properties of inverse gaussian distributions I. *Annals of Mathematical Statistics 28*, 362–377.