Solving Endogenous Regime Switching Models

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Abstract

This paper solves rational expectations models in which structural parameters switch across multiple regimes according to state-dependent (endogenous) transition probabilities. Assuming small shocks and smooth transition probabilities, we apply a perturbation approach. We first provide for conditions under which a unique bounded equilibrium exists. We then compute first- and second-order approximations. In a new-Keynesian model with monetary policy switching, we document new effects of monetary policy switching when transition probabilities depend on inflation.

Keywords: Regime switching, rational expectations models, indeterminacy, perturbation methods.

JEL: E32, E43

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1 Introduction

Many structural changes that affect the economy result from rational economic decisions. They thus are endogenous to the state of the economy at the date of the decision. Among many others, we find such endogenous regime switching in monetary policy (Davig and Leeper, 2008, 2011), financial crises (Coe, 2002), defaults (Mendoza and Yue, 2011), policy regimes (Rothert, 2009), and sudden stops (Calvo, 1998). Furthermore, some empirical literature (Kim et al., 2003) shows that regime switches may be influenced by macroeconomic fluctuations.

However, most of the literature on solving rational expectations models considers exogenous regime switching. When endogenous regime switching is studied, numerical methods (projection, value function iteration...) involving high computational costs, lack of analytical results and few results in terms of the existence and uniqueness of equilibrium are used.

In this paper, we generalise standard perturbation methods to solve a class of non-linear rational expectations models with endogenous regime switching - id est in which transition probabilities depend on state variables or shocks. We then apply our method in a new-Keynesian model with different monetary policy regimes and where transition probabilities depend on inflation. Our contribution is threefold.

First, we apply the Implicit Function Theorem for small shocks and smooth state-dependent transition probabilities. The existence and uniqueness of a bounded solution in this class of models rely on the existence of a unique bounded solution for a simplified, linear model with exogenous regime switching. We thus show how fluctuations of transition probabilities may modify determinacy conditions.

Second, we derive first- and second-order Taylor expansion of the solution when the solution is unique. The approximate solution is accurate according to three criteria: the approximate solution is close to the unknown true solution, model equation errors are small and the forecast distribution function is well approximated. Finally, we provide for a fast algorithm and a Dynare-compatible program to solve such models.

Third, we document two new effects of endogenous regime switching in a new-Keynesian model with a probability to switch to a lower inflation target and more aggressive response to inflation that increases with inflation.[1] First, shocks and regimes are correlated as inflationary shocks are associated with lower probability of the high inflation regime. Second, economic agents expect that inflationary shocks increase the plausibility of a low inflation regime in the future. They thus expect higher real rates leading to stabilize more inflationary shocks. This second mechanism is preponderant in our calibrated model, reduces the volatility in the high inflation regime and relaxes determinacy conditions.

Related Literature A substantial line of empirical studies challenges the common assumption of economic agents' time-invariant behavior. For instance, several papers analyze the sharp decreases in output and inflation volatility around the mid 80s in the US, the so-called "Great Moderation", by allowing for time-varying economic behavior. Among the competing sources of parameter changes, some papers have allowed for shifts in the parameters of monetary policy rules (Clarida et al., 2000; Lubik and Schorfheide, 2004; Bianchi, 2012), others for breaks in the variance of structural shocks (Sims and Zha, 2006; Justiniano and Primiceri, 2008; Fernández-Villaverde et al., 2010; Liu et al., 2010).

Within the context of forward looking economic agents, the possibility of future regimes switches alters agents' current decision rules (Sims, 1982) through what Leeper and Zha

[1] Such an endogenous regime switching model captures political pressures that arise when inflation exceeds too much the central banker's target.

(2003) call the *expectations formation effects*. These effects can modify determinacy conditions (Davig and Leeper, 2007; Farmer et al., 2009a; Barthélemy and Marx, 2012) as well as economic dynamics (Bianchi, 2013). In most of the literature, the expectations formation effects are state-invariant as probability of regime switching is constant over time.

Our main contribution is to introduce state-dependent probability distributions. Davig and Leeper (2008) show that a monetary policy rule that changes when certain endogenous variables intersect specified thresholds leads to substantial and state-dependent expectations formation effects. We follow this line but instead of relying on numerical methods that imply high computational costs and lack of analytical results, we develop an algebraic method that is tractable even for large-scale model.

Endogenous regime switching has also been studied from an econometric standpoint. Following the seminal paper by Hamilton (1989), Filardo (1994) and Filardo and Gordon (1998) have estimated Markov switching regressions with time-varying transition probabilities. More recently, Kim et al. (2003) have developed a technique for estimating multivariate models with endogenous regime switching, i.e. where transition probabilities depend on endogenous variables. However, these significant progresses cannot be replicated yet to estimate rational expectations models with endogenous regime switching. We think that our paper contributes to advance in such a direction.

In addition to papers already mentioned, our paper is closely related to Davig and Doh (2008), Foerster et al. (2016) and Maih (2015). These papers solve non-linear Markov switching rational expectations models following perturbation approaches. Davig and Doh (2008) refer to Woodford (2003) to linearise a non-linear new Keynesian model with Markov switching. Foerster et al. (2016) and Maih (2015) propose algorithms based on successive differentiations in the vein of Kim et al. (2008). While our approach provides complementary algorithms, we differ from existing literature by proving the existence of a unique stable solution of the initial model when shocks are small and the closeness of the approximate solution to the true solution.

This paper finally extends the seminal paper by Woodford (1986) to models with endogenous regime switching. Assuming small shocks, we relate the stability properties of the endogenous regime switching model to a linearised exogenous regime switching model. We thus bridge the gap between the non-linear endogenous regime switching models and literature on (linear and exogenous) regime-switching models (see for instance Blake and Zampolli, 2006; Davig and Leeper, 2007; Farmer et al., 2009a; Svennson and Williams, 2009; Farmer et al., 2010b; Cho, 2015; Barthélemy and Marx, 2012).

Two critical and controversial issues emerge when dealing with regime switching: the choice of the solution space and the concept of stability.

First, we do not restrict the solution space to Minimum state variables and consider all stochastic processes in the vein of Woodford (1986). We hence do not exclude equilibria depending on remote past regimes. Fortunately when the model is determinate, the unique solution only depends on current and past regimes.[2]

Second, following the influential book by Costa et al. (2005), most of the literature has turned to the Mean Square Stability concept (see Farmer et al., 2009b; Maih, 2015). However, there is, at this stage, no theoretical argument ensuring the consistency of this concept of stability with the perturbation approach for non linear Markov switching DSGE models. We thus consider standard boundedness concept that is consistent with a local approach as it prevents the equilibrium to visit (even occasionally) spaces where the model can be highly

[2]For more about the relationship between solution space and determinacy, the interested reader can refer to Barthélemy and Marx (2012).

non-linear.

The remainder of the paper is organized as follows. Section 2 exposes the class of models and derives theoretical results. Section 3 illustrates our results in a new Keynesian model with endogenous regime switching. Section 4 concludes.

2 Theory

In this section, we first expose the class of models we consider. We then apply the Implicit Function Theorem to derive determinacy conditions as well as an approximate solution. We finally provide an efficient algorithm that implements our method.

2.1 Class of models

Most of recent rational expectations macroeconomic models with regime switching can be reduced to the following system:

$$\mathbb{E}_t[f_{s_t}(z_{t+1}, z_t, z_{t-1}, \sigma v_t)] = 0, \tag{1}$$

where the index t denotes time and belongs to integers, z is a bounded vector of endogenous variables, v is an i.i.d bounded multi-dimensional stochastic process, and σ is a positive scalar. The current regime is represented by s_t in $\{1, \cdots, N\}$. For any i, f_i is a smooth function (at least C^2) and $\mathbb{E}_t$ is the expectation operator given information available at time t, namely current and past shocks and regimes. We study models for which the transition probabilities between regimes are endogenous, i.e. depend on variables and shocks.

Assumption 1 (Transition probability)**.** *The transition probability from regime i to regime j conditional on information available at time $t-1$ and about shocks at time t satisfies:*

$$Pr(s_t = j | s_{t-1} = i) = p_{ij}(z_{t-1}, \sigma v_t), \tag{2}$$

where p_{ij} is a smooth function (at least C^2) with values in $[0, 1]$.

Assumption 1 allows probabilities to depend on past endogenous variables and current shocks. This is why we call this class of models: *endogenous* regime switching models. Regularity of transition probabilities is in general necessary to apply the perturbation approach. In addition, inextricable simultaneity issues arise when we allow contemporaneous variables to appear in the transition probabilities. In Appendix A, we give two examples illustrating why we need these two assumptions.

Equilibrium definition Perturbation approach requires that the evolution of the model and of the variables remains controlled with respect to a certain norm. We thus define a stable equilibrium as follows:[3]

Definition 1 (Stationary equilibrium)**.** *A stationary rational expectations equilibrium (*s.r.e.e.*) of model (1) is a stochastic process, z_t, such that the process z_t*

(i) is uniformly bounded.

(ii) depends continuously on all the past shocks.

[3] We detail notations and definitions in Appendix B.

(iii) solves equation (1) *given transition probabilities* (2)*.*

We choose this definition because perturbation approach relies on the local behavior of the model, i.e. when variables are not too far from a reference point often called a steady state. Outside this neighborhood around the steady state, perturbation approach is no more valid and higher order expansions are helpless.[4] This definition may be economically restrictive in certain circumstances (see Cochrane, 2011, for instance). Our solution space is however close to the set of essentially bounded functions considered by Woodford (1986). We restrict to boundedness instead of essentially boundedness as a direct consequence of the endogeneity of transition probabilities: essentially boundedness relies on an intrinsic measure, which does not exist in our setup since the transition probabilities are not *a priori* given. Finally, following Farmer et al. (2007), recent papers consider mean-square stability concept. However, as far as we know this latter stability concept is not consistent with a standard perturbation approach for non linear models as it allows for large deviations from the steady state. We give two simple examples of problems that may arise when using such a stability concept in appendix A.

Steady state restrictions We define a steady state as a solution of the model when no shocks affect the economy, i.e. when the scale parameter σ is zero. We then use a perturbation approach to approximate the solution in a neighborhood of this steady state. By definition, the steady state abstracts from the volatility of shocks contrary to the *risky* steady-state (Coeurdacier et al., 2011) but allows for strong non-linearities as the model's derivatives are taken at different points.

Definition 2 (Steady state)**.** *A steady state is a* s.r.e.e. *of the model* (1) *when the scale parameter σ is zero.*

When the model contains backward-looking components, two problems arise. First the steady state may depend on all the history of past regimes, second, derivatives of the model at the steady states can be difficult to handle. Assumptions 2 and 3 exclude such intractable behaviors.

Assumption 2 (Regime-dependent steady state)**.** *There exists a regime-dependent steady state $(\bar{z}_1, \cdots, \bar{z}_N)$, such that, for any k in $\{1, \cdots, N\}$,*

$$\sum_{j=1}^{N} p_{ij}(\bar{z}_i, 0) f_i(\bar{z}_j, \bar{z}_i, \bar{z}_k, 0) = 0$$

Assumption 2 embeds cases in which the endogeneity modifies the steady-state. It however requires that the steady state depends on the current regime only. This can be extended to steady states depending on a finite number of past regimes by redefining regimes as the finite product of past regimes. What we thus rule out is the existence of history-dependent steady states. While these cases may often appear, in particular when past regimes interact with backward looking components, there is to our knowledge, no general way to settle this problem.

Finally, to simplify computations, we assume, without loss of generality, the following properties of the derivatives of the model:

[4] The local behavior is controlled by a norm ensuring completeness of the set and by the fact that the operator defining the model is bounded.

Assumption 3 (Derivatives)**.** *All the derivatives of the model, at least until order* 2*, are unaffected by past regimes at the steady state.*

This assumption can be relaxed by redefining regimes as the combination of current and relevant past regimes. Once again, we need this assumption to rule out history-dependent derivatives.

2.2 Existence and uniqueness

In this subsection, we prove the existence of a unique *s.r.e.e.* when the shocks are small enough, i.e. when the scale parameter σ is small.

Linearisation Under assumptions 1, 2 and 3, the first-order Taylor expansion of the model (1) in $(\bar{z}_1, \cdots, \bar{z}_N)$ with respect to the scale parameter, σ, is:

$$\mathbb{E}_t^0[a(s_t, s_{t+1})z_{t+1}] + b(s_t)z_t + c(s_t)z_{t-1} + \sigma d(s_t)v_t = o(\sigma) \tag{3}$$

where matrices a, b, c and d are regime-dependent matrices satisfying:

$$a(s_t, s_{t+1}) = \partial_1 f_{s_t}(\bar{z}_{s_{t+1}}, \bar{z}_{s_t}, \bar{z}_{s_{t-1}}, 0),$$

$$b(s_t) = \sum_j \left[\partial_1 p_{s_t j}(\bar{z}_{s_t}, 0) f_{s_t}(\bar{z}_j, \bar{z}_{s_t}, \bar{z}_k, 0) + p_{s_t j}(\bar{z}_{s_t}, 0)\partial_2 f_{s_t}(\bar{z}_j, \bar{z}_{s_t}, \bar{z}_k, 0)\right],$$

$$c(s_t) = \sum_j p_{s_t j}(\bar{z}_{s_t}, 0)\partial_3 f_{s_t}(\bar{z}_j, \bar{z}_{s_t}, \bar{z}_k, 0), \qquad d(s_t) = \sum_j p_{s_t j}(\bar{z}_{s_t}, 0)\partial_4 f_{s_t}(\bar{z}_j, \bar{z}_{s_t}, \bar{z}_k, 0),$$

where $\partial_j f_{s_t}(.)$ is the partial derivative of the function f_{s_t} with respect to the jth component and similarly for the probabilities. The subscript 0 in $\mathbb{E}_t^0$ denotes that the underlying transition probabilities of the expectations operator are constant:

$$\bar{p}_{ij} = p_{ij}(\bar{z}_i, 0). \tag{4}$$

Future regime s_{t+1} appears in expectations as derivatives are taken at the steady state next period. Proposition 1 links the determinacy of the initial model with the determinacy of its linearised counterpart.

Proposition 1 (Linearisation)**.** *If the linearised model*

$$\mathbb{E}_t^0[a(s_t, s_{t+1})z_{t+1}] + b(s_t)z_t + c(s_t)z_{t-1} = \Psi_t \tag{5}$$

admits a unique s.r.e.e *for any bounded stochastic process Ψ_t, then for the scale parameter σ small enough, model* (1) *admits a unique* s.r.e.e.

Proposition 1 deduces the determinacy of endogenous regime switching model from its linearised exogenous Markov Switching counterpart, model (3). Endogeneity of transition probabilities affects the current variables coefficients b, and therefore determinacy except if the steady state is constant over regimes.

Determinacy of linear regime switching We now provide for sufficient determinacy conditions for model (5) based on a companion paper Barthélemy and Marx (2012) and Cho (2015). Following this latter paper, we introduce the matrix R_k satisfying:

$$R_{k+1}(s_t) = b_{s_t} - \mathbb{E}_t^0[a_{s_t,s_{t+1}} R_k(s_{t+1})^{-1} c_{s_{t+1}}], \tag{6}$$

assuming that we can indeed build such a sequence of matrices:

Assumption 4 (Forward condition)**.** *For any integer k and any regime s_t, the sequence $R_k(s_t)$ is well-defined and admits a limit $R_\infty(s_t)$ which is invertible.*

In the absence of regime switching, this condition is linked to the cyclic reduction algorithm, see Theorem 5.9 p. 158 in Bini et al. (2012) which underlines algorithms generally used to compute the generalized Schur decomposition. In the absence of backward-looking components, Assumption 4 only requires matrix $b(s_t)$ to be invertible.

The solution of model 5 then satisfies:

(i) $z_t = -R_\infty^{-1}(s_t)c_{s_t} z_{t-1} + w_t$

(ii) w_t solves $\mathbb{E}_t^0[a_{s_t,s_{t+1}} w_{t+1}] + R_\infty(s_t) w_t = \Psi_t$

Assumptions 5 and 6 ensure the existence of a unique stable process z_t satisfying conditions (i) and (ii).

Assumption 5 (Uniqueness)**.** *The sequence*

$$\left(\sum_{i_1,\cdots,i_k} \bar{p}_{i_1 i_2} \cdots \bar{p}_{i_{k-1} i_k} |||a_{i_1,i_2} R_\infty^{-1}(i_2) \cdots a_{i_{k-1},i_k} R_\infty^{-1}(i_k)||| \right)^{1/k}$$

admits a limit ρ_+ strictly lower than one when k tends to ∞.

This condition ensures the uniqueness of a bounded equilibrium satisfying (ii) (see Barthélemy and Marx, 2012).

Assumption 6 (Stability)**.** *The joint spectral radius, ρ_-, of $\{R_\infty^{-1}(1)c_1, \cdots, R_\infty^{-1}(N)c_N\}$ is lower than* 1*.*

Where the joint spectral radius is defined as the maximal asymptotic growth rate of products of matrices. This condition ensures that the process z_t defined in point (i) is stable if w_t is stable.

Proposition 2 (Determinacy)**.** *Under assumption 4, there is a unique* s.r.e.e *for the linear model* (5) *if and only if assumptions 5 and 6 are satisfied. In addition, the unique solution is given by $z_t = -R_\infty^{-1}(s_t)c_{s_t} z_{t-1} + w_t$, where w_t depends on a weighted sum of expected future stochastic processes (Ψ_t) given in Appendix.*

Proof. See Appendix D for the complete proof. First, Assumption 5 ensures the uniqueness of a bounded process w_t. Second, Assumption 6 ensures that z_t is bounded if w_t is bounded. Third, we prove that if we relax one of the two assumptions then either multiple *s.r.e.e.* arise (when the first assumption fails) or no *s.r.e.e.* exists (when the second assumption fails). □

This Proposition is an extension of Blanchard and Kahn (1980) to linear Markov switching models. In the absence of Markov Switching, assumptions 5 and 6 result from the Blanchard and Kahn conditions. When the model is purely forward-looking, Assumption 6 is obvious as $c_i = 0$ for any i.

Settling Assumptions 6 and 5 is an undecidable problem. However, approximation of ρ_+ (Barthélemy and Marx, 2012; Ogura and Jungers, 2014) and ρ_- (Jungers, 2009) are reasonably fast in most applications.

2.3 First and second order approximations

We now apply the Implicit Function Theorem to model (1) assuming that the shocks -as measured by the scale parameter σ- are small enough and derive the Taylor expansion of the true solution with respect to the scale parameter.

Proposition 3 (Taylor expansion)**.** *Under assumptions 2 to 6 and for a scale parameter σ small enough, there exists a unique bounded equilibrium of model (1), z_t. In addition, the Taylor expansion of this solution follows:*

$$z_t = \bar{z}_{s_t} + \sigma z_t^1 + \frac{\sigma^2}{2} z_t^2 + o(\sigma^2), \tag{7}$$

where

$$z_t^1 = R_\infty^{-1}(s_t) c_{s_t} z_{t-1}^1 - R_\infty^{-1}(s_t) d(s_t) v_t,$$
$$z_t^2 = R_\infty^{-1}(s_t) c_{s_t} z_{t-1}^2 + \alpha(s_t) z_{t-1}^1 \otimes z_{t-1}^1 + \beta(s_t) z_{t-1}^1 \otimes v_t + \gamma(s_t) v_t \otimes v_t + \delta(s_t)\, Vect(\Sigma).$$

Matrices α, β, γ and δ are given in Appendix. The mathematical operator $\otimes$ denotes the Kronecker product, while Vect denotes the vectorization operator.

Proof. . We apply the Implicit Function Theorem in Banach space and Propositions 1 and 2. See Appendix E for the details of the proof. □

The second-order expansion of the solution is given in equation (7). The first (second) term, z_t^1 (z_t^2 resp.) is the first-order (second-order resp.) Taylor expansion.

This Proposition leads to multiple remarks. First, the computations only require matrix manipulations, leading to a very fast algorithm. Second, because we apply the Implicit Function Theorem we know that the approximate solution is close to the true solution. This second remark is key as it departs from most of the existing literature that does not prove theoretically that the approximate solution is close to the true one. Third, the second order Taylor expansion only depends on its past value and cross products of the first-order Taylor expansion and shocks. Consequently, we are sure that this solution, z_t^2 is bounded. We thus do not need to use pruning algorithm to ensure its convergence contrary to Kim et al. (2008). Finally, we have a priori conditions to settle determinacy that ensures us that the solution is unique considering the widest solution space.

Proposition 3 proves that the approximate equilibrium is close to the true one. Corollaries 4 and 5 refine this result by showing that (i) the approximate equilibrium induces small model equations errors with respect to the scale parameter (ii) the forecast density function using the approximate equilibrium is close to the one we would find if we knew the true solution.

Corollary 4 (Model equations errors)**.** *Under assumptions of Proposition 3, the model equations errors (defined below) of a p-order approximation of the solution, $\hat{z}_t^p$ is at least of order $p+1$:*

$$\mathbb{E}_t[f_{s_t}(\hat{z}_{t+1}^p, \hat{z}_t^p, \hat{z}_{t-1}^p, \sigma v_t)] = o(\sigma^{p+1}), \text{ with } p(s_t = j | s_{t-1} = i) = p_{ij}(\hat{z}_{s_t}^p, \sigma v_t),$$

where the different approximations are defined as follows: $\hat{z}_t^0 = z_{s_t}$, $\hat{z}_t^1 = z_{s_t} + \sigma z_t^1$ and $\hat{z}_t^2 = z_{s_t} + \sigma z_t^1 + \sigma^2/2 z_t^2$.

This result is an obvious corollary of Proposition 3 as the model is sufficiently smooth. Proposition 4 proves that the standard accuracy criterium used in the computational economics literature behave nicely in our context.

We underline that even if the solving algorithm requires to linearise probabilities around the steady-state, transition probabilities do not need to be linearised, neither for simulations nor estimations. As a consequence, probabilities remain bounded between 0 and 1 and hence interpretable.

Finally, last corollary proves that the trajectory forecasted using the approximate solution remains close to the true trajectory. In our context, forecasts based on an approximate solution involves two types of errors: on the value of endogenous variables and on the probability of regime switching. We thus choose the Lévy-Prokhorov metric to compare forecasting density functions. This metric measures the distance between two probability measures both in terms of range and level. Formally, the metric between two measures, μ and ν, is given by π as follows:

$$\pi(\mu,\nu) = \inf\{\epsilon > 0 |\ \mu(A) \leq \nu(A^{\epsilon}) + \epsilon \quad \text{and} \quad \nu(A) \leq \mu(A^{\epsilon}) + \epsilon \quad \text{for all} \quad A \in B(M)\},$$

where A is any element of the (Borel) sigma-algebra $B(M)$ associated with the measurable space M and A^{ϵ} denotes a ϵ-neighborhood of A in M.

We denote by f_q and by $\hat{f}_q^p$ the density forecast function q-period ahead and its p-order approximation, conditional on initial conditions (z^t, ε^t).

Corollary 5 (Forecast errors)**.** *Under assumptions of Proposition 3, the p-order approximate of the q-period ahead density forecast, $\hat{f}_q^p$ is close to the true density forecast, f_q in the Lévy Prokhorov sense:*

$$\pi(\hat{f}_q^p, f_q) = o(\sigma^{p+1}).$$

Proof. See Appendix F. □

Corollary 5 proves that the errors on transition probabilities and on endogenous variables are not multiplicative over time and are always small with respect to the scale parameter. This result suggests that our technique can be used for forecasting purpose as well as for computing the likelihood of such models.

2.4 Algorithm

We implement previous results using Matlab and the Dynare software (Adjemian et al., 2011). Our program checks determinacy and provides the first- and second-order Taylor expansions of the solution with respect to the scale parameter σ.[5] The program is flexible, allowing for relatively large-scale models, and fast, checking determinacy and solving the model take less than a second in most applications.[6]

The main steps of the algorithm are the following:

1. Compute all the derivatives around the steady state (Dynare)

2. Compute R_∞ and check regularity conditions (Assumption 4): if it fails, either the approach is irrelevant or the model is misspecified.

[5]Programs and readme file can be found here: https://sites.google.com/site/jeanbarthelemyeconomist/research-papers

[6]The only exception is in the limiting case where the model is close to indeterminacy. In this case, it can be useful to run the program longer to be sure of the determinacy check (see Barthélemy and Marx, 2012, for a longer discussion).

3. Compute approximations of ρ_+ and ρ_-, if they are both smaller than 1, then next step. Otherwise the model is either indeterminate or admits no bounded equilibrium.

4. Compute the first- and second-order approximations using Proposition 3.

3 An application to an endogenous monetary policy regime switching

In this section, we apply our method to a New Keynesian model in which the monetary authority follows two different Taylor rules depending on the monetary policy regime and decides to switch from one to the other depending on inflation. We first expose the model. Then we show the accuracy of our method. Finally, we detail some new effects of endogenous regime switching.

3.1 The model

Households The representative household chooses consumption C_t, hours worked L_t, and debt holding B_t to maximize lifetime utility:

$$\mathbb{E}_t \sum_{t=0}^{\infty} \xi_t \beta^t \left(\frac{C_t^{1-\tau}}{1-\tau} - L_t \right),$$

under the following budget constraint:

$$P_t C_t + Q_t B_t = B_{t-1} + W_t L_t + P_t D_t - P_t T_t,$$

where the variable P_t denotes the price level, Q_t the price of a zero-coupon bond at time t yielding 1 in period $t+1$, W_t the nominal wage per hour, D_t dividend and T_t lump-sum taxes. The shock ξ_t corresponds to a preference shock affecting the discount factor. The parameter τ measures the inverse of the intertemporal elasticity of substitution and the parameter β denotes the discount factor.

Firms A continuum of firms, denoted by the subscript $j \in [0, 1]$, produces an intermediate goods Y_{jt} using labor L_{jt} as only input. The production technology is linear in labor for simplicity:

$$Y_{jt} = L_{jt}.$$

To allow for a real effect of monetary policy, we introduce nominal rigidities *à la* Rotemberg (1982). Firms pay a real adjustment cost AC_{jt} when they adjust their price:

$$AC_{jt} = \frac{\phi}{2} \left(\frac{P_{jt}}{\pi^\star_{s_t} P_{jt-1}} - 1 \right)^2 Y_t,$$

where ϕ determines the magnitude of the adjustment cost, $\pi^\star_{s_t}$ denotes the regime-dependent steady-state inflation in regime s_t and P_{jt} represents the price set by the firm j at time t. Each intermediate goods-producing firm maximizes their expected present value of profits:

$$\sum_{s=0}^{\infty} \beta^s \lambda_{t+s} \frac{D_{jt+s}}{P_{t+s}},$$

where λ_{t+s} represents the representative's household discount factor, and D_{jt} are profits of firm j at time t. The time-t profit equals:

$$\frac{D_{jt}}{P_t} = \frac{P_{jt}Y_{jt}}{P_t} - \frac{W_t}{P_t}Y_{jt} - \frac{\phi}{2}\left(\frac{P_{jt}}{\pi^\star_{s_t}Pp_{jt-1}} - 1\right)^2 Y_{jt}.$$

Finally, assuming a Dixit-Stiglitz aggregation for intermediate goods, the demand for each intermediate good is given by:

$$Y_{jt} = \left(\frac{P_{jt}}{P_t}\right)^{-\theta_t} Y_t,$$

where θ_t is the elasticity of substitution between goods.

Monetary Policy First, following a tradition initiated by Clarida et al. (2000), we describe a regime switching as a change in the monetary policy rule. The central bank sets its interest rate according to two potential rules. The first rule, defining regime 1, is characterized by a high inflation target and a modest response to inflation. The second rule - regime 2 - describes a central bank aiming at stabilizing inflation to a lower rate more aggressively. The generic formula defining the rules is given by the following equation:

$$R_t = \left[\frac{R_{t-1}}{R^\star_{s_{t-1}}}\right]^{\rho_r} \left[R^\star_{s_t}\left(\frac{\pi_t}{\pi^\star_{s_t}}\right)^{\alpha_{s_t}}\left(\frac{Y_t}{\bar{y}}\right)^{\gamma} e_t\right]^{1-\rho_r},$$

where parameters α_{s_t} and γ measure the long-run reaction to inflation gap and output gap - as approximated by the deviation from the productivity trend. The parameter ρ_r captures the smoothing motive of interest rates. We assume for simplicity that the weight on inflation can switch between two values depending on the regime, s_t, but not the other parameters γ and ρ_r. The inflation target $\pi^\star_{s_t}$ also changes across regimes. We suppose that the targeted nominal interest rate $R^\star$ is chosen such that inflation equals its target in each regime in the absence of shocks. Finally, the monetary policy shock e_t stands for the unsystematic monetary policy component.

Second, we suppose that the monetary authority chooses the regime at each point of time in the spirit of Davig and Leeper (2008). We model this choice as transition probabilities depending on the level of inflation. The higher the level of inflation the more likely the central banker will choose the more aggressive monetary policy rule in an attempt to stabilize the economy. The probability to remain in a particular regime $j \in \{1, 2\}$ follows:

$$Pr(s_t = j | s_{t-1} = j, \pi_{t-1}) = p_{jj} + \lambda_{jj}(\pi_{t-1} - \pi^*_i)$$

where p_{jj} will be the steady state level of the probability of remaining in regime j and λ_{jj} measures the sensitivity of transition probabilities to inflation. When λ_{jj} equals zero, probabilities are constant.

Such endogenous regime switching is a shortcut to model the different intensity of political pressures to change the objective of the central banker depending on the current level of inflation. We however do not present a theory of why such political pressures emerge and instead we posit an *ad hoc* specification of transition probabilities.

We choose linear probabilities to simplify the interpretation of the sensitivity parameter λ_{jj}. To ensure that probabilities remain in $[0, 1]$, one can instead use a more complex probability function like a logit function. However, since we use a perturbation approach, such a change does not modify qualitatively the approximate solutions as only the second order is affected by such a change.

Shocks We consider three types of shocks that reproduce the three shocks estimated by Lubik and Schorfheide (2004) in a reduced form model. The preference shock, ξ_t follows a first order autoregressive process:

$$\ln \xi_t = \rho_\xi \ln \xi_{t-1} + \epsilon_t^\xi,$$

where the parameter $\rho_\xi < 1$ denotes the autocorrelation and the innovation ϵ_t^ξ follows a zero-mean Gaussian law with a standard deviation σ_ξ. We denote by $\hat{\xi}_t$ the log deviation $\ln \xi_t$.

The steady-state price markup is $u = \theta/(\theta - 1)$, while the time-t markup follows:

$$\ln u_t = (1 - \rho_u) \ln u + \rho_u \ln u_{t-1} + \epsilon_t^u,$$

where ϵ_t^u follows a zero-mean Gaussian law with a standard deviation σ_u. We denote by $\hat{u}_t$ the log deviation $\ln(u_t/u)$.

Finally, the monetary policy shock, $\hat{e}_t = \ln e_t$ follows a zero-mean Gaussian law with a standard deviation σ_e.

First-order conditions The first-order conditions and market clearing conditions lead to the following non-linear system:

$$\theta u_t C_t^\tau - \phi(\theta u_t - \theta + 1)\frac{\pi_t}{\pi_{s_t}}\left[\frac{\pi_t}{\pi_{s_t}} - 1\right] + \beta\phi(\theta u_t - \theta + 1)\mathbb{E}_t\left[\frac{Y_{t+1}}{Y_t}\frac{C_{t+1}^{-\tau}}{C_t^{-\tau}}\frac{\pi_{t+1}}{\pi_{s_{t+1}}}\left(\frac{\pi_{t+1}}{\pi_{s_{t+1}}} - 1\right)\right] = \frac{\theta}{\theta - 1},$$

$$Y_t = C_t + \frac{\phi}{2}\left[\frac{\pi_t}{\pi_{s_t}} - 1\right]^2 Y_t,$$

$$R_t = \left[\frac{R_{t-1}}{R^\star_{s_{t-1}}}\right]^{\rho_r(s_t)}\left[R^\star_{s_t}\left(\frac{\pi_t}{\pi^\star_{s_t}}\right)^{\alpha(s_t)}\left(\frac{Y_t}{\bar{y}}\right)^{\gamma(s_t)} e_t\right]^{1-\rho_r(s_t)}$$

$$\mathbb{E}_t\left[\frac{\beta R_t}{\pi_{t+1}}\left(\frac{C_t}{C_{t+1}}\right)^\tau\left(\frac{\xi_{t+1}}{\xi_t}\right)\right] = 1,$$

where $\bar{c} = \bar{y} = (1/\theta)^{1/\tau}$. The first equation is a (non-linearised) new Keynesian Phillips curve reflecting the optimal price setting of intermediate firms, the second equation reports market clearing condition; the third equation is the monetary policy rule and the last equation is the Euler equation of households.

Steady state To simplify the interpretation of inflation targets, we suppose that the targeted interest rate $R^\star_{s_t}$ satisfies:

$$R^{\star\,-1}_{s_t} = \beta\left[\frac{p_{s_t 1}(\pi^\star_{s_t})}{\pi^\star_1} + \frac{p_{s_t 2}(\pi^\star_{s_t})}{\pi^\star_2}\right].$$

The steady state is denoted by $c_t = \bar{c}$, $y_t = \bar{y}$, $\pi_t = \pi^\star_{s_t}$ and $R_t = R^\star_{s_t}$. It defines a steady state consistent with assumptions 2 and 3, that changes over regimes but is unaffected by the fluctuations of transition probabilities.[7]

[7]This is not a general property, as already mentioned in subsection 2.1, the steady state can be affected by the endogeneity of regime switching.

Perturbation approach If the linearised model satisfies Assumptions 4-6, then Proposition 3 holds, and we can compute the approximate solution.

Linearisation In the absence of endogenous regime switching, the log-linearisation of the model leads to the canonical 3-equations system. The new Keynesian Phillips curve can be rewritten as:

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \kappa(\hat{Y}_t + \frac{\hat{u}_t}{\tau}),$$

where the hat denotes the log-deviation around the steady state defined above. And the Euler equation is the standard IS curve:

$$\hat{Y}_t = E_t \hat{Y}_{t+1} - \tau^{-1}(\hat{R}_t - E_t \hat{\pi}_{t+1} + (\hat{\xi}_t - E_t \hat{\xi}_{t+1})).$$

Finally, the monetary policy rule writes:

$$\hat{R}_t = \rho_r \hat{R}_{t-1} + (1 - \rho_r)[\alpha(s_t)\hat{\pi}_t + \gamma \hat{Y}_t + \hat{e}_t].$$

These three equations are similar to those estimated by Lubik and Schorfheide (2004) over sub-samples and by Bianchi (2013) over the whole US post war period. We thus take advantage from these estimations to calibrate our model.

3.2 Calibration

Calibration of non-regime switching parameters is reported in Table 1. Structural parameters are calibrated as in Woodford (2003) and are standard. Variances and persistences of shocks are calibrated following Lubik and Schorfheide (2004). Policy parameters are also calibrated as in this latter paper. When Lubik and Schorfheide (2004) estimate two different values on the two regimes, pre- and post- Volcker, we choose a value in between and close to Davig and Doh (2008). We only allow for variation across regimes for the weight of inflation and the inflation target in the Taylor rule as these two parameters have been put forward in explaining the shift in volatility and level of inflation from the 70s to the post Volcker period.

Parameters	Calibration
Structural parameters	
Slope of the NKPC, κ	0.17
Relative risk aversion, τ	1
Discount factor, β	0.99
Elasticity between goods, θ	10
Reaction of interest rate to output, γ	0.2
Persistence parameters	
Interest rate, ρ_r	0.7
Preference, ρ_ξ	0.8
Price mark-up, ρ_u	0.8
Standard deviation parameters	
Preference, σ_ξ	0.0007
Price mark-up, σ_u	0.002
Monetary policy σ_e	0.001

Table 1: Calibration of non regime-switching parameters.

Table 2 reports the calibration of regime switching parameters. Inflation targets are set to match their average empirical counterpart considering that regime 1 is supposed to describe US Great Inflation in the 70s while regime 2 describes the post-1981 US history. The reactions to inflation as well as transition probabilities are calibrated at the posterior mean estimated by Bianchi (2013). As far as we know, the only parameters for which there is no existing literature are sensitivity parameters, $\lambda_{s_t s_{t+1}}$. As an illustration, we calibrate these parameters such that (i) the probability of remaining in regime 2 is constant (ii) the probability of exiting the high inflation regime, $s_t = 1$, is increasing with inflation. Hence, we calibrate λ_{11} equal to -30.

Parameters	Calibration
Regime 1: 1970-1980	
Response to inflation, α_1	1.1
Inflation target, $\pi_1^\star$	7% (annualized)
Probability of remaining in 1, $\bar{p}_{11}$	0.90
Probability sensitivity to inflation, λ_{11}	-30
Regime 2: Post 1981	
Response to inflation, α_2	2.3
Inflation target, $\pi_2^\star$	2% (annualized)
Probability of remaining in 2, $\bar{p}_{22}$	0.95
Probability sensitivity to inflation, λ_{22}	0

Table 2: Calibration of regime switching parameters.

3.3 Accuracy

We check accuracy by computing the errors when evaluating the model equations at the approximate solution. We report the errors of all the non-linear equations except the monetary policy equation since this equation is linear in log. In Figure 1, we display the errors when considering the first- and the second-order Taylor expansion of the solution for the Euler equation, the New Keynesian Phillips Curve and the Market clearing condition with respect to the scale parameter, σ. When σ equals one it means that the maximum size of shocks is its standard error. The last figure corresponds to the ratio between the error norm of the first- and the second-order approximations.

First, we observe that the errors of the first-order approximate solution look like quadratic while those of the second-order look like cubic. The ratio between the errors appears to be linear and converge to zero when σ tends to zero. The smaller the shocks the more accurate the second-order approximate solution. This illustrates corollary 4 and suggests that the method delivers expected accuracies.

Second, the second-order approximate solution remains more accurate than the first-order one as long as shocks are smaller than their calibrated standard deviations. On the one hand, the New Keynesian Phillips Curve generates larger errors for the second-order compared to the first-order expansions even for relatively small shocks. On the other hand, other equations errors are two times smaller for the second-order approximate solution compared to the first-order approximate solution even for relatively large shocks (three standard deviations).

These contrasted results prove that while the second order is always preferable for small shocks, the choice of the order of the approximation is not unambiguous when shocks are larger. That being said, this result is not specific to endogenous regime switching but is likely to be amplified by it as it brings a second source of non-linearities in the model.

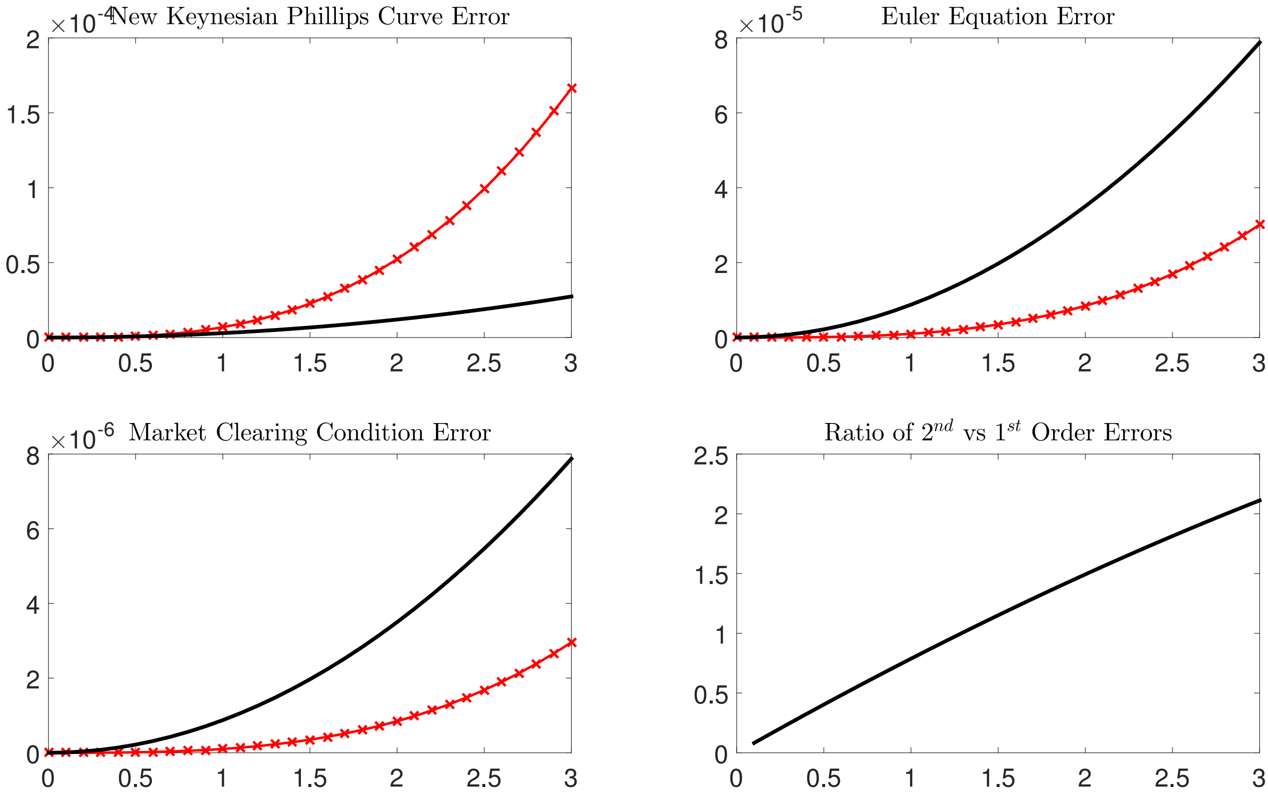

Figure 1: Model equations errors of first- and second-order approximations

Note: The first three graphs report errors when plugging the first- and second-order approximations in model equations. The black plain line (red line with crosses) reports the errors induced by first- (second- resp.) order approximation with respect to the scale parameter, σ. The fourth graph plots the ratio between the first- and second-order approximations' largest errors. To improve the speed of errors computation but without loss of generality, we compute expectations and maximum errors by assuming that shocks follow a a uniform law on the discrete support $\sqrt{3/2}\{-\sigma\sigma_\epsilon, 0, \sigma\sigma_\epsilon\}$ where σ is the scale parameter and σ_ϵ is the standard deviation of the considered shock given in Table 1.

3.4 Determinacy

Fluctuations of transition probabilities matter for determinacy. If the probability of switching toward lower inflation target increases with inflation, the economy evolves as if the reaction to inflation of the central banker were stronger than what it is actually in the high level inflation regime. Indeed, the real interest rate increases more with inflation than otherwise because expected inflation decreases with the level of current inflation. Below, we describe more formally this mechanism in a flexible price environment.

In a flex-price environment, the model can be rewritten as a Fisherian equation of inflation determination (the Euler equation in the absence of sticky prices) and a monetary policy rule. For the sake of the exposition, suppose that the monetary policy rule only responds to inflation ($\rho = \gamma = 0$). At first-order and in the absence of shocks at period t, inflation is thus determined by:

$$\alpha_{s_t}\hat{\pi}_t = E_t^0 \hat{\pi_{t+1}} + \lambda_{s_t 1}\hat{\pi}_t[\pi_1^\star - \pi_2^\star],$$

where $\hat{\pi}$ denotes the log deviation of inflation from its regime dependent steady-state, $\pi_{s_t}^\star$. If we denote by $\alpha'_{s_t} = [\alpha_{s_t} - \lambda_{s_t 1}[\pi_1 - \pi_2]]$, we then recognize a well known Fisherian equation in the presence of regime switching. The reaction to inflation is simply modified by an additive term taking into account the sensitivity of probability multiplied by the gap between the two inflation targets. This additional term reflects the negative relationship between the level of inflation at current period and expected inflation due to decreasing probability of switching toward high-inflation regime. Finally, the existence of a unique equilibrium can be analytically determined (Davig and Leeper, 2007; Farmer et al., 2009a) by the following condition:

$$p_{11}(1 - |\alpha'_2|) + p_{22}(1 - |\alpha'_1|) + |\alpha'_1\alpha'_2| > 1.$$

If this condition is satisfied, the only bounded equilibrium is $\pi_t = \pi_{s_t}$. Figure 2 plots the determinacy region with respect to the policy parameters α_1 and α_2 for different sensitivity parameters λ_{11}. We observe that the indeterminacy region (below the curves) shrinks when $\lambda_{11} < 0$ suggesting that endogenous probability and the policy response to inflation play similar role. In our calibrated model, as $\lambda_{11} = -30$, endogenous transition probabilities reinforce determinacy by allowing the real interest rate to increase more with inflation than otherwise in regime 1.

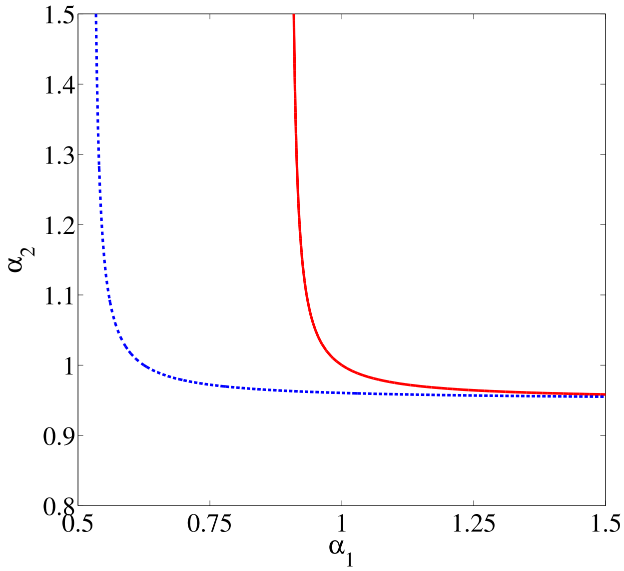

Figure 2: Determinacy and endogenous regime switching

Note: The above figure displays determinacy area with respect to central bank inflation reactions in each regimes (α_1, α_2). Above the curves, determinacy is ensured; below, multiple stable equilibria exist. The red thick line plots the determinacy frontier in the absence of endogenous regime switching $(\lambda_{11} = 0)$. The dotted curve displays the determinacy frontier when the persistence of regime 1, $Pr(s_t = 1|s_{t-1} = 1)$, falls with inflation $(\lambda_{11} = -30)$.

3.5 Macroeconomic dynamics

In this subsection, we put forward two new effects of endogenous regime switching. The first effect, that we call a selection effect, is due to the correlation between regimes and shocks. The second effect results from the expectations of the selection effect and we call it the expectations formation effect following Leeper and Zha (2003). In our calibration, the second effect dominates. Increasing expectation of a lower inflation target if inflation increases reduces inflation volatility in the high inflation regime as if the reaction to inflation in this regime were as high as in the low inflation regime.

Impulse Response Functions We define the impulse response functions as the difference between the expected dynamics of an economy hit by a 1-standard deviation shock at date t compared to the same economy without such a shock. Formally, the response of the output, y_{t+k}, to a one standard deviation preference shock (but no other shock) at date t if the economy is in regime 1 at this date writes:

$$E[y_{t+k}|\epsilon_t^\xi = \sigma^\xi, s_t = 1] - E[y_{t+k}|\epsilon_t^\xi = 0, s_t = 1]. \tag{8}$$

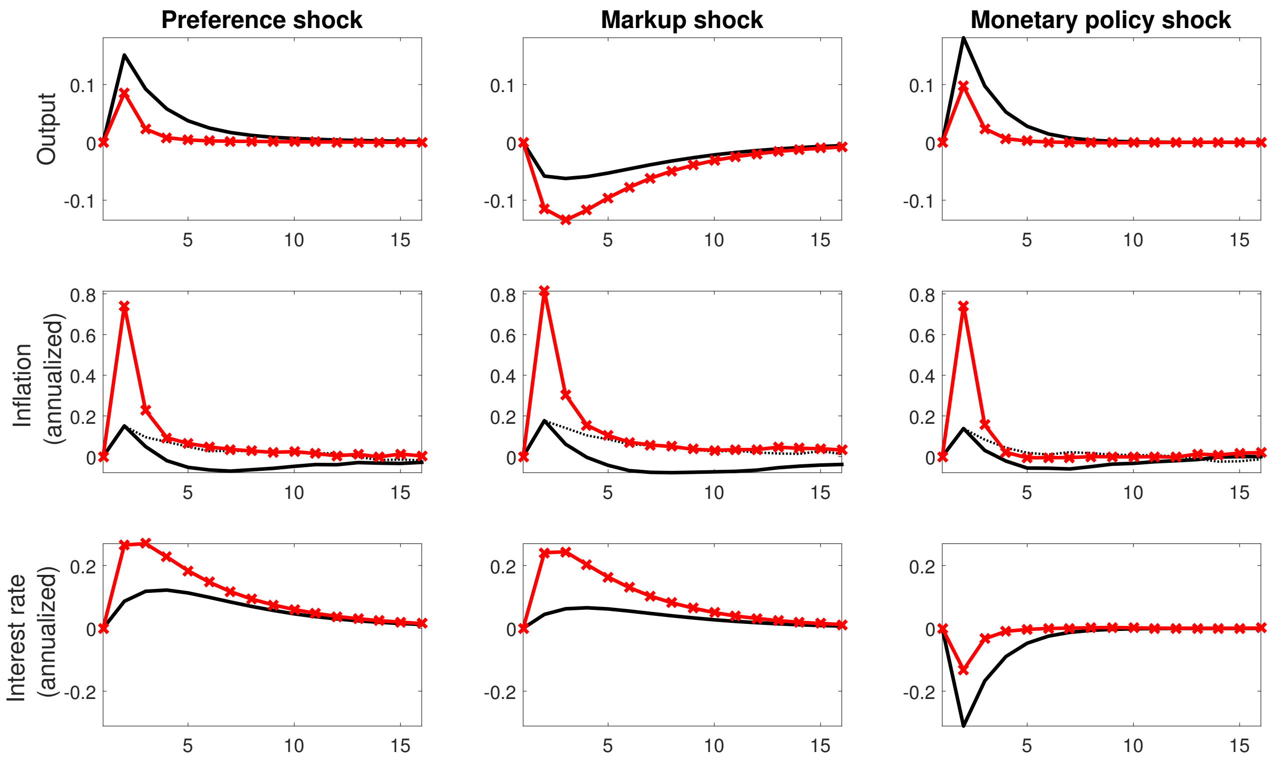

Figure 3: Impulse response functions in Regime 1 ('dovish' regime)

Note: The figures report the impulse response functions as defined in (8) to preference, markup and monetary policy shocks. Thick black lines display economic responses to shocks when transition probabilities are endogenous. Dashed thin lines display economic responses when economic agents take into account endogenous transition probabilities in the formation of their expectations but regimes are drawn from the steady-state transition probability distribution, the difference between plain and dashed lines isolates the role of the selection effect. Finally, the red thick lines with crosses display the responses of economic variables when transition probabilities are constant. The difference between dashed and thick with crosses lines isolates the endogenous expectations formation effect.

Figure 3 plots Impulse Response Functions (IRF) to shocks according to three scenarii.[8] Thick black lines display economic responses to shocks when transition probabilities are en-

[8]IRFs are computed assuming that the economy is at the ergodic distribution prior to the shock. We

dogenous. Red thick lines with crosses display the responses of economic variables when transition probabilities are constant. Dashed thin lines display economic responses when economic agents take into account endogenous transition probabilities in the formation of their expectations but regimes are drawn from the steady-state transition probability distribution (when not visible these lines coincide with the plain black lines). The difference between plain and dashed lines isolates the role of the selection effect, i.e. the effect due to the correlation between shocks and regimes. The difference between dashed and thick with crosses lines isolates the endogenous expectations formation effect.

First, the fluctuation of transition probability, p_{11}, reduces the reaction of inflation to any kind of shocks. When an inflationary shock hits the economy, economic agents expect lower inflation in the future, therefore the real interest rate is higher even if the nominal interest rate is not affected. This mechanism is present even if transition probabilities are constant but economic agents believe they are not (dashed lines) proving that it results from an expectations formation effect.

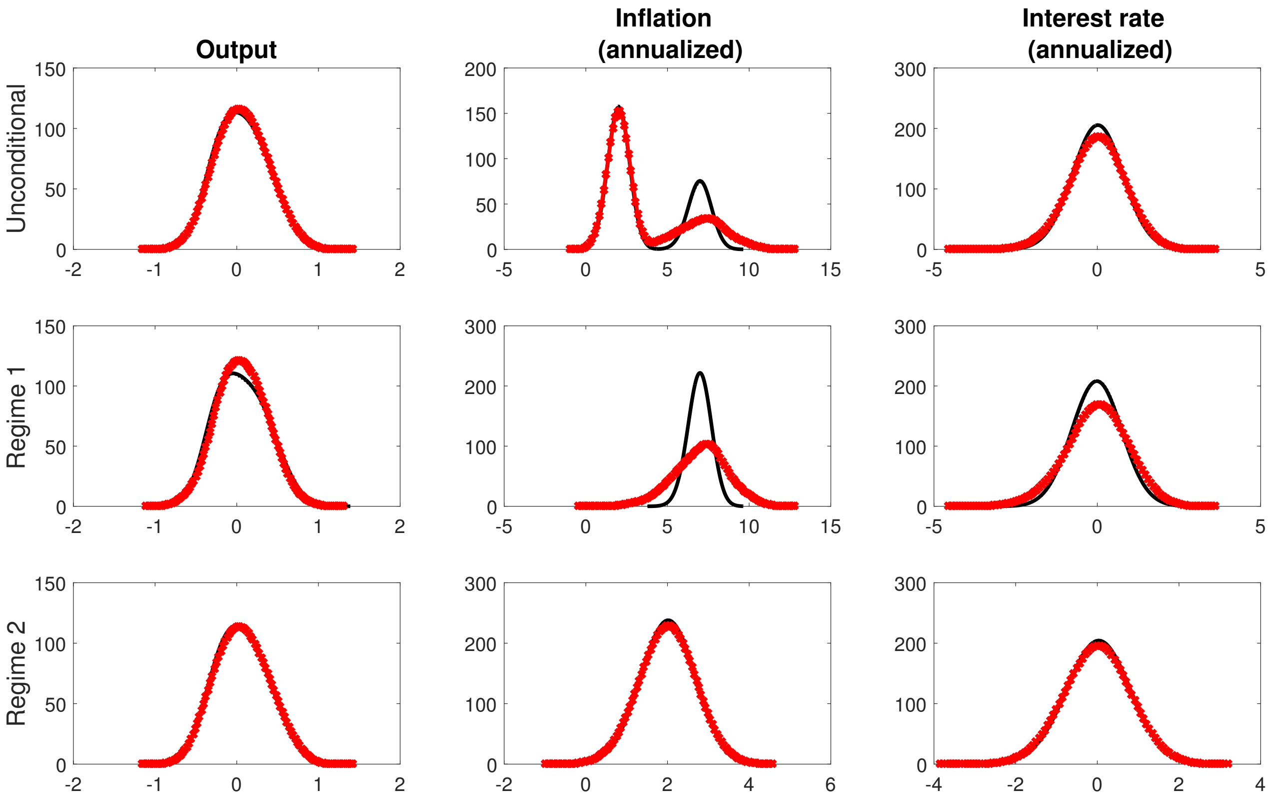

Figure 4: Ergodic distribution

Note: The figures report ergodic distribution functions of the output, inflation and the interest rate unconditionally on regimes (top), in regime 1 (middle) and regime 2 (bottom). Thick black lines report distributions when transition probabilities are endogenous. Finally, the red thick lines with crosses display the responses of economic variables when transition probabilities are constant.

Second, the selection effect dominates the medium run evolution of inflation, as the dashed and red with crosses lines overlap. The selection effect, indeed, needs time to take place as

assume that $t = 100$ in equation (8) while the economy is initially ($t = 0$) at the steady state. We have checked that this is sufficient to obtain a reasonably good approximation of the ergodic distribution. We then compute the IRF by simulating 500000 trajectories of regimes and shocks.

probabilities are only slightly modified altering the ratio of regime 1 over regime 2 only modestly each period.

Finally, we observe that the output gap is not always stabilized by the fluctuations of transition probabilities. Indeed, the rise in the real interest rate due to expectations of lower inflation target, tends to generate larger recession when a cost push shock hits the economy.

Ergodic distribution Figure 4 represents the ergodic distribution of the output, inflation and the interest rate. The selection effect only plays a little role so we only plot the endogenous regime switching case $\lambda_{11} = -30$ (in black) and the exogenous one, $\lambda_{11} = 0$ (in red with crosses). The first row reports the distribution of endogenous variables independently of the regime and the second (third) row depicts the distribution conditional on being in regime 1 (2, resp.).

First, endogenous fluctuations of transition probabilities do not matter for the low-inflation regime. In the low-inflation regime, fluctuations of transition probability in the other regime might influence economic agents decisions as they make the other regime more stable as we have seen above. This mechanism is however quantitatively irrelevant.

Second, inflation and the nominal interest rate are less volatile in the high inflation regime when regime switches endogenously, however, the output is not significantly affected. These findings directly echo results from the Impulse Response Functions.

Third, the magnitude of this effect is comparable to the effect of changing the reaction to inflation in the Taylor rule of the high-inflation regime, α_1 from 1.1 to 2.3, i.e. assuming the same response in both regimes. We compare these two cases in Figure 5 in Appendix G. Even if this latter result relies on the exact value of λ_{11}, it suggests that the threat of switching toward a low-inflation regime if inflationary shocks hit the economy helps stabilizing the economy as well as the standard reaction to inflation in the Taylor rule. We obviously do not claim that this invalidates results based on constant transition probabilities between regimes, but we think that it clearly calls for a better integration of such mechanism in empirical works to take into account expectations of regime switching in a more rational way.

4 Conclusion

In this paper, we propose a flexible tool to solve rational expectations models, with endogenous regime switching. We illustrate the flexibility and accuracy of the method but also its limits. We apply our methodology to an endogenous monetary policy regime switching model and we show that fluctuations in transition probabilities can play a significant role in the economic dynamics. We especially prove that these fluctuations can produce similar results as a more standard variation in the reaction to inflation, in terms of determinacy as well as volatility.

Finally, we hope that this methodology will help researchers to estimate models in which regime switching is considered as an economic outcome and not as a random shock as it is the case in most empirical works nowadays.

Appendices

A Key assumptions underlying Perturbation approach: four caveats

A.1 Caveat 1: Norms and model definition

In this subsection, we illustrate why we require bounded shocks and smooth models.

Consider the following model:

$$exp(x_t^3) = exp(\sigma\epsilon_t),$$

where x_t is an endogenous variable, ϵ_t is an exogenous variable and σ is a scale parameter. This model is smooth and if we assume that shocks ϵ_t are bounded, we find a unique bounded equilibrium $x_t = (\sigma\epsilon_t)^{1/3}$ whatever the distribution of shocks. Assuming that the scale parameter is sufficiently small, a perturbation approach will lead to approximations of this solution.

Now, let us assume that we are looking for a Mean Square Stable equilibrium instead and that we do not want to assume bounded shocks. We show in this example that the model is not well defined for mean-square stable functions around the steady state, this prevents the economist from using a perturbation approach, whereas there exists a unique mean square stable solution.

The steady state of the previous model is $x_t = 0$. Applying a perturbation approach around this reference point requires that the model is defined for variables close to this steady state. Let us consider $\tilde{x}_t = \gamma\epsilon$ where ϵ is a zero mean Gaussian shock and γ a scalar. For γ small enough, the spread between $\tilde{x}$ and the steady-state is mean square stable. In addition, this variable is in the $\mathcal{L}-2$ neighborhood of x but the model is not defined at $x_t = \tilde{x}$ as the exponential of a cubic Gaussian shock is not mean square stable.

This very naive example proves that a perturbation approach requires to consider simultaneously the same concept of stability for the model and for the variables.

A.2 Caveat 2: Norms and differentiation

In this subsection, we illustrate why standard differentiation (perturbation approach) is consistent with boundedness concept but not necessarily with other stability concept (here mean square stability as in Caveat 1).

Let us consider the following model:

$$\alpha\mathbb{E}_t(\ln(1 + x_{t+1})) = \ln(x_t + 1) - \sigma^3\epsilon_t^3$$

where α is a scalar strictly smaller than one, x_t is an endogenous variable strictly greater than -1, ϵ_t is a shock and σ is the scale parameter.

Remarking that zero is a steady state of this model, standard linearisation leads to:

$$\alpha E_t x_{t+1} = x_t.$$

The only bounded solution is $x_t^0 = 0$. The exact solution of the model is $x_t = \exp(\sigma^3\varepsilon_t^3) - 1$ as long as the shock is symmetric and zero mean. Implicit function theorem tells us that if the shock is bounded then the solution is also close to the true solution.

Considering (unbounded) Gaussian shock and mean square stability concept is however once again misleading. Indeed, if the economist linearises the model as above, he can be tempted to conclude that the mean square stable solution, $x_t^0 = 0$, is a good approximation of the true solution. However, in this case there is no mean square stable solution of the original problem.[9]

This example reminds us that usual linearisation uses derivatives with respect to $\mathcal{C}^\infty$ norm and is consistent if shocks are bounded and model is smooth but can lead to misleading results if shocks are not bounded and the economist uses another stability concept. The technical reason in this example is that derivatives of the model cannot be properly defined in $\mathcal{L}^2$.

A.3 Caveat 3: Contemporaneous probabilities

Let us consider the model

$$x_t = 0 \qquad \text{if} \quad s_t = 1,$$
$$x_t = 1 \qquad \text{if} \quad s_t = 2,$$

with transition probabilities:

$$Pr(s_t = 1 | s_{t-1} = 1, 2) = x_t.$$

This model does not have any solution. The model (including probabilities) is fully linear but transition probability depends on contemporaneous endogenous variables raising an inextricable simultaneity issue.

A.4 Caveat 4: Continuous probabilities

We consider

$$\alpha_{s_t} \mathbb{E}_t x_{t+1} = x_t$$

where $\alpha_1 = 1/2$, and $\alpha_2 = 2$, with transition probabilities:

$$p(s_t = 1 | s_{t-1} = 1, 2) = 1 \text{ if } \sigma\varepsilon_t = 0, \ p(s_t = 2 | s_{t-1} = 1, 2) = 0 \text{ if } \sigma\varepsilon_t \neq 0$$

There is a unique solution for $\sigma = 0$, but it is no more the case for any $\sigma > 0$. Indeed, the non-continuity of transition probabilities prevent from using a perturbation approach to determine the existence of a unique bounded solution locally.

B Definitions and Notations

B.1 Formalism

We consider the model

$$\mathbb{E}_t[f_{s_t}(z_{t+1}, z_t, z_{t-1}, \sigma v_t)] = 0,$$

where the transition probability from regime i to regime j satisfies Assumption 1. u_t represents the concatenation of regimes and shocks (s_t, v_t). The set U^∞ represents the set of infinite sequences $u^t = (u_t, u_{t-1}, \cdots)$.[10] We describe a solution of the model as a continuous function ϕ of all the past shocks and regimes, satisfying:

[9] We can completely solve this model as it is linear in log.

[10] For more details about this formalism, see Woodford (1986).

$$\mathcal{N}(\phi,\sigma)(s^t,v^t) = \sum_j \int p_{s_tj}(\phi(u^t),\sigma v) f_j(\phi(js^t,vv^t),\phi(s^t,v^t),\phi(s^{t-1},v^{t-1}),\sigma v_t) f(v)dv$$

The conditional probability distribution of the stochastic process u_t is then described, for any (s,v) :

$$\mu(s,v,\phi,u^{t-1},\sigma) = p_{s_{t-1}s}(\phi(u^{t-1}),\sigma v) f(v)dv, \tag{9}$$

The key complexity here is that this conditional probability distribution depends on the equilibrium itself, defined through function ϕ.

B.2 Definition of a stationary rational equilibrium

As in the case without Markov Switching, the idea is to start with a solution to the model when $\sigma = 0$ and to apply a perturbation approach for small positive σ value. The following definition depicts precisely the solution space we consider:

A stationary rational expectations equilibrium (s.r.e.e.) of model (1) is a *continuous* function $\phi : U^\infty \to F$ such that:

1. $||\phi||_\infty = \sup_{U^\infty} \|\phi(u^t)\| < \infty$.
2. If u is a U valued stochastic process whose conditional probability distribution is μ (see Equation 9), then $z_t = \phi(u^t)$ is a solution of Equation (1).

We restrict our analysis to continuous and bounded functions, and we denote by $\mathcal{B}$, the set of such functions. Precisely, $\mathcal{B}$ is the set of functions ϕ on U^∞ such that, for all s, the map $v \mapsto \phi(s,v)$ is continuous and such that $(s,v) \mapsto \phi(s,v)$ is bounded. If the model $\mathcal{N}$ cannot be defined on the whole set $\mathcal{B}$ (for instance because a variable has to be positive), we can always restrict $\mathcal{B}$ to a neighborhood of the steady state to avoid caveats 1 and 2. In this case, all the proofs are done on the restriction of $\mathcal{B}$ instead of $\mathcal{B}$ itself.

B.3 Implicit Function Theorem

We will prove the existence and uniqueness of a *s.r.e.e.* when the continuous shocks are small enough (small σ) by applying the Implicit Function Theorem to an operator acting on the Banach space of bounded and continuous functions, $\mathcal{B}$, see Abraham et al. (1988). We start with recalling this result.

Theorem 6. *[**Abraham et al. (1988)**] Let E, F, G be 3 Banach spaces, let $U \subset E, V \subset F$ be open and $f : U \times V \to G$ be C^r, $r \geq 1$. For some $x_0 \in U$, $y_0 \in V$ assume $D_y f(x_0,y_0) : F \to G$ is an isomorphism. Then there are neighborhoods U_0 of x_0 and W_0 of $f(x_0,y_0)$ and a unique C^r map $g : U_0 \times W_0 \to V$ such that, for all $(x,w) \in U_0 \times W_0$*

$$f(x,g(x,w)) = w.$$

C Proof of Proposition 1

In our framework, the problem can be rewritten as finding the zeros of an operator $\mathcal{N}$ acting on a bounded function ϕ in $\mathcal{B}$, the scale parameter σ with values in $\mathcal{B}$, such that, for any u^t:

$$\mathcal{N}(\phi,\sigma)(u^t) = \int_V \sum_j p_{s_t j}(\phi(u^t), \sigma v_{t+1}) f_{s_t}(\phi(uu^t), \phi(u^t), \phi(u^{t-1}), \sigma v_t) dv_{t+1}. \tag{10}$$

As we see in Appendix B, to apply the Implicit Function Theorem we have to check that:

1. $\mathcal{B}$ with the norm $\|\|_\infty$ is a Banach space.
2. $\mathcal{N}$ is C^1 on $\mathcal{B}$.
3. The function ϕ_0 such that $\phi_0(s^t, v^t) = z_{s_t}$ satisfies $\mathcal{N}(\phi_0, 0) = 0$.
4. $D_\phi \mathcal{N}(\phi_0, 0)$ is invertible.

The first point is immediate, $\mathcal{B}$ with the norm $\|\|_\infty$ is a Banach space as a product of Banach spaces. Point 2. results from the regularity of f, p_{ij}, and Lebesgue's dominated convergence Theorem. When $\sigma = 0$, we have that

$$\mathcal{N}(\phi, 0)(u^t) = \sum_j p_{s_t j}(\phi(u^t), 0) \int_V f_{s_t}(\phi(uu^t), \phi(u^t), \phi(u^{t-1}), 0) dv_{t+1}$$

Thus point 3. results from Assumption 2. Concerning point 4., the differential of $\mathcal{N}$ in $(\phi_0, 0)$ is described by the following Lemma:

Lemma 7. *Under assumptions 2 and 3, the differential $D_\phi \mathcal{N}(\phi_0, 0)$ satisfies: For any H in $\mathcal{B}$,*

$$\begin{gathered} D_\phi \mathcal{N}(\phi_0, 0) H = \\ \sum_j p_{s_t j}(z_{s_t}, 0) \partial_1 f_{s_t}(z_j, z_{s_t}, z_{s_{t-1}}, 0) \int_V H(js^t, vv^t) dv \\ + \left(\sum_j f_{s_t}(z_j, z_{s_t}, z_{s_{t-1}}, 0) \partial_1 p_{s_t j}(z_{s_t}, 0) + \sum_j p_{s_t j}(z_{s_t}, 0) \partial_1 f_{s_t}(z_j, z_{s_t}, z_{s_{t-1}}, 0) \right) H(s^t, v^t) \\ + \sum_j p_{s_t j}(z_{s_t}, 0) \partial_3 f_{s_t}(z_j, z_{s_t}, z_{s_{t-1}}, 0) H(s^{t-1}, v^{t-1}) \end{gathered}$$

This Lemma shows that the differential is exactly equivalent to the linearised model given in Proposition 1, this ends the proof of Proposition 1.

D Proof of Proposition 2

We first recall the construction of R_k by forward iteration, then we prove the proposition.

Suppose that the process z_t is a *s.r.e.e.* of (5), then, for any $k > 0$, the process z_t should also solve:

$$\mathbb{E}_t^0[R_k(s_t, \cdots, s_{t+k}) z_{t+k}] + R_k(s_t) z_t + c_{s_t} z_{t-1} = \mathbb{E}_t^0 \sum_{p=0}^{k} \Lambda_k^p(s^{t+p}) \Psi_{t+p} \tag{11}$$

where M_k, R_k and Λ_k^p are defined recursively by

$$M_1(s_t, s_{t+1}) = a_{s_t,s_{t+1}}, \qquad M_{k+1}(s_t, \cdots, s_{t+k+1}) = -a_{s_t,s_{t+1}} R_k(s_{t+1})^{-1} M_k(s_{t+1}, \cdots, s_{t+k+1})$$
$$R_1(s_t) = b_{s_t}, \qquad R_{k+1}(s_t) = b_{s_t} - \mathbb{E}_t[a_{s_t,s_{t+1}} R_k(s_{t+1})^{-1} c_{s_{t+1}}],$$

and

$$\Lambda_k^0 = \mathbb{1}, \qquad \Lambda_{k+1}^{p+1}(s^{t+p+1}) = -a_{s_t,s_{t+1}} R_k(s_{t+1})^{-1} \Lambda_k^p(s^{t+p+1}), \text{ for } p \geq 0 \text{ and } k \geq 0.$$

We now prove Proposition 2 assuming that assumption 4 is satisfied.

If assumptions 5 and 6 are satisfied, thanks to assumption 5, w_t is bounded, and thanks to assumption 6, z is bounded. It remains to check that z is the unique solution. By definition, R_∞ is solution of

$$R_\infty(s_t) = b_{s_t} - \mathbb{E}_t^0[a_{s_t,s_{t+1}} R_\infty(s_{t+1})^{-1} c_{s_{t+1}}]$$

Thus, defining $w_t = z_t + R_\infty^{-1}(s_t) C_{s_t} z_{t-1}$, w_t is solution of

$$\mathbb{E}_t^0[a_{s_t,s_{t+1}} w_{t+1}] + R_\infty(s_t) w_t = \Psi_t \tag{12}$$

According to Barthélemy and Marx (2012), we know that there exists a unique solution if assumption 5 is satisfied, and that in this case,

$$w_t = -R_\infty^{-1}(s_t) h_t - \mathbb{E}_t^0 \left[\sum_{k=1}^{\infty} \prod_{j=0}^{k-1} a_{s_{t+j},s_{t+j+1}} R_\infty(s_{t+j+1})^{-1} h_{t+k} \right]$$

Assuming that 1. is satisfied, we notice that assumptions 5 and 6 are necessary conditions for determinacy.

If assumption 6 is satisfied, and 5 is not, then, for any bounded solution z_t, $w_t = z_t + R_\infty^{-1}(s_t) C_{s_t} z_{t-1}$ is bounded and solution of (12). Thus, according to Barthélemy and Marx (2012), we know that if assumption 6 fails, there exist several bounded solutions for (12), therefore the model is indeterminate.

If assumption 5 is satisfied, and assumption 6 is not, then convergence of $R_k(s_t)$ and assumption 5 imply that, for any $\varepsilon > 0$, there exist a constant C and an integer k_0 such that, for $k \geq k_0$:

$$\sum_{i_1,\cdots,i_k} \bar{p}_{i_1 i_2} \cdots \bar{p}_{i_{k-1} i_k} |||A_{i_1,i_2} R_{k-1}^{-1}(i_2) \cdots A_{i_{k-1},i_k} R_1^{-1}(i_k)||| \leq C(\rho_+ + \varepsilon)^k \tag{13}$$

This implies that, if z_t is a bounded solution, then, according to (13), the sequence defined by $\mathbb{E}_t[M_k(s_t, \cdots, s_{t+k}) z_{t+k}]$ tends to zero, and the sequence $\mathbb{E}_t \sum_{p=0}^{k} \Lambda_k^p(s^{t+p}) \Psi_{t+p}$ is convergent, for any Ψ. In particular, if z_t is a bounded solution of (5), with $\Psi_t = 0$, then $z_t = -R_\infty^{-1}(s_t) C_{s_t} z_{t-1}$, which is not bounded since assumption 6 is not satisfied.

E Proof of Proposition 3

E.1 First-order

By application of the implicit function Theorem around $(\phi_0, 0)$, we know that the solution satisfies

$$\phi(\sigma) = \phi_0 + \sigma z^1 + o(\sigma)$$

where z_t^1 is solution of

$$D_\phi \mathcal{N}(\phi_0, 0) z^1 = -D_\sigma \mathcal{N}(\phi_0, 0)$$

Elementary computations lead to:

$$D_\sigma \mathcal{N}(\phi_0, 0) = \sum_{j=1}^{N} p_{s_t j}(z_{s_t}, 0) \partial_4 f_{s_t}(z_j, z_{s_t}, z_{s_{t-1}}, 0) \varepsilon_t$$

$$D_\phi \mathcal{N}(\phi_0, 0) h = \mathbb{E}_t^0 [a(s_t, s_{t+1}) h_{t+1}] + b(s_t) h_t + c(s_t) h_{t-1}$$

Thus,

$$z^1(s^t) = R_\infty^{-1}(s_t) c_{s_t} z^1(s^{t-1}) - R_\infty^{-1}(s_t) \sum_{j=1}^{N} p_{s_t j}(z_{s_t}, 0) \partial_4 f_{s_t}(z_j, z_{s_t}, z_{s_{t-1}}, 0) \varepsilon_t$$

E.2 Second-order

To obtain the second-order approximation, we derive twice the implicit equation defining the solution $\phi(\sigma)$:

$$\mathcal{N}(\phi(\sigma), \sigma) = 0 \tag{14}$$

For a multivariate model, $\mathcal{N} = \begin{bmatrix} \mathcal{N}_1 \\ \mathcal{N}_2 \\ \vdots \\ \mathcal{N}_p \end{bmatrix}$, we denote by

$$D_{\phi,\phi} \mathcal{N} = \begin{bmatrix} \text{Vect}(D_{\phi,\phi} \mathcal{N}_1)' \\ \text{Vect}(D_{\phi,\phi} \mathcal{N}_2)' \\ \vdots \\ \text{Vect}(D_{\phi,\phi} \mathcal{N}_p)' \end{bmatrix}$$

Thus, the second derivative of equation (14) is

$$D_{\phi,\phi} \mathcal{N}(\phi_0, 0) z^1 \otimes z^1 + 2 D_{\phi,\sigma}(\phi_0, 0) z^1 + D_{\sigma,\sigma} \mathcal{N}(\phi_0, 0) + D_\phi \mathcal{N}(\phi_0, 0) z^2 = 0$$

This implies that z_t^2 is solution of:

$$\mathbb{E}_t [a_{s_t, s_{t+1}} z_{t+1}^2] + b_{s_t} z_t^2 + c_{s_t} z_{t-1}^2 = h_t$$

where

$$h = -D_{\phi,\phi} \mathcal{N}(\phi_0, 0) z^1 \otimes z^1 - 2 D_{\phi,\sigma}(\phi_0, 0) z^1 - D_{\sigma,\sigma} \mathcal{N}(\phi_0, 0)$$

As a first step, we rewrite

$$z_t^1 = \Omega(s_t) z_{t-1}^1 + \Delta(s_t) \varepsilon_t$$

and compute

$$h_t = \gamma_1(s_t) z_t^1 \otimes z_t^1 + \gamma_2(s_t) z_t^1 \otimes z_{t-1}^1 + \gamma_3(s_t) z_{t-1}^1 \otimes z_{t-1}^1 + \gamma_4(s_t) z_t^1 \otimes \varepsilon_t$$

$$+ \gamma_5(s_t) z_{t-1}^1 \otimes \varepsilon_t + \gamma_6(s_t) \varepsilon_t \otimes \varepsilon_t + \gamma_7(s_t) \text{Vect}(\Sigma)$$

where

$$\gamma_1(s_t) = -\sum_j \big[f_{s_t}(z_j, z_{s_t}, z_{s_{t-1}}, 0)\partial_{11}p_{s_tj}(z_{s_t}, 0) + p_{s_tj}(z_{s_t}, 0)\partial_{11}f_{s_t}(z_j, z_{s_t}, z_{s_{t-1}}, 0)\Omega(j) \otimes \Omega(j)$$

$$+ p_{s_tj}(z_{s_t}, 0)\partial_{22}f_{s_t}(z_j, z_{s_t}, z_{s_{t-1}}, 0)\big]$$

$$-2\sum_j \big[\partial_1 f_{s_t}(z_j, z_{s_t}, z_{s_{t-1}}, 0) \otimes \partial_1 p_{s_tj}(z_{s_t}, 0)\Omega(j) \otimes \mathbb{1}_n + \partial_2 f_{s_t}(z_j, z_{s_t}, z_{s_{t-1}}, 0) \otimes \partial_1 p_{s_tj}(z_{s_t}, 0)+$$

$$p_{s_tj}(z_{s_t}, 0)\partial_{12}f_{s_t}(z_j, z_{s_t}, z_{s_{t-1}}, 0)\Omega(j) \otimes \mathbb{1}_n\big]$$

$$\gamma_2(s_t) = -2\sum_j \big[\partial_1 p_{s_tj}(z_{s_t}, 0) \otimes \partial_3 f_{s_t}(z_j, z_{s_t}, z_{s_{t-1}}, 0) + p_{s_tj}(z_{s_t}, 0)\partial_{23}f_{s_t}(z_j, z_{s_t}, z_{s_{t-1}}, 0)\Omega(j) \otimes \mathbb{1}_n$$

$$+ p_{s_tj}(z_{s_t}, 0)\partial_{13}f_{s_t}(z_j, z_{s_t}, z_{s_{t-1}}, 0)\big]$$

$$\gamma_3(s_t) = -\sum_j p_{s_tj}(z_{s_t}, 0)\partial_{33}f_{s_t}(z_j, z_{s_t}, z_{s_{t-1}}, 0)$$

$$\gamma_4(s_t) = -2\sum_j \big[\partial_1 p_{s_tj}(z_{s_t}, 0) \otimes \partial_4 f_{s_t}(z_j, z_{s_t}, z_{s_{t-1}}, 0) + p_{s_tj}(z_{s_t}, 0)\partial_{14}f_{s_t}(z_j, z_{s_t}, z_{s_{t-1}}, 0)\Omega(j) \otimes \mathbb{1}_p$$

$$p_{s_tj}(z_{s_t}, 0)\partial_{24}f_{s_t}(z_j, z_{s_t}, z_{s_{t-1}}, 0)\big]$$

$$\gamma_5(s_t) = -2\sum_j p_{s_tj}(z_{s_t}, 0)\partial_{34}f_{s_t}(z_j, z_{s_t}, z_{s_{t-1}}, 0)$$

$$\gamma_6(s_t) = -\sum_j p_{s_tj}(z_{s_t}, 0)\partial_{44}f_{s_t}(z_j, z_{s_t}, z_{s_{t-1}}, 0)$$

$$\gamma_7(s_t) = -\sum_j f'_{s_t}(z_j, z_{s_t}, z_{s_{t-1}}, 0) \otimes \partial_{22}p_{s_tj}(z_{s_t}, 0)$$

We define the $np \times np$ permutation matrix $T_{z\varepsilon}$ by

$$T_{z\varepsilon}(i, p * ((i-1)[n]) + 1 + E((i-1)/n)) = 1, \quad \forall i \in \{1, \cdots, np\}$$

$T_{z\varepsilon}$ is such that:

$$\varepsilon_t \otimes z^1_{t-1} = T_{z\varepsilon} z^1_{t-1} \otimes \varepsilon_t$$

This leads to:

$$h_t = -[\alpha_1(s_t) z^1_{t-1} \otimes z^1_{t-1} + \alpha_2(s_t) z^1_{t-1} \otimes \varepsilon_t + \alpha_3(s_t)\varepsilon_t \otimes \varepsilon_t + \alpha_4(s_t)\text{Vect}(\Sigma)]$$

where

$$\alpha_1 = \gamma_1 \Omega(s_t) \otimes \Omega(s_t) + \gamma_2 \Omega(s_t) \otimes \mathbb{1} + \gamma_3$$

$$\alpha_2 = \gamma_1 \Omega(s_t) \otimes \Delta(s_t) + \gamma_4 \Omega(s_t) \otimes \mathbb{1} + \gamma_5 + [\gamma_1 \Delta(s_t) \otimes \Omega(s_t) + \gamma_2 \Delta(s_t) \otimes \mathbb{1}]\, T_{z\varepsilon}$$

$$\alpha_3 = \gamma_6 + \gamma_1 \Delta(s_t) \otimes \Delta(s_t) + \gamma_4 \Delta(s_t) \otimes \mathbb{1}$$

$$\alpha_4 = \gamma_7$$

We introduce w^2 such that

$$z^2_t = \Omega(s_t) z^2_{t-1} + w^2_t$$

The process w^2 is solution of the equation:

$$R_\infty(s_t)^{-1}\mathbb{E}^0_t[A_{s_t,s_{t+1}} w^2_{t+1}] + w^2_t = -R_\infty(s_t)^{-1} h_t$$

We look for w_t^2 under the form

$$w_t^2 = \alpha(s_t)z_{t-1}^1 \otimes z_{t-1}^1 + \beta(s_t)z_{t-1}^1 \otimes \varepsilon_t + \gamma(s_t)\varepsilon_t \otimes \varepsilon_t + \delta(s_t)\text{Vect}(\Sigma) \tag{15}$$

Then, α, β, γ and δ satisfy:

$$\sum_j R_\infty(s_t)^{-1}A_{s_t,j}\alpha(j)[\Omega^*(s_t) \otimes \Omega^*(s_t)] + \alpha(s_t) = -R_\infty(s_t)^{-1}\alpha_1(s_t)$$

$$\sum_j R_\infty(s_t)^{-1}A_{s_t,j}\alpha(j)[\Omega^*(s_t) \otimes \Delta^*(s_t) + \Delta^*(s_t) \otimes \Omega^*(s_t)T_{z\varepsilon}] + \beta(s_t) = -R_\infty(s_t)^{-1}\alpha_2(s_t)$$

$$\sum_j R_\infty(s_t)^{-1}A_{s_t,j}\alpha(j)[\Delta^*(s_t) \otimes \Delta^*(s_t)] + \gamma(s_t) = -R_\infty(s_t)^{-1}\alpha_3(s_t)$$

$$\sum_j R_\infty(s_t)^{-1}A_{s_t,j}(\gamma(j) + \delta(j)) + \delta(s_t) = -R_\infty(s_t)^{-1}\alpha_4(s_t)$$

Introducing

$$\Upsilon_j = \begin{bmatrix} \text{Vect}[-R_\infty(1)^{-1}\alpha_j(1)] \\ \text{Vect}[-R_\infty(2)^{-1}\alpha_j(2)] \\ \vdots \\ \text{Vect}[R_\infty(N)^{-1}\alpha_j(N)] \end{bmatrix}$$

The vector $\text{Vec\_}\alpha = \begin{bmatrix} \text{Vect}(\alpha(1)) \\ \cdots \\ \text{Vect}(\alpha(N)) \end{bmatrix}$ is solution of

$$\text{Vec\_}\alpha = (\mathbb{1} + M_\Omega)^{-1}U_1$$

where

$$M_\Omega = [p_{ij}(\Omega^*)'(i) \otimes (\Omega^*)'(i) \otimes R_\infty(i)^{-1}A_{i,j}]$$

β and γ are explicitly given by α, and $\text{Vec\_}\delta$ is solution of

$$\text{Vec\_}\delta = (\mathbb{1} + M_{I_p})^{-1} * (U_4 - M_{I_p}\text{Vec\_}\gamma)$$

where

$$M_{I_p} = [p_{ij}\mathbb{1} \otimes \mathbb{1} \otimes R_\infty(i)^{-1}A_{i,j}]$$

F Proof of Corollary 5

Proof. The case $q = 1$ follows directly from Taylor expansion (7) and boundedness of shocks. Without loss of generality, we consider $q = 2$.
Fix M any subset of $\{(z_t, \varepsilon_t)\}$, then

$$f_2(M) = \int_{z_t \in M} \int_{\varepsilon_1, \varepsilon_2} f_1(f_1(z_t, \varepsilon_1), \varepsilon_2) d\varepsilon_1 d\varepsilon_2 dz_t$$

Moreover, we have that

$$\hat{f}_2^p(M) = \int_{z_t \in M} \int_{\varepsilon_1, \varepsilon_2} \hat{f}_1^p(\hat{f}_1^p(z_t, \varepsilon_1), \varepsilon_2) d\varepsilon_1 d\varepsilon_2 dz_t$$

Thus

$$\|f_2(M) - \hat{f}_2^p(M)\| \leq C\sigma^{p+2}$$

which leads to the fact that

$$\pi(\hat{f}_2^p, f_2) = o(\sigma^{p+1}).$$

For higher forecasting horizons, the same reasoning applies, this ends the proof of Corollary 5. □

G Figure

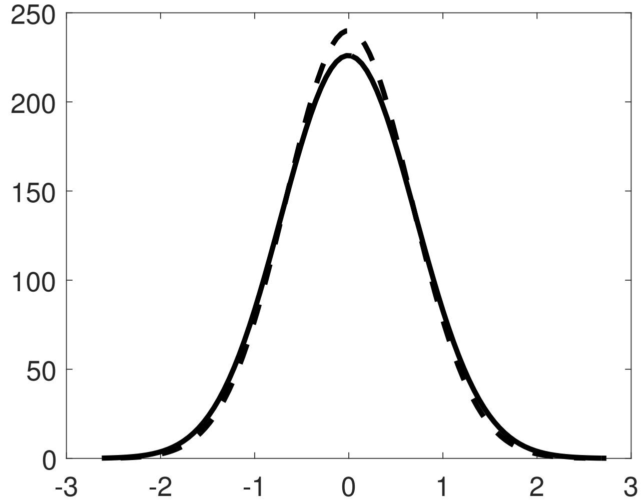

Figure 5: Comparison of distribution of inflation in each regime

Note: The figure reports demeaned ergodic distribution of the inflation in regime 1 (thick black line) and regime 2 (dotted line).

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