The emergence of money: a dynamic analysis

Maurizio Iacopetta

Macroeconomic Dynamics, **23**, 2019, 2573–2596. Printed in the United States of America.
doi:10.1017/S1365100517000815

ARTICLES

THE EMERGENCE OF MONEY: A DYNAMIC ANALYSIS

MAURIZIO IACOPETTA
SKEMA Business School, Université Côte d'Azur (GREDEG)
and
OFCE, Sciences-Po Paris

This paper studies the role of liquidity in triggering the emergence of money in a Kiyotaki-Wright economy. A novel method computes the dynamic Nash equilibria of the economy by setting up an iteration of the agents' profile of (pure) strategies and of the distribution of commodities across agents. The analysis shows that the evolving state of liquidity can spark the acceptance of a high-cost-storage commodity as money or cause the disappearance of a commodity money. It also reveals the existence of multiple dynamic equilibria with pure strategies. Several simulations clarify how history and the coordination of beliefs matter for the selection of a particular equilibrium.

Keywords: Speculative Strategy, Dynamic Nash Equilibrium, Liquidity Differential

1. INTRODUCTION

In their well-known paper "On Money as a Medium of Exchange," Kiyotaki and Wright (1989) (KW henceforth) study the conditions for the existence of a steady-state Nash equilibrium in which a low-return object is accepted as money. The literature, however, has yet to clarify under what conditions the economy could converge to such an equilibrium and what may the set of strategies look like along the transition. Does convergence occur from any initial condition? If a low-return object plays the role of money in the steady state, does it also do so along a dynamic equilibrium? Conversely, can such an object be used as money temporarily and then permanently lose this function? To answer these types of questions, this paper develops a methodology that generates dynamic Nash equilibria in a generalized KW environment.

I am grateful to Federico Bonetto for his help at several stages of the paper. Two anonymous Referees, and an anonymous Associate Editor of this journal gave me valuable suggestions on how to clarify the analysis and to simplify the exposition. I wish also to thank Zakaria Babutsidze, Nobuyuki Hanaki, and seminar and conference participants at the 2016 Meetings of the Society for Computational Economics Conference (Bordeaux), Luiss University (Rome), and University of Göttingen for comments. All remaining errors are mine. Address correspondence to: Maurizio Iacopetta, SKEMA Business School, 60 Rue Fedor Dostoievski, 06902 Valbonne, France; e-mail: maurizio.iacopetta@skema.edu

 1365-1005/17

The algorithm builds directly on the concept of Nash equilibrium with many players as in Fudenberg and Levine (1988). The main idea is to obtain an equilibrium such that, given the actions of all other players, no player can make any gain by changing his action. The algorithm creates a sequence of rounds with the aim of finding a convergence toward an equilibrium for the distribution of assets and trading strategies. In particular, each round of the iterative scheme consists of two steps. In the first step, given the initial state of the economy, it derives a time pattern of the distribution of goods that results from decentralized meetings, under an educated guess on the profile of trading strategies. In the second step, the algorithm verifies whether any agent has an incentive to deviate from such a guess on the profile of strategies. The follow up round uses the information on the deviation gains, if any, to formulate a new guess.

The analysis sheds new light on the issue of the emergence of money. The steady-state analysis would suggest that the conditions for an economy to adopt a low-return object as money are related to the cost of storing commodities, the matching rate, and the specification of agents' utility. The dynamic analysis yields a different insight: It is the evolution of the degree of the liquidity of commodities that may induce some agents to accept a low-return commodity as money.[1] The paper also discusses equilibria where the reverse phenomenon occurs: A low-return commodity is initially used as money, and then, as the economy moves toward its long-run equilibrium, it loses this function due to a gradual decline in its degree of marketability.

The algorithm is also useful to explore issues of multiple equilibria because it generates patterns starting from any arbitrary initial condition and because it admits multiple solutions. It is known that one version of the KW economy, called Model B, exhibits multiple pure strategies steady states. Little is known, however, about the initial conditions that could lead to any of these states. As Krugman (1991), Matsuyama (1991), and Fukao and Benabou (1993) note, the presence of multiple steady states may or may not be associated with multiple equilibria.[2] The algorithm reveals that in fact in Model B there are multiple dynamic equilibria.

Previous studies have investigated the dynamics of KW with different approaches. Marimon et al. (1990) and Başçı (1999) explore the question of whether artificially intelligent agents can learn to play equilibrium strategies. A similar question is tested in a number of controlled laboratory experiments with real people [Brown (1996), Duffy and Ochs (1999, 2002)]. Matsuyama et al. (1993), Wright (1995), Luo (1999), and Sethi (1999) approach the issue through evolutionary dynamics. Duffy (2001) studies the fitness of a hypothetical learning algorithm against the outcome of laboratory experiments. In contrast to these studies, here the rational expectations hypothesis is maintained throughout the dynamics. Each agent views the strategies used by others as being beyond his control. The algorithm aims at revealing equilibria in Markov strategies, in which strategies are permitted to depend only on the current cross-agent distribution of goods, and from any arbitrary initial condition.

Some have focussed on the dynamics of KW with mixed strategies. Kehoe et al. (1993) showed that the model features a large multiplicity of dynamic equilibria that includes cycles, sunspots, and other non-Markovian equilibria. Renero (1998), however, argued that it is hard to find an initial condition from which an equilibrium pattern converges to a mixed strategy steady-state equilibrium. More recently, Oberfield and Trachter (2012) found that, in a symmetric environment, as the frequency of search increases, cycles and multiplicity in mixed strategy tend to disappear. The algorithm discussed in this paper does not explicitly address this debate, as it is designed to study the emergence of money in a pure strategy environment.

The remainder of the paper is organized as follows. The next section briefly describes the economic environment, illustrates the evolution of the distribution of inventories under a given profile of strategies, and defines the best response function. The section that follows studies the properties of the dynamical system. Section 4 describes the numerical algorithm. Sections 5 provides simulations that illustrate the emergence and disappearance of money, and discusses multiple dynamic equilibria in Model A and Model B of KW. Section 6 concludes.

2. THE MODEL ECONOMY

The economy is essentially the same as that described in KW, except for three differences. First, agents are not necessarily equally divided among the three types. Second, to facilitate the description of the dynamics, time is continuous and the matching technology is governed by a Poisson process. Third, the ranking of the storage costs across the three goods may change.[3] The overall size of the population is N. There are three types of infinitely lived agents, indexed by N_i, where $i = 1, 2, 3$. The fraction of each type is $\theta_i = \frac{N_i}{N}$. A type i agent consumes only good i and can produce only good $i + 1$ (mod. 3).[4] Production occurs immediately after consumption. Agent i's instantaneous utility from consuming a unit of good i and the disutility of producing good $i + 1$ are denoted by U_i and D_i, respectively, with $U_i > D_i > 0$, and their difference is $u_i = U_i - D_i$. Restrictions on U_i, D_i, and c_i are imposed so that no agent wants to drop out of the economy, no one finds it optimal to dispose of a commodity, and an individual always wants to consume his preferred good and produce a new one rather than holding it.

At each instant in time, an individual can hold one and only one unit of any type i good at a cost c_i, measured in units of utility.[5] The discount rate is denoted by $\rho > 0$. A pair of agents is randomly and uniformly chosen from the population to meet for a possible trade. After a pair is formed, the waiting time for the next pair to be called is governed by a Poisson process of parameter α. A bilateral trade occurs if, and only if, it is mutually agreeable. Agent i always accepts good i but never holds it because, provided that u_i is sufficiently large, there is immediate consumption (see KW, Lemma 1, p. 933). Therefore, agent i enters the market with either one unit of good $i + 1$ or one unit of good $i + 2$.

The proportion of all agents of type i that hold good j at time t is denoted by $p_{i,j}(t)$.[6] Then, the vector $\tilde{\mathbf{p}}(t) = \{p_{i,j}(t)\}$, for $i = 1, 2, 3$, and $j = 1, 2, 3$, describes the state of the economy at time t (henceforth, i and j go from 1 to 3). However, because $p_{i,i}(t) = 0$,

$$p_{i,i+1}(t) + p_{i,i+2}(t) = \theta_i, \tag{1}$$

for any $t > 0$, and the state of the economy can be represented in a more parsimonious way by $\mathbf{p}(t) = [p_{1,2}(t), p_{2,3}(t), p_{3,1}(t)]$. An individual i has only to decide whether to exchange his production good $i + 1$ for good $i + 2$. Agent i's choice in favor of such a trade (called "indirect trade") is denoted with $\sigma_i(t) = 1$ and that against it with $\sigma_i(t) = 0$. Agent i has to select a time path, $\sigma_i(t)$, that maximizes his expected stream of present and future net utility, given other agents' strategy paths, $\mathbf{s}(t) = [s_1(t), s_2(t), s_3(t)]$, and $\mathbf{p}(t)$, for any $t > 0$, where $s_i(t)$ has the same interpretation as $\sigma_i(t)$ but refers to the symmetric strategies of the ensemble of type i agents.[7]

2.1. Distribution of Assets and Value Functions

For a given profile of strategies, $\mathbf{s}(t)$, the evolution in the stock of good $i + 1$ held by agents of type i is given by

$$\dot{p}_{i,i+1} = \alpha\{p_{i,i+2}[p_{i+1,i}(1 - s_{i+1}) + p_{i+2,i} + p_{i+2,i+1}(1 - s_i)] - p_{i,i+1}p_{i+1,i+2}s_i\}. \tag{2}$$

The terms inside the brackets before the minus sign calculate the frequency with which type i agents are called for a match while holding good $i + 2$ and end up with good $i + 1$. Specifically, $p_{i+1,i}(1 - s_{i+1})$ is the probability of acquiring good i from type $i + 1$ agents—the good is then consumed and good $i + 1$ is immediately produced afterward, $p_{i+2,i}$ is the probability of acquiring the consumption good i from type $i + 2$ agents, and $p_{i+2,i+1}(1 - s_i)$ is the probability of obtaining good $i + 1$ from type $i + 2$ agents. Finally, the term $p_{i,i+1}p_{i+1,i+2}s_i$ accounts for the probability that an agent of type i, who holds good $i + 1$, ultimately has good $i + 2$. The behavior of $p_{i,i+2}$ is derived through (1). The system that describes the evolution of the distribution of inventories is denoted by $F(\mathbf{p}(t))$.

Consider now a representative agent of type i who has to compute her best profile of strategies, given a pattern of inventories, $\mathbf{p}(t)$, and a pattern of strategies for other agents, $\mathbf{s}(t)$—including those of her own type. Let $V_{i,j}(t)$ be the value function when holding good j at time t. When $j = i + 1$, we have that

$$\begin{aligned} V_{i,i+1}(t) = \max_{\{\sigma_i(\upsilon)\}_{\upsilon \geq t}} \int_t^{\infty} \alpha e^{-\alpha(\upsilon - t)} \Bigg(& e^{-\rho(\upsilon-t)}\{[p_{i,i+2}\sigma_i(1 - s_i) + p_{i+1,i+2}\sigma_i]V_{i,i+2} \\ & + [1 - p_{i,i+2}\sigma_i(1 - s_i) - p_{i+1,i+2}\sigma_i]V_{i,i+1} + (p_{i+1,i} + p_{i+2,i}s_{i+2})u_i\} \\ & - \frac{1 - e^{-(\upsilon - t)\rho}}{\rho} c_{i+1}\Bigg) d\upsilon, \end{aligned} \tag{3}$$

where the term $\alpha e^{-\alpha(\upsilon-t)}d\upsilon$ measures the probability that an agent of type i is called to form a match for the first time after time t in the time interval $(\upsilon, \upsilon + d\upsilon)$. The term after the discount factor is the probability that this agent engages in indirect trading—in which case, she is left with $V_{i,i+2}$ as his continuation value. Otherwise, she ultimately has good $i+1$ either because no trade occurs or because she acquired her consumption good—an event that occurs with probability $p_{i+1,i} + p_{i+2,i}s_{i+2}$—and then produces good $i+1$. The last term is the cost on holding good $i+1$ from time t to time υ. The appendix derives a similar expression for $V_{i,i+2}(t)$.

Let $\Delta_i(\upsilon) \equiv V_{i,i+1}(\upsilon) - V_{i,i+2}(\upsilon)$, and let $\tilde{\sigma}_i(\upsilon; \mathbf{s}(\upsilon), \mathbf{p}(\upsilon))$ denote the optimal (or best) response profile of strategies of representative agent i to other players' strategies, $\mathbf{s}(\upsilon)$, and the pattern of inventories, $\mathbf{p}(\upsilon)$, for $\upsilon > t$. Then, it must be that

$$\tilde{\sigma}_i(\upsilon;\ \mathbf{s}(\upsilon),\ \mathbf{p}(\upsilon)) = \begin{cases} 1 & \text{if } \Delta_i(\upsilon) < 0 \\ 0 & \text{if } \Delta_i(\upsilon) > 0 \end{cases}, \tag{4}$$

for any $\upsilon \geq t$. In case $\Delta_i(\upsilon) = 0$, the agent is indifferent between the two alternatives, and the choice is taken at random. Hence, the formulation of the problem corresponds to a Markov decision process in which the representative agent optimizes over a sequence of functions, $\tilde{\sigma}_i(.)$. The disutility of production is large enough that it is never optimal to throw good $i+2$ away to produce good $i+1$, that is $D_i > -\Delta_i(t)$, for any $t > 0$ (see KW, p. 939 for a similar restriction).

2.2. Definition of Dynamic Nash Equilibrium

Given an initial distribution of inventories, $\mathbf{p}(0) = \mathbf{p}_0$, a *Dynamic Nash Equilibrium (DNE)* is a path of strategies, $\mathbf{s}^*(t)$, together with a distribution of inventories, $\mathbf{p}^*(t)$, such that for all $t > 0$:

i. $\mathbf{p}^*(t)$ and $\mathbf{s}^*(t)$ satisfy the dynamics equations (2) with the initial condition $\mathbf{p}^*(0) = \mathbf{p}_0$ and subject to the constraint (1);
ii. for all $t > 0$, every agent maximizes his or his expected utility given the strategy profiles of the rest of the population; and
iii. $\tilde{\sigma}_i(t; \mathbf{s}^*(t), \mathbf{p}^*(t)) = s_i^*(t)$ for all $t > 0$.

To obtain the DNE, it is useful first to explore the evolution of the distribution of inventories, $\mathbf{p}(t)$, for any arbitrary set of time-constant strategies, $\mathbf{s}$, and then to search for all the possible steady-state Nash equilibria.

3. CONVERGENCE TO A STEADY STATE

It is common to compute Nash equilibria by studying simultaneously the evolution of state and choice variables. In this environment, where there is a multiplicity of state and choice variables, it is easier to follow an alternative two-step procedure. First, consider the behavior of the assets $\mathbf{p}(t)$ for a set of strategies $\mathbf{s}(t)$ that is not necessarily the one that supports the Nash equilibrium. Then, verify which, among the different strategies, does support a Nash equilibrium.

Let $\mathbf{s}(t)$ be an *arbitrary* given profile of strategies. Following equation (2), the evolution of the distribution of assets, $\mathbf{p}(t)$, is (the time index is dropped)

$$\dot{p}_{1,2} = \alpha\{p_{1,3}[p_{2,1}(1-s_2) + p_{3,1} + p_{3,2}(1-s_1)] - p_{1,2}p_{2,3}s_1\}, \tag{5}$$

$$\dot{p}_{2,3} = \alpha\{p_{2,1}[p_{3,2}(1-s_3) + p_{1,2} + p_{1,3}(1-s_2)] - p_{2,3}p_{3,1}s_2\}, \tag{6}$$

$$\dot{p}_{3,1} = \alpha\{p_{3,2}[p_{1,3}(1-s_1) + p_{2,3} + p_{2,1}(1-s_3)] - p_{3,1}p_{1,2}s_3\}. \tag{7}$$

PROPOSITION 1. *Under any time-constant profile of strategies,* $\mathbf{s}$*, with the possible exception of* $\mathbf{s}=(1,1,1)$*,* $\mathbf{p}(t)$ *converges to a stationary distribution,* $\mathbf{p}$*, from any initial* $p_{i,i+1}(0)$ *that satisfies* $0 \leq p_{i,i+1}(0) \leq \theta_i$ *and (1), for* $i = 1, 2, 3$.[8]

Proof. See the appendix.

3.1. Steady States

The conditions under which a stationary distribution is a Nash equilibrium depends, among other things, on the relative size of the three groups of agents and the ordering of the storage cost, c_i. To facilitate the comparison with KW, consider for now the case in which there is an equal share of the population of each type: $\theta_i = \frac{1}{3}$, for $i = 1, 2, 3$. Even under such a restriction, a number of steady states may arise, depending on the ordering of the storage cost. In general, for a given steady-state distribution to be a Nash equilibrium, the sign of Δ_i has to be consistent with the profile of strategies assumed for that particular steady-state distribution, namely $\Delta_i > 0$ when $s_i = 0$ and $\Delta_i < 0$ when $s_i = 1$.

Model A. When $c_1 < c_2 < c_3$, which corresponds to the so-called Model A of KW, the steady-state (pure strategies) equilibrium is given by

$$\mathbf{s} = (0, 1, 0) \text{ and } \mathbf{p} = \frac{1}{3}\left(1, \frac{1}{2}, 1\right), \tag{8}$$

if

$$\frac{c_3 - c_2}{u_1\alpha} > p_{3,1} - p_{2,1} = \frac{1}{6} \tag{9}$$

and by

$$\mathbf{s} = (1, 1, 0) \text{ and } \mathbf{p} = \frac{1}{3}\left(\frac{1}{2}\sqrt{2}, \sqrt{2} - 1, 1\right),$$

if

$$\frac{c_3 - c_2}{u_1\alpha} < p_{3,1} - p_{2,1} = \frac{\sqrt{2}}{3}(\sqrt{2} - 1) \tag{10}$$

where the $p_{i,j}$ in the inequalities (9) and (10) are evaluated in the respective steady states. Following KW, these two equilibria are called the fundamental and speculative equilibrium, respectively, or as (0,1,0) and (1,1,0) equilibrium. Table 1

TABLE 1. Steady-state equilibria, Model A

Strategies		Inventories		Money	
F	S	F	S	F	S
(0,1,0)	(1,1,0)	$\frac{1}{3}(1, \frac{1}{2}, 1)$	$\frac{1}{3}(\frac{\sqrt{2}}{2}, \sqrt{2}-1, 1)$	1	1,3

Note: The table reports the set of strategies, the distribution of inventories, and the type of good that plays the role of money, under the fundamental (F) and the speculative (S) steady-state equilibrium for a Model A economy ($c_1 < c_2 < c_3$).

summarizes the main features of the two equilibria and the appendix details the derivation of (9) and (10).

4. THE ALGORITHM

The preceding section established that from any initial condition, there exists a time-constant strategy profile that converges to a steady-state NE. Proposition 1 ensures, for instance, that the economy that follows the (0,1,0) set of strategies eventually converges to a stationary distribution that coincides with the steady-state distribution implied by the fundamental strategy. A constant strategy profile, however, needs not be an NE all along the dynamic pattern. Along the transition path, one or more groups of agents might find optimal to switch trading strategies, possibly multiple times. To address this issue, the algorithm builds directly on the concept of open-loop NE with many players as in Fudenberg and Levine (1988).[9]

The law of motion of the state variable, $\mathbf{p}(t)$, is a function of the profile of strategies, $\mathbf{s}(t)$, (both are multidimensional objects) that can change over time in discrete steps. Agents use the value functions $V_{i,j}$ as criteria to select their optimal patterns of strategies. These can be derived analytically in the steady state but not along a transition path. The algorithm computes the NE policies and the distribution of goods iteratively. It uses two properties of the system. One property was demonstrated by Proposition 1: For any interesting profile of strategies, the state variable, $\mathbf{p}(t)$, converges toward a fixed point, not necessarily an NE. The second property is that along a given pattern of $\mathbf{p}(t)$, the numerical value functions converge to their theoretical values when integrated backward in time starting from a neighborhood of the steady state. According to (4), what matters for any agent i's decision is only the *sign* of Δ_i. After some algebra (see the appendix), one obtains

$$\dot{\Delta}_i = (\alpha\chi_i + \rho)\Delta_i + \omega_i, \tag{11}$$

where $\chi_i \equiv p_{i,i+2}\sigma_i(1-s_i) + p_{i+1,i+2}\sigma_i + p_{i+1,i}(1-s_{i+1}) + p_{i+2,i} + (p_{i,i+1}s_i + p_{i+2,i+1})(1-\sigma_i) > 0$ and $\omega_i \equiv -\alpha[p_{i+1,i}s_{i+1} - p_{i+2,i}(1-s_{i+2})]u_i - (c_{i+2} - c_{i+1})$. For a given pattern of χ_i, the solution of Δ_i can be obtained numerically by integrating (11) backward in time, starting from a neighborhood of the steady state Δ_i^*, where this satisfies $(\alpha\chi_i + \rho)\Delta_i^* + \omega_i = 0$.

These two properties suggest that one can verify the consistency of the value function of any particular agent of type i along a specific trajectory, $\mathbf{p}(t)$, with the profile of strategies that are used to obtain such a trajectory, $\mathbf{p}(t)$.

4.1. Iteration on the Profile of Strategies

The algorithm sets up an iteration on the profile of strategies, $\mathbf{s}(t)$, and on the distribution of assets, $\mathbf{p}(t)$. The value function, $V_{i,j}(t)$, of an agent of type i holding good j serves as a device to update the guess on the profile of strategies and to determine when the algorithm has converged. As only pure strategies are considered, an agent i has a binary choice at each point in time.[10] The algorithm seeks the convergence of the best response $\tilde{\sigma}_i(t;\ \mathbf{s}(t),\ \mathbf{p}_0)$ of a particular agent i, named a_i, to the profile followed by other individuals of his type, $s_i(t)$. When a_i does not have an interest in deviating from a strategy and this coincides with that followed by the rest of type i agents, a DNE is found. The algorithm works in three steps.

Step 1. It integrates the distribution of inventories, $F(\mathbf{p}(t))$, from some $\mathbf{p}_0$, under a guess $\mathbf{s}^{(0)}(t)$. The integration is stopped at some time T sufficiently large that $|F(\mathbf{p}(T))| < 10^{-6}$. An obvious initial guess is $\mathbf{s}^{(0)}(t) = \mathbf{s}^{ss}$, where $\mathbf{s}^{ss}$ is the steady-state Nash profile of strategies. (For some $\upsilon > \bar{\upsilon}$ with $\bar{\upsilon}$ sufficiently large, one can expect that $\tilde{\sigma}_i(\upsilon) = s_i^{(0)}(\upsilon)$ for $\upsilon > \bar{\upsilon}$.) Let $\mathbf{p}^{(0)}(t)$ be the inventory solution under such a guess.

Step 2. It computes the best response of a_i on the trajectory $\mathbf{p}^{(0)}(t)$. Her Δ_i is computed integrating (11) backward in time, beginning from the initial condition $[\Delta_i(\mathbf{s}^{ss}, \mathbf{p}^{(0)}(T)),\ \mathbf{p}^{(0)}(T)]$.[11] At the end of this step, the algorithm delivers a trajectory, $\Delta^{(0)}(t)$, and more importantly, the corresponding best response of a_i, $\tilde{\sigma}_i^{(0)}(t)$.

Step 3. It verifies the consistency between $s_i^{(0)}(t)$ and $\tilde{\sigma}_i^{(0)}(t)$. If these are different, $\tilde{\sigma}_i^{(0)}(t)$, becomes the new guess in the next round, namely $s_i^{(1)}(t) = \tilde{\sigma}_i^{(0)}(t)$, and the procedure restarts from step one. The method allows the profile of strategies to change at any point in time.

The algorithm repeats the iteration until convergence between $s_i^{(n+1)}(t)$ and $\tilde{\sigma}_i^{(n)}(t)$ is achieved or until a maximum number of iterations is reached. If the iteration converges to a fixed point, say $\mathbf{p}^*(t)$ and $\mathbf{s}^*(t)$, then $\mathbf{p}^*(t)$ and $\mathbf{s}^*(t)$ are the distribution of assets and the trading strategies, respectively, of a Markov-perfect NE. The procedure verifies that at such a fixed point, the value function of any agent is at its maximum value, given the actions of the rest of the agents.[12]

5. NUMERICAL EXAMPLES AND EXTENSIONS

This section provides some applications of the algorithm. It shows how the emergence and the disappearance of money may occur along the dynamic equilibrium. It also discusses the relationship between multiple equilibria and multiple steady states. Finally, it illustrates the dynamics of an extension of KW [Wright (1995)] where agents are unevenly distributed across the three types.

TABLE 2. Baseline parameters

	Population			Utility	Storage costs		
	θ_1	θ_2	θ_3	u_i	c_1	c_2	c_3
A1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	100	1	4	9
A2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	20	1	4	9
A3	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$	100	1	4	9
A4	0.2	0.2	0.6	100	0	1	10
B	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	100	9	4	1

Note: The parameters in A1–A3 and B are taken from Duffy and Ochs (1999, Table 1, p. 851) and in A4 are similar to Wright [1995, Fig. 1(b) and (c)]. In all economies $\delta = 0.1$ and $\alpha = 1$.

5.1. The Emergence of Money

Section 3.1 clarified that for the existence of a speculative steady-state equilibrium, where commodity 3 plays the role of money, $p_{2,1} - p_{3,1} < \frac{c_2 - c_3}{\alpha u_1}$ [condition (10)] must hold. This section argues, however, that such a condition conveys limited information to evaluate the existence of commodity money when the economy is not in steady state. First, there are cases for which $p_{2,1} - p_{3,1} > \frac{c_2 - c_3}{\alpha u_1}$ and yet Nash speculative strategies are played along the dynamic equilibrium. Second, and more importantly, commodity 3 may not be accepted as money in a dynamic equilibrium, despite the validity of (10) in steady state. The main result is that money emerges endogenously along the transition. Such a process is in line with Menger (1892)'s idea that the emergence of money is triggered by changes in the relative marketability of commodities: "When the relatively most saleable commodities have become 'money,' the event has in the first place the effect of substantially increasing their originally high salableness. ... On the other hand, he who brings other wares than money to market, finds himself at a disadvantage more or less." [Menger (1892, p. 250)]

Figure 1a illustrates a numerical example that captures Menger's observation. Its left plot depicts the evolution of the distribution of inventories in a phase diagram of $p_{2,1} - p_{3,1}$ against $p_{1,2}$, of the A1 economy in Table 2. Clearly, the lower $p_{2,1} - p_{3,1}$, the greater the likelihood that a type 1 agent obtains his consumption good when carrying good 3 than when carrying good 2. Because at the initial point of the trajectory shown in the left plot of Figure 1(a), the difference between $p_{2,1}$ and $p_{3,1}$ is relatively high, it is optimal for type 1 agents to play fundamental strategies. The dynamics follow the laws (5–7) with $\mathbf{s}= (0, 1, 0)$ (all agents follow fundamental strategies), generating a pattern that initially points in the direction of the fundamental (F) steady-state equilibrium. Nevertheless, along the transition good 3 becomes more marketable relative to good 2; as a result the value of $\Delta_1(t)$ declines until it changes sign from positive to negative, implying that type 1 agents switch from fundamental to speculative strategies. The figure shows a threshold

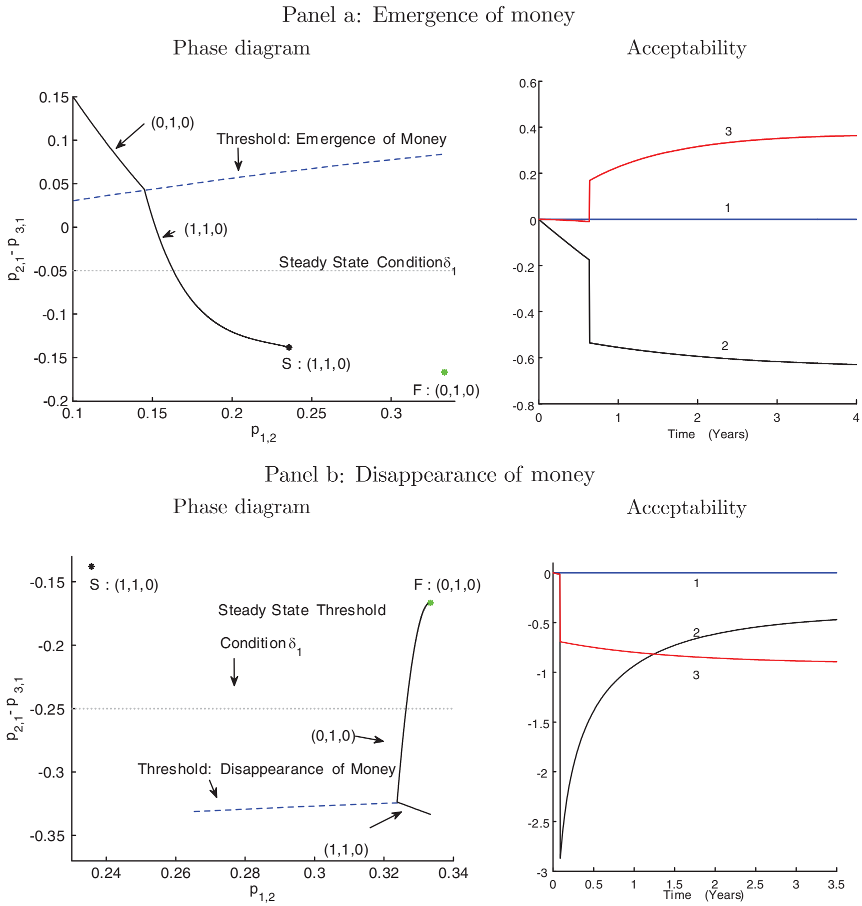

FIGURE 1. The emergence and disappearance of commodity money. In all simulations $\theta_i = \frac{1}{3}$, $\rho = 0.1$, $c = (1, 4, 9)$. In Panel a, $u_i = 100$; in Panel b, $u_i = 20$ (see rows A1 and A2 of Table 2). Type 1 agents optimally switch from fundamental to speculative strategies at the threshold "Emergence of Money" (Panel a) and from speculative to fundamental strategies at the threshold the "Disappearance of Money" (Panel b). The initial distribution, $\mathbf{p}_0$, is $\frac{1}{3}(0.30.25\ 0.3)$ and $\frac{1}{3}(1\ 1\ 1)$ in Panel a and b, respectively. The dotted horizontal line $\delta_1 = \frac{c_2 - c_3}{\alpha u_1}$. The points F and S denote the steady-state distribution of assets under the fundamental and speculative strategies, respectively. The numbers inside the acceptability plots identify the type of good. The changes in acceptability are calculated with respect to the starting level.

switch line on the $(p_{2,1} - p_{3,1},\ p_{1,2})$ space that slopes upward: At higher initial levels of $p_{1,2}$, corresponding to lower levels of $p_{1,3}$, there is an anticipation that there will be greater demand for commodity 3 on the part of type 1 agents. Such anticipation accelerates the switch.

The right plot of Figure 1(a) shows the behavior of a liquidity index, l_i, known as "acceptability." This measures the probability that an object is traded given that someone offers it.[13] It demonstrates that after the switch, the acceptability of commodity 3, which has just become money, increases dramatically. Conversely, that of commodity 2 declines substantially. An agent holding good 2 finds himself at a disadvantage relative to the situation before the switch because type 1 agents no longer accept it in trade, now preferring good 3. This result echoes that obtained by Araujo and Minetti (2011), who demonstrate that the intensification of trade is associated with the issuing of inside money. In contrast to Arajuo and Minetti (2011), however, the emergence of a new asset as money comes in an environment where all trades are anonymous and nonmonitored. What does change is the "thickness" of the market along the dynamics. When a group of individuals comes to realize that a high-storage-cost asset is more marketable, it accepts it in indirect trade. Therefore, pair-meetings that formerly resulted in no trade now result in trade.

5.2. The Disappearance of Money

That commodity money can lose the function of money is a well-documented historical event [see the excellent review by the anthropologist Quiggin (1949)] that has not yet been the subject of theoretical investigation. The dynamics of the KW model would explain such a phenomenon with changes in the relative degrees of liquidity across goods along the transition. Intuitively, if the emergence of money is due to the higher marketability of commodity 3, its disappearance could, in principle, be caused by a decline in its rate of acceptability along the transition. Figure 1(b) illustrates a numerical example in which commodity 3 permanently loses its role as money. The set of parameters of this economy is in row A2, Table 2. Its left plot shows the phase diagram of the liquidity differential $p_{2,1} - p_{3,1}$ against $p_{1,2}$, and its right plot the time evolution of the acceptability index l_i introduced in the previous example.

The set of parameters that characterize the A2 economy supports the fundamental long-run equilibrium (see condition 9)—the benefits derived from the marketability of good 3 are not large enough relative to the storage cost, c_3. Despite the large values of c_3 and the maintenance of condition (9) in steady state, commodity 3 can be a medium of exchange if its level of marketability is significantly higher than in the steady state. In the initial state, all agents are endowed with their production goods, that is, $\mathbf{p}_0 = \frac{1}{3}(1\ 1\ 1)$. The dynamic solution calls for type 1 agents to play speculative strategies at the beginning. As the left plot of Figure 1(b) shows, along the transition $p_{2,1} - p_{3,1}$ increases. Therefore, the marketability of good 3 drops and type 1 agents move from speculative to fundamental strategies. The acceptability of good 2 and 3, shown in the right plot of Figure 1(b), drops substantially after the switch, although that of good 2 regains some of the lost ground as the number of type 2 agents holding good 1 increases over the transition. Conversely, the liquidity of type 1 commodity money is not affected much by the strategy switch, as type 2 agents keep playing their role of intermediation between type 3 and type 1 agents.

Finally, note that the threshold that indicates when type 1 agents switch strategies slopes upward [left plot, Figure 1(b)], similarly to that built for the emergence of money [left plot, Figure 1(a)]. The implication with respect to the switching time is, however, different. The greater the initial level of good 2 in the hands of type 1 agents, $p_{1,2}$, the longer it takes for the switch to take place.

5.3. Uneven Distribution of Agents

Altering the proportion of agents across types has important consequences on the type of strategies that support a Nash equilibrium. For instance, in an effort to induce a higher fraction of individuals to play a speculative strategy, Duffy (2001) also considers an experimental design in which the share of type 3 individuals is larger than $\frac{1}{3}$. Wright (1995) noted that multiple steady-state equilibria may emerge when agents are unevenly distributed across the three types. First, consider the effects that altering the distribution of agents across types have on the emergence of money in economies with unique equilibria. Figure 2(a) compares the threshold level of liquidity, $p_{2,1} - p_{3,1}$, that induces type 1 agents to switch from fundamental to speculative strategies—the threshold for the emergence of money—in an A3 economy where $\theta = (\frac{2}{9}, \frac{1}{3}, \frac{4}{9})$ and an A1 economy where agents are equally split across the three types (see Table 2, rows A3 and A1, respectively). Either economy has a unique (1,1,0) steady-state equilibrium, with an asset distribution $\mathbf{p} = (p^*_{1,2}, p^*_{2,3}, \theta_3)$, where $p^*_{2,3} = \frac{1}{2}[-(\theta_1 + \theta_3) + \sqrt{(\theta_1 + \theta_3)^2 + 4\theta_1\theta_2}]$ and $p^*_{2,1} = \frac{\theta_1\theta_3}{\theta_3 + p^*_{2,3}}$ (derivations are in the appendix).

The patterns of the two economies are constructed to start from comparable initial positions. Initially, in both economies, liquidity favors a $s_1 = 1$ strategy. While money emerges in both economies, in the A1 economy the threshold level of liquidity for the emergence of money is closer to the steady state than that of the A3 economy. Additional simulations, not illustrated, show that as the share of θ_3 tends to $\frac{1}{3}$ in the A3 economy, the difference in switching time between two economies gradually converges to zero. They also indicate that further reductions of θ_3 below the value of $\frac{1}{3}$ may cause the A3 economy's steady-state equilibrium (1,1,0) to disappear altogether and to give rise to the (0,1,0) steady state. Conversely, increasing the level of the population θ_3 from the initial value of $\frac{4}{9}$, may generate multiple steady states {see also Wright [1995, Fig. 1(a)–(c)]}.

5.4. Multiple Equilibria

Previous studies have focussed on parameter sets of the model that imply a unique steady state, partly because this provides sharper predictions about agents' behavior. Nevertheless, it has also been argued that the self-referential nature of liquidity could be the source of multiplicity [Ennis (2001), Trejos and Wright (2013)]. This section expands the dynamic analysis to specifications of KW that

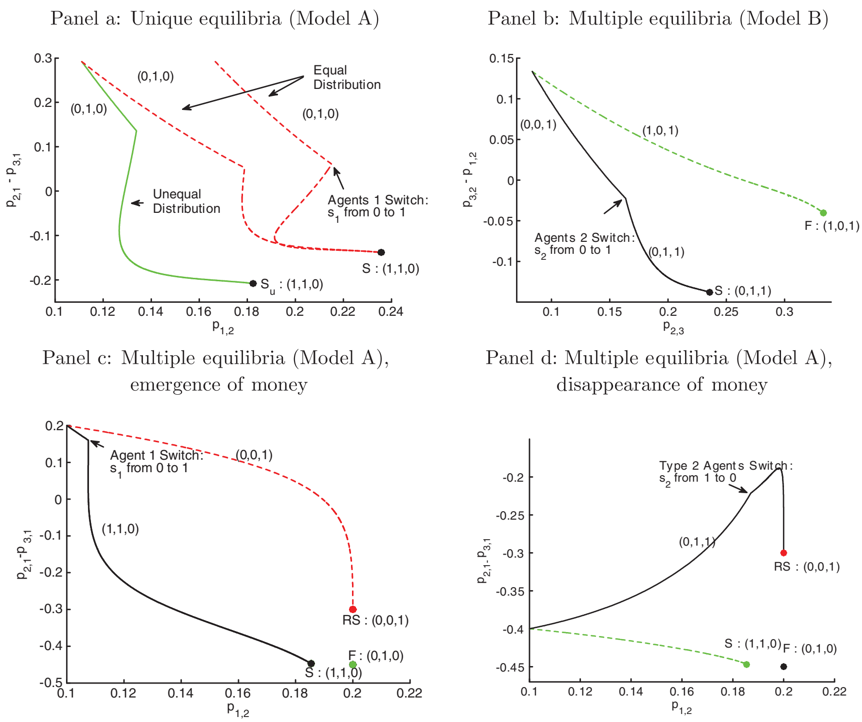

FIGURE 2. Multiple equilibria in Models A and B. In Panel a, for the unequal economy $\theta_1 = \frac{2}{9}, \theta_2 = \frac{1}{3}$, and $\theta_3 = \frac{2}{9} c = (1, 4, 9)$, the initial position is $p_0 = (\frac{\theta_1}{2}, \frac{\theta_2}{8}, 0) = (\frac{1}{9}, \frac{1}{24}, 0)$. For the economy with $\theta_i = \frac{1}{3}$, one initial position is equivalent in values $[p_0 = (\frac{1}{9}, \frac{1}{24}, 0)]$ and the other is equivalent in proportions $[p_0 = (\frac{\theta_1}{2}, \frac{\theta_2}{8}, 0) = (\frac{1}{6}, \frac{1}{24}, 0)]$. In Panel b, $c = (9, 4, 1)$, and $\theta_i = \frac{1}{3}$ (see Table 2, rows A1, A3, and B, respectively). In Panel c, $c = (0, 1, 10)$, and $\theta_1 = \theta_2 = 0.2$ (row A4, Table 2). Welfare comparisons are in Table 3.

generate multiple steady states. The Model B of KW, for instance, where type i agents can produce good $i + 2$ instead of good $i + 1$, is known to give rise to multiple steady states under the same storage cost specification of Model A and with $\theta_i = \frac{1}{3}$ for all i. An equivalent Model B economy can be obtained by maintaining the same specialization in production of Model A—type i produces good $i + 1$—but reversing the storage cost ordering, namely $c_3 < c_2 < c_1$ [see, for instance, Lagos et al. (2017)]. Another situation of multiple steady states has been pointed out by Wright (1995) for a Model A economy, i.e., $c_1 < c_2 < c_3$, in which type 3 agents are more than half of the population. In such a case, the (1,1,0) equilibrium may coexist with (0,0,1) {see Wright [1995, Fig. 1(a)–(c)]}.

Krugman (1991), Matsuyama (1991), and Fukao and Benabou (1993) note, however, that the presence of multiple steady states is not necessarily associated with multiple equilibria. It could be that once the initial condition is specified, there is a unique pattern that leads to one of the steady-state equilibria. In such a case, it is history that matters. Conversely, from some initial conditions, individuals may coordinate on either of the two steady states. In this case, expectations determine which equilibrium emerges.

Model B. To get further insights on whether multiple steady-state equilibria are associated with multiple equilibria,[14] consider first a Model B environment obtained by rearranging the ranking of the storage cost as follows: $c_3 < c_2 < c_1$. The fundamental and the speculative steady states are characterized by the set of strategies (1,0,1) and (0,1,1), respectively. In the (1,0,1) equilibrium, good 2 and good 3 are used as commodity money. In the (0,1,1) equilibrium, good 1 and good 2 play the role of money. The (1,0,1) equilibrium always exists. The (0,1,1) exists if

$$p_{3,2} - p_{1,2} < \frac{c_3 - c_1}{\alpha u_2}, \tag{12}$$

and

$$p_{2,1} > \frac{c_2 - c_3}{\alpha u_1}, \tag{13}$$

are satisfied, where $p_{1,2}$, $p_{3,2}$, and $p_{1,2}$ are evaluated on the (0,1,1) steady state. Condition (12) is the marketability requirement from the perspective of type 2 agents: When the demand for good 1 is sufficiently large relative to good 3, good 1 becomes commodity money, as long as the number of type 2 agents supplying good 1 is large enough (condition 13). Table 3 summarizes the main features of the two steady-state equilibria. The appendix derives the conditions (12) and (13) and explains why the (1,0,1) equilibrium always exist.

It is easy to find regions in the inventory space from which, given an initial condition $\mathbf{p}_0$, an equilibrium pattern converges to the (1,0,1) steady-state equilibrium and another one to a (0,1,1) equilibrium. The algorithm delivers one or the other pattern depending on the initial guess on the set of strategies $\mathbf{s}^{(0)}(t)$. Figure 2(b) illustrates a numerical example of a Model B economy characterized by the set of parameters in Table 2, row B. It shows that two qualitative different patterns originate from a certain initial condition, on a phase diagram $p_{3,2} - p_{1,2}$ against $p_{2,3}$. Along the pattern that converges to the fundamental steady state (dashed line), the set of strategies is (1,0,1) at all times. Conversely, the pattern converging to the speculative steady state (0,1,1) features the emergence of money. In the early phase of the transition, the gap $p_{3,2} - p_{1,2}$ is relatively large and the Nash set of strategies is (0,0,1). As this gap declines, the high-storage good 1 becomes more marketable. When Δ_2 switches signs from positive to negative, type 2 agents engage in indirect trade and good 1 becomes money. The phenomenon is similar to that illustrated in Figure 1(a) for Model A except that type 2 agents take the place type 1 agents and that the high-storage good that becomes money is now good 1 instead of good 3.

Model A, unequal distribution. The A4 economy specified in Table 2 exhibits multiple steady states. There is a (1,1,0) speculative equilibrium and a (0,0,1)

TABLE 3. Steady-state equilibria, Model B

Strategies		Inventories		Money	
F	S	F	S	F	S
(1,0,1)	(0,1,1)	$\frac{1}{3}(\sqrt{2}-1, 1, \frac{\sqrt{2}}{2})$	$\frac{1}{3}(1, \frac{\sqrt{2}}{2}, \sqrt{2}-1)$	3, 2	1, 2

Note: The table reports the set of strategies, the distribution of inventories, and the type of good that plays the role of money, under the fundamental (F) and the speculative (S) steady-state equilibrium for the Model B economy ($c_3 < c_2 < c_1$). The two equilibria may coexist.

reverse speculative equilibrium—"reverse" because agents have their strategies flipped relative to the speculative equilibrium. The conclusion reached for the Model B economy, namely that steady states are associated with multiple dynamic equilibria, also applies to the Model A economy. The example of Figure 2(c) depicts the adjustment process of an A4 economy starting from an initial condition, $\mathbf{p}_0$, where the level of liquidity is unfavorable to good 3. Two patterns originate from such a point. On the pattern that converges to the (1,1,0) steady state, money emerges along the transition—type 1 agents switch from $s_1 = 0$ to $s_1 = 1$. Conversely, on the (0,0,1)-bound path, all agents keep the same set of strategies (0,0,1) forever. Nevertheless, when the initial condition $\mathbf{p}_0 = (\frac{\theta_1}{2}, 0, \theta_3)$ is unfavorable to the liquidity of good 3, as in Figure 2(d), no strategy switch is observed on the (1,1,0)-bound path. This time, it is on the (0,0,1)-bound pattern—this starts from the same initial condition $\mathbf{p}_0 = (\frac{\theta_1}{2}, 0, \theta_3)$—that a switch of strategy is observed: s_2 changes from 1 to 0 and the low-storage-cost good ceases to play the role of money.

5.5. Welfare

The previous analysis has clarified that the presence of multiple steady states goes along with multiple equilibria. Therefore, when multiple steady states exist, it is expectations rather than history that determine which path an economy undertakes. A natural question is whether a Pareto ranking of multiple equilibria can be obtained. To answer this question, a payoff criterion must be established. It is convenient to focus on two criteria that are easily related to the previous analysis. One is $V_{i,j}$ that represents the payoff of type i conditioned on holding the particular good j. The other is payoff to type i, *not conditioned* on holding good j. This can be calculated as an average of $V_{i,j}$ at a point in time across type i agents:

$$W_i(t) = \frac{1}{\theta_i}[p_{i,i+1}V_{i,i+1} + (\theta_i - p_{i,i+1})V_{i,i+2}].$$

From this expression, it also follows that the payoff of the whole society would be

$$W(t) = \sum_i \theta_i W_i(t).$$

TABLE 4. Welfare comparison

Model B

	Average welfare				$V_{i,j}$					
	W_1	W_2	W_3	W	$V_{1,2}$	$V_{1,3}$	$V_{2,3}$	$V_{2,1}$	$V_{3,1}$	$V_{3,2}$
Steady state	−8.0	−10.6	9.1	−4.4	−6.9	−12.9	−11.9	3.1	13.2	−3.0
$\mathbf{p}_0 = \frac{1}{3}(0.3, 0.25, 0.3)$	3.7	−27.0	22.5	−4.1	7.5	2.1	−33.2	−24.4	20.3	23.4

Model A, unequal distribution

	Average welfare				$V_{i,j}$					
	W_1	W_2	W_3	W	$V_{1,2}$	$V_{1,3}$	$V_{2,3}$	$V_{2,1}$	$V_{3,1}$	$V_{3,2}$
Steady state	71.8	23.6	53.7	50.2	75.4	30.0	30.6	21.6	23.0	93.0
$\mathbf{p}_0 = (\frac{\theta_1}{2}, 0, \theta_3)$	110.9	23.8	14.6	51.4	148.7	81.1	19.9	23.8	14.6	40.0
$\mathbf{p}_0 = (\frac{\theta_1}{2}, 0, 0)$	76.7	−19.7	−23.9	−4.9	78.3	75.1	7.2	−19.7	−46.5	−23.9

Note: The values are percentage differences of the W_i and $V_{i,j}$, evaluated at p_0 and at the steady state, of the (0,1,1) over the (1,0,1) bound equilibrium for the B economy, and of the (0,0,1) over the (1,1,0) bound equilibrium for the A4 economy [see Figure 2(b)–(d)].

First, consider the Model B economy in Table 1, row B. Table 4 reports the percentage gap of W_i, W, and $V_{i,j}$ between the (0,1,1) and the (1,0,1) steady state. The average welfare is lower in the (0,1,1) equilibrium, but only by about 4.4%. The table also indicates that the average payoff, W_i, is not uniformly lower in the (0,1,1) equilibrium for all types i. While on average type 1 and type 2 agents are worse off, type 3 agents are better off. A closer inspection of $V_{i,j}$ reveals diverging interests even within type 2 and type 3 agents, depending on their holdings.

A welfare comparison between equilibria away from the steady state for the same B economy, also reveals gains for some and losses for others. For instance, in the position $\mathbf{p}_0 = \frac{1}{3}(0.3, 0.25, 0.3)$—the subject of the numerical example in Figure 2(b)—while type 1 and type 3 enjoy a higher payoff on the (0,1,1) than on the (1,0,1) path, type 2 agents would prefer that the economy undertake the (1,0,1) path.

The welfare calculations of the A3 economy, also reported in Table 3, give sharper predictions—at least as long as the economy is close to the steady state: The (1,1,0) equilibrium is Pareto dominated by the (0,0,1), both when considering the conditional and the unconditional payoffs, $V_{i,j}$ and W_i, respectively. The Pareto dominance is also observed in some regions of the inventory space, as, for instance, around $\mathbf{p}_0 = (\frac{\theta_1}{2}, 0, \theta_3)$—the initial condition of paths in Figure 2(d). It is, nevertheless, possible to find situations where no Pareto dominance can be established. The point $\mathbf{p}_0 = (\frac{\theta_1}{2}, 0, 0)$—the initial condition of the two paths in Figure 2(c)—is one example.

In sum, although welfare analysis gives some hints on who may lose or gain from coordinating on one or another equilibrium, additional institutional mechanisms are needed to understand which groups are more likely to tilt the economy toward a particular equilibrium.

6. CONCLUSION

This paper studied dynamic Nash equilibria in KW by means of an algorithm designed to create an iteration over the patterns of strategies and of the distribution of commodities. Numerical simulations revealed two main results. First, when multiple steady states exist, two paths that start from the same initial condition approach two different steady states. This implies that two initially identical economies may generate different flows of production and consumption because of differences in beliefs. Second, the simulations showed that, while some DNE trigger the emergence of commodity money, others do not. The algorithm could characterize scenarios of unique DNE in which the evolving state of liquidity triggers the emergence of a high-storage cost good as money, hence highlighting the role of history.

Future research could clarify the modifications to the original framework that could discipline agents' beliefs in the presence of multiple DNE. One obvious modification is the addition of fiat money. Mechanisms that slow down the transmission of information across agents [see, for instance, Araujo and Camargo (2006)] may also provide a tool to select among equilibria. Another important aspect that is left open for further research is the usefulness of computing a DNE to interpret historical accounts about the emergence of money. The simulation shown in Figure 2(a), for instance, suggests that the distribution of the population with respect to specialization in production and consumption affects the waiting time before a high-storage-cost commodity is used as money. It would be interesting to assess the relevance of such a conjecture by reviewing the adoption of money across different civilizations in the premodern world.

NOTES

1. The literature on money has recently emphasized the aspect of liquidity [see Lagos et al. (2017)].

2. It could be that once the initial condition is specified, a unique pattern leads the economy to one of the steady-state equilibria. Alternatively, it is possible that from some initial conditions, individuals coordinate on any of the steady states (see Section 5.4 of this paper).

3. For a similar generalization also see Lagos et al. (2017, Sec. 2).

4. Where no confusion arises, I will use the loose language of calling an agent of type i simply agent or individual i.

5. While in KW c_i is assumed positive, such a restriction is not necessary for computing the dynamics of the model. A commodity with negative storage cost can be interpreted as an asset that fetches a positive return.

6. The notation used here follows Wright (1995), except that, to simplify the description of the dynamical system, $p_{i,j}$ measures type i agents holding good j as a fraction of the overall population rather than as a share of type i agents.

7. The distinction between the trading strategy of a particular individual of type i, σ_i, and that of the rest of type i agents, s_i, facilitates the explanation of the solution algorithm that aims to seek a convergence between σ_i and s_i.

8. The appendix clarifies that, while proving convergences with $\mathbf{s}=(1,1,1)$ is challenging, this set of strategies does not support any steady-state Nash equilibrium—at least when agents are equally divided among the three types.

9. While the steady-state results were presented for $\theta_i = \frac{1}{3}$ to facilitate the comparison with KW, no such restriction is imposed on the construction of the algorithm. Section 5.4, for instance, discusses the dynamics of economies where agents are unevenly distributed across the three types.

10. Kehoe et al. (1993) build cyclical equilibria in a similar environment under sets of parameters that do not admit pure strategy equilibria. On this point, see also Renero (1998), and more recently Oberfield and Trachter (2012).

11. $\Delta_i(\theta^{ss}, \mathbf{p}^{ss})$ could also be used as the initial point. In principle, on $\Delta_i(\theta^{ss}, \mathbf{p}^{ss})$, the system stays still, but if computed numerically, there is always a small machine error that allows the integration to begin. In the experiments, the difference—in norm— between the two points is smaller than 10^{-5} when $|F(\mathbf{p}(t)| < 10^{-6}$.

12. All the programming is done in Matlab. The files are available upon request.

13. It is calculated as the ratio between the frequency of trade t_i (the number of times good i is traded in a unit of time) and the frequency with which good i is offered in a period of time, o_i. The appendix derives t_i and o_i.

14. A recent work on the subject by Oberfield and Trachter (2012) studies how the number of equilibria is related to the frequency of search in a symmetric environment.

REFERENCES

Araujo, L. and B. Camargo (2006) Information, learning, and the stability of fiat money. *Journal of Monetary Economics* 53, 1571–1591.

Araujo, L. and R. Minetti (2011) On the essentiality of banks. *International Economic Review* 52, 679–691.

Başçi, E. (1999) Learning by imitation. *Journal of Economic Dynamics and Control* 23, 1569–1585.

Brown, P. M. (1996) Experimental evidence on money as a medium of exchange. *Journal of Economic Dynamics and Control* 20, 583–600.

Duffy, J. (2001) Learning to speculate: Experiments with artificial and real agents. *Journal of Economic Dynamics and Control* 25, 295–319.

Duffy, J. and Jack Ochs (1999) Emergence of money as a medium of exchange: An experimental study. *American Economic Review* 89, 847–877.

Duffy, J. and Jack Ochs (2002) Intrinsically worthless objects as media of exchange: Experimental evidence. *International Economic Review* 43, 637–674.

Ennis, H. (2001) On random matching, monetary equilibria, and sunspots. *Macroeconomic Dynamics* 5, 132–142.

Fudenberg, D. and D. Levine (1988) Open-loop and closed-loop equilibria in dynamic games with many players. *Journal of Economic Theory* 44, 1–18.

Fukao, K. and R. Benabou (1993) History versus expectations: A comment. *Quarterly Journal of Economics* 108, 535–542.

Kehoe, M. J., N. Kiyotaki, and R. Wright (1993) More on money as a medium of exchange. *Economic Theory* 3, 297–314.

Kiyotaki, N. and R. Wright (1989) On money as a medium of exchange. *Journal of Political Economy* 97, 927–954.

Krugman, P. (1991) History versus expectations. *Quarterly Journal of Economics* 106, 651–667.

Lagos, R., G. Rocheteau, and R. Wright (2017) Liquidity: A new monetarist perspective. *Journal of Economic Literature* 55, 371–440.

Luo, G. (1999) The evolution of money as a medium of exchange. *Journal of Economic Dynamics and Control* 23, 415–458.

Marimon, R., E. McGrattan, and T. Sargent (1990) Money as a medium of exchange in an economy with artificially intelligent agents. *Journal of Economic Dynamics and Control* 14, 329–373.

Matsuyama, K. (1991) Increasing returns, industrialization, and indeterminacy of equilibrium. *Quarterly Journal of Economics* 106, 617–650.

Matsuyama, K., N. Kiyotaki, and A. Matsui (1993) Toward a theory of international currency. *Review of Economic Studies* 60, 283–307.

Menger, K. (1892) On the origin of money. *Economic Journal* 2, 239–255.

Oberfield, E. and T. Trachter (2012) Commodity money with frequent search. *Journal of Economic Theory* 147, 2332–2356.

Quiggin, A. H. (1949) *A Survey of Primitive Money: The Beginning of Currency*. New York: Barnes and Noble.

Renero, J. M. (1998) Unstable and stable steady-states in the Kiyotaki-Wright model. *Economic Theory* 11, 275–294.

Sethi, R. (1999) Evolutionary stability and media of exchange. *Journal of Economic Behavior and Organization* 40, 233–254.

Trejos A. and R. Wright (2016) Search-based models of money and finance: An integrated approach. *Journal of Economic Theory* 164, 10–31.

Wright, R. (1995) Search, evolution, and money. *Journal of Economic Dynamics and Control* 19, 181–206.

APPENDIX

A.1. PROOF OF PROPOSITION 1

The claim is that *under any time-constant profiles of strategies,* **s***, with the possible exception of* **s**=(1, 1, 1), the system (5)–(7), under the constraint (1), converges globally to the unique steady state.

Case (0,1,0). Equation (5) reduces to $\dot{p}_{1,2} = \alpha p_{1,3}\theta_3$, implying that the line $p_{1,2} = \theta_1$ is globally attractive. Similarly, in equation (7), $\dot{p}_{3,1} = \alpha p_{3,2}(p_{1,3} + \theta_2)$, the line $p_{3,1} = \theta_3$ is globally attractive. Finally, along these lines, the system collapses to

$$\dot{p}_{2,3} = (\theta_2 - p_{2,3})\theta_1 - p_{2,3}\theta_3,$$

that converges globally to $\frac{\theta_2\theta_1}{\theta_3+\theta_1}$. In summary, under the profile of strategies (0,1,0), the distribution of inventories converges globally to the stationary distribution $(\theta_1, \frac{\theta_2\theta_1}{\theta_3+\theta_1}, \theta_3)$.

Case (0,0,1) and *Case (1,0,0).* One can verify that the stationary distribution converges to $(\theta_1, \theta_2, \frac{\theta_1\theta_3}{\theta_1+\theta_2})$ and $(\frac{\theta_1\theta_3}{\theta_2+\theta_3}, \theta_2, \theta_3)$, respectively, using the same observations as in the previous case.

Case (1,1,0). Equation (7) becomes $\dot{p}_{3,1} = \theta_2(\theta_3 - p_{3,1})$. Consequently, $\theta_3 = p_{3,1}$ is an invariant set. The Jacobian, J, of the system of the two remaining equations (5) and (6) along the line $\theta_3 = p_{3,1}$ is

$$J = \alpha \begin{matrix} -(\theta_3 + p_{2,3}) & -p_{1,2} \\ (\theta_2 - p_{2,3}) & -(\theta_3 + p_{1,2}) \end{matrix} .$$

The determinant is positive, and the trace is negative; therefore, both eigenvalues are negative. The system, having two dimensions, is globally stable. To find the stationary distribution, set (5) and (6) to zero. They yield $p_{1,2} = \frac{\theta_1\theta_3}{\theta_3+p_{2,3}}$ and $p_{1,2} = \frac{\theta_3}{\theta_2/p_{2,3}-1}$, respectively. The two lines necessarily cross once and only once for $p_{2,3}$ in the interval $[0, \theta_2]$. The fixed point is $(p^*_{1,2}, p^*_{2,3}, \theta_3)$, where $p^*_{2,3} = \frac{1}{2}[-(\theta_1 + \theta_3) + \sqrt{(\theta_1 + \theta_3)^2 + 4\theta_1\theta_2}]$ and $p^*_{2,1} = \frac{\theta_1\theta_3}{\theta_3+p^*_{2,3}}$.

Cases (1,0,1) and (0,1,1). A Jacobian with similar properties to that of the (1,1,0) case can be obtained when the profiles of strategies are (1,0,1) or (0,1,1). The fixed point with (1,0,1) is $(p^{\#}_{1,2}, \theta_2, p^{\#}_{3,1})$, where $p^{\#}_{1,2} = \frac{1}{2}[-(\theta_3 + \theta_2) + \sqrt{(\theta_3 + \theta_2)^2 + 4\theta_3\theta_1}]$ and $p^{\#}_{3,1} = \frac{\theta_1\theta_3}{p^{\#}_{1,2}+\theta_2}$. Similarly, under (0,1,1), the fixed point is $(\theta_1, p^{\boxtimes}_{2,3}, p^{\boxtimes}_{3,2})$, where $p^{\boxtimes}_{3,1} = \frac{1}{2}[-(\theta_2 + \theta_1) + \sqrt{(\theta_2 + \theta_1)^2 + 4\theta_2\theta_3}]$ and $p^{\boxtimes}_{2,3} = \frac{\theta_1\theta_2}{p^{\boxtimes}_{3,1}+\theta_1}$.

Case (0,0,0). The system converges globally to $\mathbf{p} = (\theta_1,\theta_2,\theta_3)$. In this stationary state, agents keep their production goods.

The following table summarizes these results.

Strategies	Assets distribution	*Strategies*	Assets distribution
(0,1,0)	$(\theta_1, \frac{\theta_2\theta_1}{\theta_3+\theta_1}, \theta_3)$	*(1,0,1)*	$(p^{\#}_{1,2}, \theta_2, p^{\#}_{3,1})$
(1,0,0)	$(\frac{\theta_1\theta_3}{\theta_2+\theta_3}, \theta_2, \theta_3)$	*(0,1,1)*	$(\theta_1, p^{\boxtimes}_{2,3}, p^{\boxtimes}_{3,2})$
(0,0,1)	$(\theta_1, \theta_2, \frac{\theta_1\theta_3}{\theta_1+\theta_2})$	*(0,0,0)*	$(\theta_1, \theta_2, \theta_3)$
(1,1,0)	$(p^*_{1,2}, p^*_{2,3}, \theta_3)$		

While proving stability for the (1,1,1) case is challenging, because the inventory system cannot be reduced to two dimensions, such set of strategies does not support a Nash equilibrium in Model A when the population is equally split across types (see Section A.3).

A.2. DERIVATION OF EQUATION (12)

Equation (3) describes $V_{i,i+1}(t)$. The following does the same for $V_{i,i+2}(t)$

$$\begin{aligned}V_{i,i+2}(t) = \max_{\{\sigma_i(\upsilon)\}_{\upsilon\geq t}} \int_t^\infty \alpha e^{-\alpha(\upsilon-t)} \Bigg(& e^{-\rho(\upsilon-t)}\{[p_{i+1,i}(1-s_{i+1}) + p_{i+2,i}](V_{i,i+1}+u_i) \\ & + (p_{i,i+1}s_i + p_{i+2,i+1})(1-\sigma_i)V_{i,i+1} \\ & \times [1 - p_{i+1,i}(1-s_{i+1}) - p_{i+2,i} - (p_{i,i+1}s_i + p_{i+2,i+1})(1-\sigma_i)]V_{i,i+2}\} \\ & - \frac{1-e^{-(\upsilon-t)\rho}}{\rho} c_{i+2}\Bigg) d\upsilon, \end{aligned} \tag{A.1}$$

Taking derivatives of equations (3) and (A.1) with respect to time yields

$$\begin{aligned}\dot{V}_{i,i+1} = & -\alpha\{[p_{i,i+2}\sigma_i(1-s_i) + p_{i+1,i+2}\sigma_i]V_{i,i+2} \\ & + [1 - p_{i,i+2}\sigma_i(1-s_i) - p_{i+1,i+2}\sigma_i]V_{i,i+1} \\ & + (p_{i+1,i} + p_{i+2,i}s_{i+2})u_i\} + c_{i+1} + (\alpha+\rho)V_{i,i+1}, \end{aligned} \tag{A.2}$$

and

$$\begin{aligned}\dot{V}_{i,i+2} = & -\alpha([p_{i+1,i}(1-s_{i+1}) + p_{i+2,i}](V_{i,i+1}+u_i) + [p_{i,i+1}s_i + p_{i+2,i+1}](1-\sigma_i)V_{i,i+1} \\ & + [1 - p_{i+1,i}(1-s_{i+1}) - p_{i+2,i} - (p_{i,i+1}s_i + p_{i+2,i+1})(1-\sigma_i)]V_{i,i+2}) \\ & + c_{i+2} + (\alpha+\rho)V_{i,i+2}, \end{aligned}$$

respectively.

The last two expressions can also be written as

$$\begin{aligned}\dot{V}_{i,i+1} = & -\alpha\{[p_{i,i+2}\sigma_i(1-s_i) + p_{i+1,i+2}\sigma_i](-\Delta_i) + V_{i,i+1} \\ & + [p_{i+1,i} + p_{i+2,i}s_{i+2}]u_i\} + c_{i+1} + (\alpha+\rho)V_{i,i+1}, \end{aligned}$$

and

$$\dot{V}_{i,i+2} = -\alpha\{[p_{i+1,i}(1-s_{i+1}) + p_{i+2,i} + (p_{i,i+1}s_i + p_{i+2,i+1})(1-\sigma_i)]\Delta_i$$
$$+ [p_{i+1,i}(1-s_{i+1}) + p_{i+2,i}]u_i + V_{i,i+2}\} + c_{i+2} + (\alpha+\rho)V_{i,i+2}.$$

Subtracting side-by-side, we obtain

$$\dot{\Delta}_i = (\alpha\chi_i + \rho)\Delta_i + \omega_i,$$

where $\chi_i \equiv p_{i,i+2}\sigma_i(1-s_i) + p_{i+1,i+2}\sigma_i + p_{i+1,i}(1-s_{i+1}) + p_{i+2,i} + (p_{i,i+1}s_i + p_{i+2,i+1})(1-\sigma_i)$ and $\omega_i \equiv -\alpha([p_{i+1,i}s_{i+1} - p_{i+2,i}(1-s_{i+2})]u_i) - (c_{i+2} - c_{i+1})$. The above expression corresponds to (11). Because the term $\alpha\chi_i + \rho > 0$, for any given pattern of the asset distribution, (11) is unstable.

A.3. STEADY-STATE NASH EQUILIBRIA FOR MODEL A AND B

The stationary distribution of inventories is derived from (2) under the assumption that $\theta_1 = \theta_2 = \theta_3 = \frac{1}{3}$. The key condition for determining whether a stationary distribution is a NE is the sign of Δ_i. From (11), it follows that because $\alpha\chi_i + \rho > 0$, $\Delta_i > 0$ if $-\omega_i > 0$, namely if

$$p_{i+1,i}s_{i+1} - p_{i+2,i}(1-s_{i+2}) > \frac{c_{i+1} - c_{i+2}}{\alpha u_i}.$$

Consistency requires that $s_i = 0$ (1) with $\Delta_i > 0$ (< 0). This consistency condition is reviewed below both for the Model A and the Model B economy when $\theta_i = \frac{1}{3}$.

Model A ($c_1 < c_2 < c_3$). There are two unique NE: (0,1,0) and (1,1,0). The (0,1,0) equilibrium requires that

$$p_{2,1} - p_{3,1} > \frac{c_2 - c_3}{\alpha u_1} \tag{A.3}$$

$$-p_{1,2} < \frac{c_3 - c_1}{\alpha u_2}, \tag{A.4}$$

and

$$0 > \frac{c_1 - c_2}{\alpha u_3}, \tag{A.5}$$

with $p_{2,1} = \frac{1}{3} - p_{2,3}$, $p_{1,2} = \frac{1}{3}$, $p_{2,3} = \frac{1}{6}$, and $p_{3,1} = \frac{1}{3}$. Conditions (A.4) and (A.5) are clearly verified. From (A.3), it follows that the (0,1,0) equilibrium exists if $\frac{c_3 - c_2}{u_1\alpha} > \frac{1}{6}$.

For the (1,1,0) equilibrium the stationary distribution is $\mathbf{p} = \frac{1}{3}(\frac{\sqrt{2}}{2}, \sqrt{2} - 1, 1)$. The above three conditions for the existence of NE are therefore replaced by

$$p_{2,1} - p_{3,1} < \frac{c_2 - c_3}{\alpha u_1},$$

$$0 < \frac{c_3 - c_1}{\alpha u_2},$$

and

$$p_{1,3} > \frac{c_1 - c_2}{\alpha u_3},$$

respectively, with $p_{2,1} = \frac{1}{3} - p_{2,3}$, $p_{1,2} = \frac{\sqrt{2}}{6}$, $p_{2,3} = \frac{\sqrt{2}-1}{6}$, and $p_{3,1} = \frac{1}{3}$. It is evident that under the constraint $c_1 < c_2 < c_3$ the last two conditions are always satisfied. The first condition says that in a NE, type 1 agents play speculative if

$$-\frac{\sqrt{2}-1}{6} < \frac{c_2 - c_3}{\alpha u_1}.$$

As for the remaining six sets of strategies, θ, there is at least one inconsistency between the value of θ_i and the sign of Δ_i. Consider, for instance, the (1,1,1) case mentioned at the end of the first section of this appendix. For such a set of strategy to support a Nash strategy, $\Delta_i < 0$, for $i = 1, 2, 3$. Specifically, the requirement would be that $p_{i+1,i}s_{i+1} - p_{i+2,i}(1 - s_{i+2}) < \frac{c_{i+1}-c_{i+2}}{\alpha u_i}$ for all i, that is

$$p_{2,1} < \frac{c_2 - c_3}{\alpha u_1},$$

$$p_{3,1} < \frac{c_3 - c_1}{\alpha u_2},$$

and

$$p_{1,3} < \frac{c_1 - c_2}{\alpha u_3}.$$

Clearly, the first condition can never be satisfied because $c_3 > c_2$. Similar inconsistencies between θ_i and the sign of Δ_i are found for the remaining five sets of strategies (0,0,0), (0,0,1), (0,1,1), (1,0,1), and (1,0,0). To summarize, only (1,1,0) and (0,1,0) may support a pure strategy NE.

Model B. One fundamental NE always exists. This could coexist with another equilibrium in which two types of agents play speculative strategies (multiple equilibria). Consider the case in which $c_3 < c_2 < c_1$. Then, the fundamental NE is (1,0,1). It exists for any set of parameters—the inequalities $\Delta_1 < 0$, $\Delta_2 > 0$, and $\Delta_3 < 0$ hold in the (1,0,1) steady state for any set of parameters.

To verify the statement about the existence of the (1,0,1) NE, one needs to check that $p_{i+1,i}s_{i+1} - p_{i+2,i}(1 - s_{i+2}) < \frac{c_{i+1}-c_{i+2}}{\alpha u_i}$ holds for $i = 1, 3$ and that the reverse inequality holds for $i = 2$. Specifically, the set of conditions is

$$0 < \frac{c_2 - c_3}{\alpha u_1},$$

$$p_{3,2} > \frac{c_3 - c_1}{\alpha u_2},$$

and

$$p_{1,3} - p_{2,3} < \frac{c_1 - c_2}{\alpha u_3}.$$

The top and the middle inequalities hold for any $p_{i,j}$. The bottom one is also verified because in the (1,0,1) equilibrium $p_{1,3} - p_{2,3} < 0$ [see Table 3, inventories (F)].

Turning now to the conditions for the existence of the (0,1,1) NE equilibrium, the inequalities to be verified are as follows:

$$p_{2,1} > \frac{c_2 - c_3}{\alpha u_1}$$

$$p_{3,2} - p_{1,2} < \frac{c_3 - c_1}{\alpha u_2}$$

$$p_{1,3} < \frac{c_1 - c_2}{\alpha u_3},$$

where the various $p_{i,j}$ are to be evaluated in the inventories (S) of Table 3. The bottom inequality is always verified because $p_{1,3} = 0$. The top and the middle inequalities are equivalent to $\frac{1}{3}(1 - \frac{\sqrt{2}}{2}) > \frac{c_2 - c_3}{\alpha u_1}$ and $-\frac{1}{3}(\sqrt{2} - 1) < \frac{c_3 - c_1}{\alpha u_2}$, respectively.

A.4. LIQUIDITY INDEX: ACCEPTABILITY

Let $o_i(\upsilon)d\upsilon$ be the probability that good i is offered (but not necessarily traded) on the market between time υ and $\upsilon + d\upsilon$. Then,

$$o_i(\upsilon) = \alpha p_{i+2}[p_{i+1} + (\theta_i - p_i) + (p_i + \theta_{i+2} - p_{i+2})s_{i+2}] + \\ \alpha(\theta_{i+1} - p_{i+1})[p_i + (\theta_{i+2} - p_{i+2}) + (p_{i+1} + \theta_i - p_i)(1 - s_{i+1})].$$

Let $t_i(\upsilon)d\upsilon$ the probability that good i is traded on the market between time υ and $\upsilon + d\upsilon$. Then,

$$t_i(\upsilon) = \alpha\{p_{i+2}[\theta_i - p_i(1 - s_{i+2}) + s_{i+1}p_{i+1}] + (\theta_{i+1} - p_{i+1})[p_i + \\ (\theta_i - p_i)(1 - s_{i+1}) + (\theta_{i+2} - p_{i+2})(1 - s_{i+2})]\}.$$

The acceptability of commodity i is

$$l_i(\upsilon) = \frac{t_i(\upsilon)}{o_i(\upsilon)}.$$

This indicates how willing people are to accept commodity i, once it is being offered.