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DYNAMICS OF ASSETS, LIQUIDITY, AND INEQUALITY IN ECONOMIES WITH DECENTRALIZED MARKETS

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Dynamics of Assets, Liquidity, and Inequality in Economies with Decentralized Markets

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Abstract

An algorithm for computing Dynamic Nash Equilibria (DNE) in an extended version of Kiyotaki and Wright (1989) (hereafter KW) is proposed. The algorithm computes the equilibrium profile of (pure) strategies and the evolution of the distribution of three types of assets across three types of individuals.

It has two features that together make it applicable in a wide range of macroeconomic experiments: (i) it works for any feasible initial distribution of assets; (ii) it allows for multiple switches of trading strategies along the transitional dynamics.

The algorithm is used to study the relationship between liquidity, production, and inequality in income and in welfare, in economies where assets fetch different returns and agents have heterogeneous skills and preferences.

One experiment shows a case of reversal of fortune. An economy endowed with a low-return asset takes over a similar economy endowed with a high-return asset because, in the former economy, a group of agents abandon a rent-seeking trading behavior and increase their income by trading and producing more intensively. A second experiment shows that a reduction of market frictions leads both to higher income and lower inequality. Other experiments evaluate the propagation mechanism of shocks that hit the assets' returns.

A key result is that trade and liquidity tend to squeeze income inequality.

Keywords: Trading Strategies, Liquidity, Matching, Decentralized Markets.
JEL codes: C61, C63, E41, E27, D63.

*Correspondence to: maCorrespondence: 65 Rue F. Dostoïevski, 06902 Valbonne, France; Tel.: +33 (4)89737107; E-mail: maurizio.iacopetta@sciencespo.fr. In the elaboration of the manuscript I have used materials of an ongoing joint work with Federico Bonetto. I have also benefited from comments from seminar participants at the Poole School of Management, North Carolina State University, and at the University of Rome, Tor Vergata, and from discussions with Todd Keister and Ricardo Lagos. All remaining errors are mine.

1 Introduction

A method for computing dynamic equilibria in a model with genuine heterogenous agents that trade in decentralized markets, as exemplified in Kiyotaki and Wright (1989) (hereafter KW), is proposed. Although the original KW model has been enriched and simplified in many ways (see Lagos et al., 2014) the implications of its dynamics with respect to the movement of aggregate variables, such as consumption, assets, production, and income inequality, have not been investigated.

Most of the dynamic general equilibrium theory is based on the idea that individuals trade their goods and services in centralized markets. By assuming that agents trade with each other, the KW types of models offer an alternative view of how macroeconomic phenomena emerge from individual interactions. A prominent feature of this approach is that the links between the behavior of the macroeconomic series and micro-level decisions are more articulated than in Walrasian models.

Performing macroeconomic experiments in decentralized market economies has been elusive, especially when agents are allowed to carry assets over time. The main difficulty is that the evolution of the assets and the trajectories of trading decisions are intertwined and multidimensional. This paper demonstrates that dynamic equilibria can be obtained numerically, despite the fact that agents always trade with each other and that their optimal decisions incorporate information on future distribution of traded assets. It complements the work of Molico (2006), and Molico and Chiu (2010, 2011) who studied macroeconomic aggregates in decentralized settings combining analytic and computational methods. Although similar in spirit, the present study emphasizes the role of the liquidity of real assets[1] rather than that of fiat money, and deals with the evolution over time of the distribution of wealth in transitional dynamics rather than in the steady state.

The proposed algorithm computes the evolution of the distribution of assets and the profile of trading strategies in separate steps, according to an iterative guess procedure. One important feature of the algorithm is that it does not require any ad-hoc assumptions on the agents' ability to process or access to information: agents are rational, forward-looking, with full knowledge of the distribution of assets and of the trading strategies of other individuals.[2] The main result is that, despite the lack of competitive markets, equilibria can be computed

[1] There is growing consensus that real assets play an important role in facilitating trade (see section 10 of Lagos et al. (2014) and citations therein).

[2] Matsuyama et al. (1993), Wright (1995), Luo (1999) and Sethi (1999) search for equilibria in a similar environment using evolutionary dynamics. Marimon et al. (1990) and Basçı (1999) explore how artificially-intelligent agents can learn to play equilibrium in the model. Brown (1996) and Duffy and Ochs (1999), Duffy (2001), and Duffy and Ochs (2002) in a number of laboratory experiments, ask the extent to which agents adopt "speculative" strategies when the theory indicates these should prevail.

in all interesting parametrizations of the economy.

The algorithm builds directly on the concept of (open loop) Nash equilibrium. The computational task is to calculate the law of motion for the distribution of different types of assets, across heterogenous individuals, and to determine the best response of each individual. Because the focus is on pure strategies[3] and goods are not divisible, there is a finite number of choices and the distribution of assets is discrete.[4] Therefore, at each point in time, the state of the economy and agents' choices are represented by a finite-dimensional vector.

A key methodological insight is that optimal trading strategies can be determined by integrating backward in time the value functions (in differences), starting from a neighborhood of the steady state. The information about the potential gains in deviating from the predetermined profile of strategies is then used in the following iteration when a new time-pattern of the wealth distribution is calculated. In particular, each round of the iterative scheme consists of two steps. In a first step, given the initial state of the economy, a time-pattern of the asset distribution is derived from the outcome of decentralized meetings. In the second step, the algorithm verifies whether any agent has an incentive to deviate from such profile of strategies. The algorithm is able to uncover patterns in which one or more types of agents switch trading strategies.

Because the algorithm works for any feasible initial state, it gives ample freedom in designing macroeconomic experiments whose results are interesting on their own. A first experiment parallels the development of two similar economies that differ only for the rate of return of one type of assets. Starting from the same initial position, the high-return economy is taken over by the other economy. The reversal occurs because in the low-return economy a group of agents abandon a rent-seeking trading behavior, and instead prefer to increase their income by trading and producing more intensively. A second experiment reveals that an amelioration of market frictions leads to a more equitable distribution of income, as assets become more liquid. Additional experiments put at the center stage of the analysis the heterogeneity of skills and assets in interpreting the propagation mechanism of shocks that hit the returns of assets.

A common theme of all experiments is the relationship between income and welfare inequality and liquidity. There are two main reasons for inequality: (i) assets fetch different returns but every individual is allowed to carry only one type of asset at a time; (ii) assets

[3]For a discussion of mixed strategies in a similar environment see Kehoe et al. (1993) and Renero (1998).

[4]The KW dynamics are not directly comparable to the ones generated in Trejos and Wright (1995), where the key evolving variable is the value of fiat money, nor to the ones based on Lagos and Wright (2005) where the dynamics are only over two periods. They are also quite different from those in Boldrin, Kiyotaki, and Wright (1993) where the only state variable is the share of the population engaged in production or in trade activity but there are no storable assets.

differ in their degree of liquidity. Although individuals have equal opportunities in accessing the decentralized market, the odds that a match leads to the maximum gain (which comes by acquiring the consumption goods) varies both across individuals and across time. One question that will be explored then is how liquidity can squeeze or magnify inequality. A general pattern that emerges is that when a shock induces some agents to play speculative strategies, income inequality shrinks because more frequent market interactions tend to correct inequality due to the variance of returns.

The rest of the paper is organized as follows. The next section briefly describes the economic environment, illustrates the evolution of the distribution of inventories under a given profile of strategies, and defines the best response functions of three types of representative agents. The section that follows, studies the properties of the dynamical system and details the numerical algorithm. Section (4) defines macroeconomic indicators. Section (5) proposes four macroeconomic experiments (two more are in the Appendix): One illustrates a case of fortune reversal between two economies. A second one follows the responses to a shock that improves the matching rate. A third experiment studies the fluctuations in liquidity and production caused by a shock to one of the assets' rates of return. The last experiment deals with multiple switches. Section (6) summarizes the results and suggests further applications of the algorithm.

2 The Model Economy

There are only minor differences with respect to the decentralized economy described in KW: time is continuous; the ranking on the returns across assets is allowed to change; and agents are not necessarily equally distributed across types. A brief description follows. The economy is populated by three types of individuals, denoted by $1, 2$, and 3. There is a large number of agents of each type, N_i, for $i = 1, 2, 3$. The overall size of the population is N. The fraction of each type is $\mu_i = N_i/N$. People live forever. An agent of type i derives utility exclusively from consuming good i and can produce only good $i + 1$ (mod. 3). Production takes place immediately after consumption. Agent i's[5] instantaneous utility from consumption and the disutility of producing good $i + 1$ are denoted by U_i and D_i, respectively, and their difference is $u_i = U_i - D_i$. There is a capacity constraint. At each instant of time an individual can hold only one unit of some storable good i that offers an instantaneous return r_i, measured in units of utility (the terms good, commodity, inventory, and asset will be used interchangeably). The return of good i is the same for all agents of

[5] When no confusion arises, I will use the loose language of calling an agent of type i simply as agent or individual i.

any type. The discount rate is denoted by $\rho > 0$.

A pair of agents is randomly and uniformly chosen from the population to meet for a possible trade. After a pair is formed, the waiting time for the next pair to be called is governed by a Poisson process with intensity α. This implies that the probability that an agent is called for a (first) match before time t is $1 - e^{-\alpha t}$. A meeting does not necessarily mean that the two parties trade. A bilateral trade occurs if and only if it is mutually agreeable. If both agents want what the other has, they swap goods. Otherwise, they part company and keep the same good in the inventory as they wait for the next call. Agent i always accepts good i. Furthermore, because agent i consumes his consumption good immediately upon reception, she never carries good i. Therefore, agent i always enters the market either with one unit of good $i+1$, or with one unit of good $i+2$.

The proportion of type i agents that hold good j at time t is denoted by $p_{i,j}(t)$. Then, the vector $\tilde{\mathbf{p}}(t) = \{p_{i,j}(t)\}$ for $i = 1, 2, 3$ and $j = 1, 2, 3$ describes the state of the economy at time t (from now on, it is understood that i and j go from 1 to 3). But since $p_{i,i}(t) = 0$,

$$p_{i,i+1}(t) + p_{i,i+2}(t) = \mu_i. \tag{1}$$

for any $t > 0$, the state of the economy can be represented in a more parsimonious way by $\mathbf{p}(t) = \{p_{1,2}(t), p_{2,3}(t), p_{3,1}(t)\}$. To simplify the notation, sometimes $p_{i,i+1}$becomes p_i. An individual i has only to decide whether to exchange his production good for the other type of good. Agent i's choice in favor of indirect trade is denoted with $\tau_i(t) = 1$ and against it with $\tau_i(t) = 0$. Agent i has to select a time path $\tau_i(t)$ that maximizes her expected stream of present and future net utility, given other agents' paths of strategies, $\boldsymbol{\theta}(t) = [\theta_1(t), \theta_2(t), \theta_3(t)]$, and $\mathbf{p}(t)$, for any $t > 0$.

2.1 Distribution of Inventories and Value Functions

For a given profile of strategies $\boldsymbol{\theta}(t)$, the evolution in the stock of good $i+1$ held by agents of type i is given by[6]

$$\dot{p}_{i,i+1} = \alpha\{p_{i,i+2}[p_{i+1,i}(1 - \theta_{i+1}) + p_{i+2,i} + p_{i+2,i+1}(1 - \theta_i)] - p_{i,i+1}[p_{i+1,i+2}\theta_i]\}. \tag{2}$$

The terms inside the brackets before the minus sign calculate the probability that a type i agent is called for a match while holding good $i+2$, and ends up in the position of carrying good $i+1$. Such an event materializes either because of barter or because the agent leaves the meeting with good i, consumes it, and then immediately produces good $i+1$. The following expression accounts for the probability that an agent of type i who holds good $i+1$ ends up

[6] Duffie and Sun (2012) show that in a similar matching environment frequency coincides with probability. See in particular their Theorem 1.

with good $i+2$. The behavior of $p_{i,i+2}$ is derived through (1). The ensemble of the system that describes the evolution of the inventories' distribution is denoted by $F(\mathbf{p}(t))$.

Consider now a representative agent of type i that has to compute her best profile of strategies, given a pattern of inventories $\mathbf{p}(t)$ and a pattern of strategies for other agents $\boldsymbol{\theta}(t)$ – including those of her own type. Let $V_{i,j}(t)$ be value function when carrying good j at time t. When $j = i+1$, we have that

$$\begin{aligned} V_{i,i+1}(t) = \max_{\{\tau_i(s)\}_{s\geq t}} \int_t^\infty \alpha e^{-\alpha(s-t)} \{ & e^{-r(s-t)} ([p_{i,i+2}\tau_i(1-\theta_i) + p_{i+1,i+2}\tau_i] V_{i,i+2} + \\ & + [1 - p_{i,i+2}\tau_i(1-\theta_i) - p_{i+1,i+2}\tau_i] V_{i,i+1} + [p_{i+1,i} + p_{i+2,i}\theta_{i+2}] u_i) + \\ & + \frac{1-e^{-(s-t)r}}{r} r_{i+1} \} ds, \end{aligned} \tag{3}$$

where the term $\alpha e^{-\alpha(s-t)} ds$ measures the probability that an agent of type i is called to form a match for the first time after time t in the time interval $(s, s+ds)$. The term after the discount factor is the probability that this agent goes through indirect trading – in which case, she is left with $V_{i,i+2}$ as continuation value. Else, she ends up with good $i+1$ either because no trade takes place or because she acquired her consumption good – an event that occurs with probability $p_{i+1,i} + p_{i+2,i}\theta_{i+2}$ – and then produces good $i+1$. The last term is the return for holding the asset $i+1$ from time t to time s. An additional equation for $V_{i,i+2}(t)$, reported in the Appendix, completes the description of the optimization problem.

Let $\Delta_i(s) \equiv V_{i,i+1}(s) - V_{i,i+2}(s)$, and let $\tilde{\tau}_i(s; \boldsymbol{\theta}(s), \mathbf{p}(s))$ denote the optimal (or best) response profile of strategies of representative agent i to other players' strategies $\boldsymbol{\theta}(s)$ along the pattern of inventories $\mathbf{p}(s)$ for $s > t$. Then, it must be that

$$\tilde{\tau}_i(s; \boldsymbol{\theta}(s), \mathbf{p}(s)) = \begin{cases} 1 & \text{if } \Delta_i(s) < 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

for any $s \geq t$. Hence, the way the problem has been formulated corresponds to a Markov decision process in which the representative agent optimizes over a sequence of functions $\tilde{\tau}_i(t)$ that allows the ex-post decision on indirect trading to vary with the current state of the inventory distribution, and the pattern of strategies of other agents.

2.2 Dynamic Nash Equilibrium

Given an initial distribution of inventories $\mathbf{p}(0) = \mathbf{p}_0$, a *Dynamic Nash Equilibrium (DNE)* is a path of strategies $\boldsymbol{\theta}^*(t)$ together with a distribution of inventories $\mathbf{p}^*(t)$ such that for all $t > 0$:

i. $\mathbf{p}^*(t)$ and $\boldsymbol{\theta}^*(t)$ satisfy the dynamics equations (2) with the initial condition $\mathbf{p}^*(0) = \mathbf{p}_0$, and subject to the constraint (1);

ii. For all $t > 0$, every agent maximizes his or her expected utility given the profile strategies of the rest of the population;

iii $\tilde{\tau}_i(t; \boldsymbol{\theta}^*(t), \mathbf{p}^*(t)) = \theta_i^*(t)$ for all $t > 0$.

It is the objective of the next section to explain how to compute a DNE.

The coming section outlines an algorithm that searches for the equilibrium strategies $\boldsymbol{\theta}^*(t)$ and for the distribution of assets $\mathbf{p}^*(t)$. The following section will then use it to compute aggregate time series along the transitional dynamics, and to study the impulse responses to shocks that affect the matching rate, and asset returns.

3 The Algorithm

An informal description of the algorithm is followed by a more formal presentation. Then an alternative algorithm is discussed. The algorithm builds directly on the concept of open-loop Nash equilibrium with many players as in Fudenberg and Levine (1988).[7] The idea is to obtain an equilibrium such that, given the actions of all other players, no player can make any gain by changing her action. The law of motion of the endogenous multi-dimensional state variable of the economy, $\mathbf{p}(t)$, is a function of the profile of strategies $\boldsymbol{\theta}(t)$, also a multi-dimensional object, that can change over time in discrete steps. The value functions $V_{i,j}$ are the criteria that agents follow to decide their optimal patterns of strategies. These can be obtained analytically on the steady state, but are difficult to determine analytically for non steady state equilibria. The algorithm computes the Nash equilibrium policies, and the distribution of assets iteratively. It uses two properties of the system. First, for any interesting profile of strategies, the state variable, $\mathbf{p}(t)$, converges towards a fixed point (which is not necessarily an Nash equilibrium). Second, along a given pattern of $\mathbf{p}(t)$, the numerical value functions converge to their theoretical values when integrated backward in

[7] The design of the algorithm is quite different from others used in the macroeconomic literature. The main concern of the one proposed by Krussel and Smith (1998) is to find a parsimonious way of conveying the essential information of the state of the economy to agents. They assume that agents are boundedly rational in their perceptions of how the state variables evolves over time but are sophisticated enough so that the errors that they make because they are not fully rational become negligible. The objective of the algorithm in Marimon et al. (1998) is to understand under which conditions artificial agents operating in a KW economy follow a predefined classifier system can select the 'speculative' equilibrium. The present algorithm is conceptually close to the one proposed by Pakes and McGuire (1994, 2000), in that it searches for a policy function in a dynamic environment. But the domain of applications and the type of issues that the algorithm solves are different. In particular, the challenge for Pakes and McGuire (1994, 2000) (as much as for Krussel and Smith, 1998) is to deal with the high-dimensionality of the state variable. In the present work, the main challenge is finding the initial point of the value functions associated to a particular state of the economy.

time. It is then possible to verify whether the value function of a representative agent along a specific trajectory $\mathbf{p}(t)$ is consistent with the profile of strategies that are used to obtain such a trajectory $\mathbf{p}(t)$. The term consistency means that no agent has an incentive to deviate at any point in time from the designated profile of strategies. The algorithm generates a sequence of rounds that seeks convergence towards an open-loop Nash equilibrium for $\mathbf{p}(t)$ and $\boldsymbol{\theta}(t)$.

After a brief description of the two properties, the steps of the iterative procedure will be detailed.

3.1 Convergence to Stationary Distributions

Eq. (2) says that the system of the distribution of assets is given by (the time index is dropped):

$$\dot{p}_{1,2} = \alpha\{p_{1,3}[p_{2,1}(1-\theta_2) + p_{3,1} + p_{3,2}(1-\theta_1)] - p_{1,2}p_{2,3}\theta_1\}, \tag{5}$$

$$\dot{p}_{2,3} = \alpha\{p_{2,1}[p_{3,2}(1-\theta_3) + p_{1,2} + p_{1,3}(1-\theta_2)] - p_{2,3}p_{3,1}\theta_2\}, \tag{6}$$

$$\dot{p}_{3,1} = \alpha\{p_{3,2}[p_{1,3}(1-\theta_1) + p_{2,3} + p_{2,1}(1-\theta_3)] - p_{3,1}p_{1,2}\theta_3\}. \tag{7}$$

Proposition 1. For seven out of eight time-constant profiles of strategies, $\mathbf{p}(t)$ *converges to a stationary distribution, from any initial position.*[8]

Proof. See Appendix

Next, the question of which of the stationary distributions is a Nash equilibrium is explored. Because the answer depends crucially on the relative size of the three groups of agents, the attention is restricted to an economy in which agents are uniformly distributed. To prove that a given steady state distribution is a NE, one needs to verify that the sign of Δ_i is consistent with the profile of strategies assumed for that particular steady state distribution.

Proposition 2. When the population is equally split across the three types, the following six steady state Nash Equilibria (NE) exist:

[8] For the profile of strategies (1,1,1), proving that $\mathbf{p}(t)$ converges to a fixed point is more challenging. But such profile of strategy happens not to be a steady state Nash equilibrium (see Proposition 2).

Strategies	Assets Distribution	Strategies	Assets Distribution
(0,1,0)	$\frac{1}{3}[1, \frac{1}{2}, 1]$	(1,1,0)	$\frac{1}{3}[a, b, 1]$
(1,0,0)	$\frac{1}{3}[\frac{1}{2}, 1, 1]$	(1,0,1)	$\frac{1}{3}[b, 1, a]$
(0,0,1)	$\frac{1}{3}[1, 1, \frac{1}{2}]$	(0,1,1)	$\frac{1}{3}[1, a, b]$

where $a = \frac{1}{2}\sqrt{2}$ *and* $b = \sqrt{2} - 1$. *In some cases a pair of NE coexist.*

Proof. See Technical Appendix.

Table (1) details which of these equilibria exists under every return configuration.

3.2 Reverse Integration

This section examines the dynamic properties of the value functions of the representative agent i. These properties are important because they will be used by the algorithm to verify whether a particular distribution of assets and of strategies is an open-loop Nash equilibrium. According to (4), what matters for representative i's decision is only the *sign* of Δ_i. After some algebra, one obtains that

$$\dot{\Delta}_i = (\alpha\chi_i + r)\Delta_i + \omega_i, \tag{8}$$

where $\chi_i \equiv p_{i,i+2}\tau_i(1-\theta_i) + p_{i+1,i+2}\tau_i + p_{i+1,i}(1-\theta_{i+1}) + p_{i+2,i} + (p_{i,i+1}\theta_i + p_{i+2,i+1})(1-\tau_i) > 0$ and $\omega_i \equiv -\alpha[p_{i+1,i}\theta_{i+1} - p_{i+2,i}(1-\theta_{i+2})]u_i + (r_{i+2} - r_{i+1})$.

For a given pattern of χ_i the solution of Δ_i can be obtained numerically by integrating (8) backward in time,[9] starting from a neighborhood of the steady state Δ_i^*, where this satisfies $(\alpha\chi_i + \delta)\Delta_i^* + \omega_i = 0$.[10]

It is important to recognize that the distribution of inventories $\mathbf{p}(t)$ and the value functions in differences $\Delta_i(t)$ could be studied together as a whole system. In particular, it is possible to generate equilibrium trajectories by simply applying the principle of backward induction as long as the profile of strategies of the population are kept consistent with the sign of $\Delta_i(t)$ at each point in time. The problem with this procedure is that it is hard to build a trajectory that goes through a designated point in the space of the assets' distribution. In models with unidimensional state variable, backward induction works well– at least when

[9] The mechanism is illustrated in a figure contained in the Technical Appendix.

[10] See Technical Appendix for details. This technique has been used in different contexts. Gear and Kevredikis (2008) explains how it can be applied in more general situations. Pakes and McGuire (1994) and Pakes and McGuire (2000), develop a reverse integration algorithm for industrial organizations' problems. In the context of growth models, Brunner and Strulik (2000) explains the construction of manifolds by means of backward integration.

Table 1: Steady State Equilibria, Strategies, and Money

	Returns	F	S	Assets (F)	Assets (S)	M (F)	M (S)
			Panel A				
R1	$r_3 < r_2 < r_1$	(0,1,0)	(1,1,0)	$\frac{1}{3}[1, \frac{1}{2}, 1]$	$\frac{1}{3}[a, b, 1]$	1	1,3
R2	$r_2 < r_1 < r_3$	(1,0,0)	(1,0,1)	$\frac{1}{3}[\frac{1}{2}, 1, 1]$	$\frac{1}{3}[b, 1, a]$	3	2,3
R3	$r_1 < r_3 < r_2$	(0,0,1)	(0,1,1)	$\frac{1}{3}[1, 1, \frac{1}{2}]$	$\frac{1}{3}[1, a, b]$	2	1,2
			Panel B				
R4	$r_2 < r_3 < r_1$	(1,1,0)	(1,0,1)	$\frac{1}{3}[a, b, 1]$	$\frac{1}{3}[b, 1, a]$	1, 3	2, 3
R5	$r_3 < r_1 < r_2$	(0,1,1)	(1,1,0)	$\frac{1}{3}[1, a, b]$	$\frac{1}{3}[a, b, 1]$	1, 2	1, 3
R6	$r_1 < r_2 < r_3$	(1,0,1)	(0,1,1)	$\frac{1}{3}[b, 1, a]$	$\frac{1}{3}[1, a, b]$	3, 2	1, 2

- Note: $a = \frac{1}{2}\sqrt{2}$ and $b = \sqrt{2}-1$. The F and S columns contain the triplet $(\theta_1, \theta_2, \theta_3)$ that describes the fundamental and the speculative steady state strategy, respectively. The following two columns are the assets' stationary distributions: $[p_{1,2}, p_{2,3}, p_{3,1}]$. The last two columns indicate which asset is traded indirectly – or acts as 'money'– in the fundamental and speculative equilibrium, respectively. In rows R1 through R3 the equilibria are unique; only one type of agent plays speculative strategies in the S equilibrium. In the R4-R6 rows the two equilibria may coexist; two types of agents play speculative strategies in the S equilibrium.

the dynamics are not cyclical or chaotic – because it can be stopped when the state variable reaches a desired level. But as the dimension of the manifold expands, guiding the system towards a particular point on the state space (that is on the initial condition) becomes a hurdle. In fact, if the system has Liapunov exponents of a different order of magnitudes, some regions of the manifold cannot even be reached. Conversely, the method proposed here gives total control on the initial condition, a feature that turns out to be essential for most interesting macroeconomic experiments.

3.3 Contraction Iteration on the Profile of Strategies

The algorithm sets up an iteration on the profile of strategies $\boldsymbol{\theta}(t)$ and on the distribution of assets $\mathbf{p}(t)$. The value function $V_{i,j}(t)$ of the representative agents i holding good j serves as device to update the guess on the profile of strategies, and to determine when the

algorithm has converged. Since only pure strategies are considered,[11] a representative agent i has a binary choice at each point in time. The algorithm seeks for the convergence of the representative agent i's best response, $\tilde{\tau}_i(t;\boldsymbol{\theta}(t),\mathbf{p}_0)$, to the profile followed by the rest of individuals of her type $\theta_i(t)$. When the representative individual i does not have an interest in deviating from a strategy that coincides with that followed by the rest of type i agents, a Dynamic Nash Equilibrium is found. The algorithm works as follows.

Step 1. The distribution of inventories $F(\mathbf{p}(t))$ is integrated forward in time starting from some $\mathbf{p}_0$, under a guess $\boldsymbol{\theta}^{(0)}(t)$. The integration is stopped at some time T large enough so that $|F(\mathbf{p}(T))| < 10^{-6}$. An obvious initial guess is $\boldsymbol{\theta}^{(0)}(t) = \boldsymbol{\theta}^{ss}$, where $\boldsymbol{\theta}^{ss}$ is the steady state Nash profile of strategies. (For some $s > \bar{s}$ with $\bar{s}$ sufficiently large, one can expect that $\tilde{\tau}_i(s) = \boldsymbol{\theta}_i^{(0)}(s)$ for $s > \bar{s}$). Let $\mathbf{p}^{(0)}(t)$ be the inventory solution under such a guess.

Step 2. The algorithm computes the best response of a representative agent i, on the trajectory $\mathbf{p}^{(0)}(t)$. His Δ_i is computed integrating (8) backward in time, starting from the initial condition $(\Delta_i(\boldsymbol{\theta}^{ss},\mathbf{p}^{(0)}(T)),\mathbf{p}^{(0)}(T))$.[12] At the end of this step, one obtains a trajectory $\Delta^{(0)}(t)$, and, more importantly, the corresponding best response $\tilde{\tau}_i^{(0)}(t)$ of the representative agent i.

Step 3. The consistency between $\theta_i^{(0)}(t)$ and $\tilde{\tau}_i^{(0)}(t)$ is verified. If these are different, $\tilde{\tau}_i^{(0)}(t)$ becomes the new guess on the next round, namely $\theta_i^{(1)}(t) = \tilde{\tau}_i^{(0)}(t)$, and the procedure restarts from step one. The method allows the profile of strategies to change at any point in time.

The iteration is repeated until convergence between $\theta_i^{(n+1)}(t)$ and $\tilde{\tau}_i^{(n)}(t)$ is achieved, or until a maximum number of iterations is reached. If the iteration converges to a fixed point, say $\mathbf{p}^*(t)$ and $\boldsymbol{\theta}^*(t)$, then $\mathbf{p}^*(t)$ and $\boldsymbol{\theta}^*(t)$ are the distribution of assets and the trading strategies, respectively, of a Markov-perfect Nash equilibrium. The procedure checks that at such fixed point the value function of any agent is at its maximum value, given the action of the rest of the agents.

3.4 Issues with Convergence, Uniqueness, and Alternative Algorithms

In some instances, after a fast convergence between $\theta_i^{(n+1)}(t)$ and $\tilde{\tau}_i^{(n)}(t)$, the algorithm gets stuck on a cycle that goes back and forth between two close points. This problem is usually fixed by reducing the size of the time step (the hypothesis being that the theoretical

[11] Kehoe et al. (1993) build cyclycal equilibria in a similar environment under sets of parameters that do not admit pure strategies equilibria.

[12] $\Delta_i(\boldsymbol{\theta}^{ss},\mathbf{p}^{ss})$ could also be used as initial point. In principle on $\Delta_i(\boldsymbol{\theta}^{ss},\mathbf{p}^{ss})$ the system stays still, but if computed numerically, there is always a small machine error which allows the integration to start. In the experiments the difference – in norm – between the two points is smaller than 10^{-5} when $|F(\mathbf{p}(t)| < 10^{-6}$.

switch in strategies falls between t and $t+1$). In all experiments that started with a time invariant initial guess for the profile of strategies that corresponds to one of the Nash steady states, convergence was always obtained in less than ten iterations. The greater the gap between the initial guess and the steady state Nash strategies, the more it would take the algorithm to converge and, in some cases, it would fail to do so. For example, if for a given set of parameters a unique Nash steady state equilibrium of the type (1,1,0) exists, convergence may not occur with an initial guess (0,0,1). One may speculate that, sometimes, the iteration does not converge because the system may exhibit dynamics that the algorithm is not designed to capture, such as chaotic dynamics. Although such a possibility cannot be ruled out, in absence of indications that the model could in fact produce unusual behavior, I estimated that it is not convenient to expand the algorithm to pick up trajectories other then those converging to canonical fixed points. The algorithm delivers, however, multiple dynamic equilibria. In fact, if two steady states coexist (Table 1, Panel B), the economy can evolve towards either of the two, possibly following a common path for a while.

I conclude by mentioning a variation of the current algorithm. The iteration of the current algorithm is achieved on the entire path of distribution of strategies, allowing therefore any number of switches by any type of agents. In environments where only one switch is feasible or interesting to investigate, an algorithm that seeks convergence over 'the waiting time' before a type of agents changes her current trading strategies could do just as well. With such an alternative algorithm, in each round of the iteration there would be a forward integration of the assets distribution under a guess on the switching time, followed by a backward integration of the value functions of the representative agent i. At the end of the round, one would obtain the optimal switching time for the representative agent i – her reaction function – that forms the basis for the next guess.

4 Macroeconomic Indices

The dynamics of the economy will be characterized through the behavior of liquidity, production, and inequality of income and welfare. A formal definition of these quantities follows.

Liquidity. The model delivers several liquidity indices that are comparable to those used in empirical macroeconomics. First, the stock of each of the three assets, x_i, can be interpreted as 'market thickness' - a measure of how easy it is to find an asset on the market. A second measure of liquidity is the 'frequency of trade', t_i, that measures the number of times good i is traded in a unit of time. The ratio $\frac{t_i}{x_i}$ is sometimes called velocity of circulation. KW proposes also the level of 'acceptability' of an asset in a trade, $a_i = \frac{t_i}{o_i}$, as an index of liquidity, where o_i is the frequency with which good i is offered in a period of time. The variable a_i is, by construction, bounded between zero and one. It captures the

Table 2: Baseline Parameters

Population	**Discount**	**Matching**	**Utility**		**Returns**	
μ_i	δ	α	u_i	r_1	r_2	r_3
$\frac{1}{3}$	0.03	1	1	0.21	0.2 or 0.1	0

- Note: When $r_2 = 0.2$ the economy converges to the fundamental equilibrium, whereas when $r_2 = 0.1$ it converges to the speculative equilibrium. The steady state capital income share is 0.37 and 0.51, for the fundamental and speculative equilibrium, respectively.

probability that an asset is traded, given that someone offers it.[13]

Production. It is easier to first deal with the flow of consumption. Let $c_i ds$ be the fraction of agents of type i in the overall population that consumes goods between times s and $s+ds$. Then, the rate of consumption for type i individuals is

$$c_i = p_{i,i+1}[p_{i+1,i} + p_{i+2,i}\tau_{i+2}] + p_{i,i+2}[p_{i+1,i}(1-\tau_{i+1}) + p_{i+2,i}].$$

As production immediately follows consumption, c_i, also represents the rate of production of good $i+1$. Aggregate production is $\sum_i \mu_i c_i$. In steady state, $c_i = c_{i+1}$. Clearly, the level of c_i is affected by the distribution of skills. If a good is produced by a small fraction of the population, the economy is trapped in a low-production equilibrium.

Income. Since prices are all set to one, the average flow of income generated by agents of type i holding good $i+1$ and $i+2$ is

$$g_{i,i+1} = (p_{i+1,i} + p_{i+2,i}\theta_{i+2}) + r_{i+1}$$

and

$$g_{i,i+2} = \ (p_{i+1,i}(1-\theta_{i+1}) + p_{i+2,i}) + r_{i+2},$$

respectively. One could view r_{i+1} and r_{i+2} as capital income, and the rest as labor income. National income is $\sum_i \sum_j p_{i,j} g_{i,j}$.

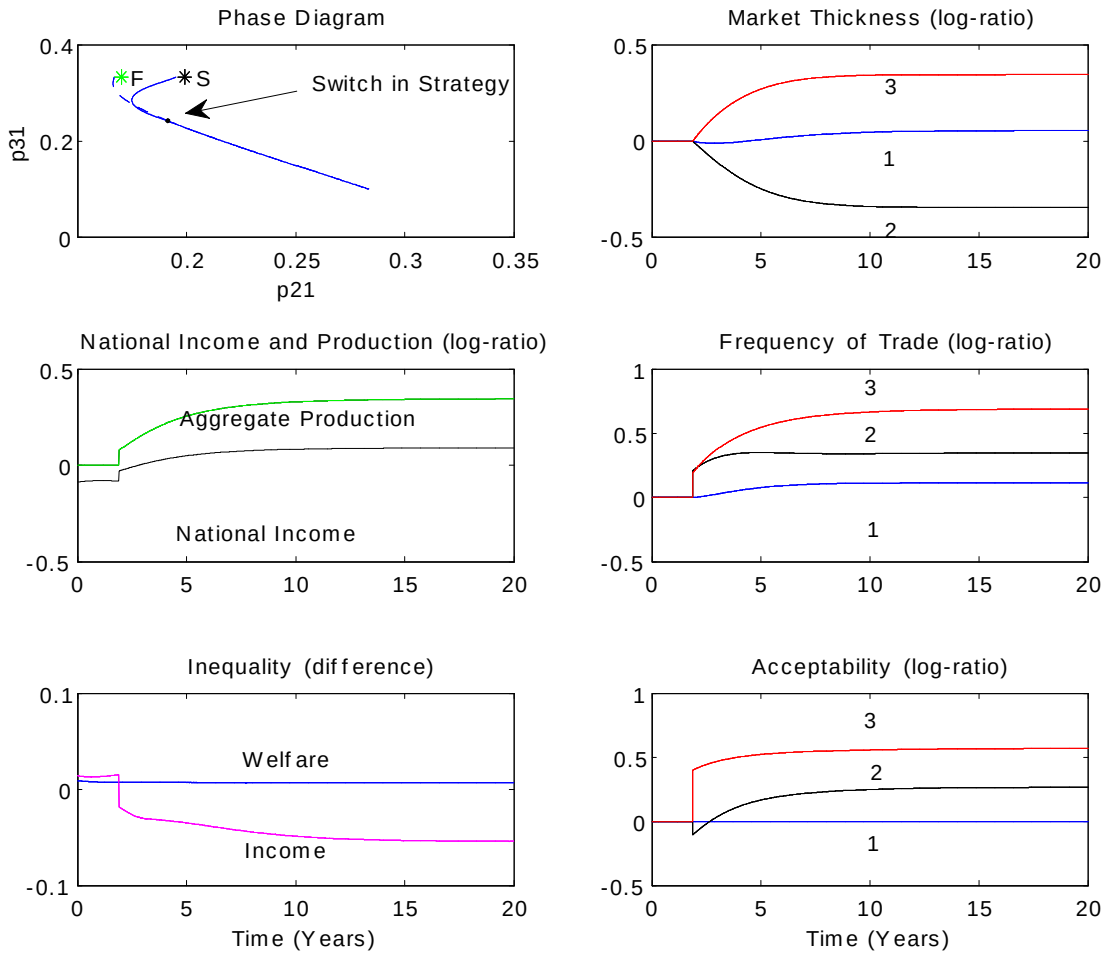
Inequality. Income and welfare inequality are computed with a Gini coefficient that measures the area comprised between a 45° line and the Lorenz curve. The population is split into six income groups according to $g_{i,j}$, and into six welfare groups on the basis of the value functions $V_{i,j}$.

5 Experiments

This section proposes four experiments on economies of the type listed Panel A of Table (1), which are characterized by a unique steady state equilibrium. Because the focus is

[13] The Technical Appendix details how t_i, x_i, and o_i are formally derived.

Figure 1: Reversal of Fortune



- Note: For parameters see Table (2). The top-left plot is a partial view of the evolution of the assets' distribution of two similar economies that share the same initial condition. The economy with $r_2 = 0.1$ (0.2) converges to the speculative (fundamental) Nash equilibrium. The curves of the remaining plots are ratios or differences of the speculative economy's time series relative to the ones of the fundamental economy. The numbers inside the three right plots identify the type of asset traded.

on inequality that arises from changes in liquidity, the utility u_i is normalized to 1 for all individuals.

The first experiment illustrates a case of reversal between two economies that differ only for the level of the rate of return on one of the assets. Starting from the same initial distribution of assets, a similar set of strategies is adopted for a while in the two economies. At the beginning of the transition, the national income is larger in the economy endowed with the high-yield asset. Over time, however, trade intensifies relatively more in the economy endowed with the low-yield asset. As a result, its production expands more rapidly and national income surpasses that of the other economy. Furthermore, in the low-yield asset economy, inequality declines because a greater share of income comes from market interactions.

A second experiment shows that a better matching rate is associated not only with larger production and income, but also with a more equitable distribution of income.

A third macroeconomic experiment studies the aftermath of a crisis triggered by a negative shock to an asset return: liquidity and national income drop, and income inequality shoots up.

A final experiment deals with a more radical return shock that causes multiple switches along the adjustment process. The asset with the lowest yield turns into the one with the highest yield. Along the adjustment process two groups of agents, at different times, switch their trading strategies. Consequently, the amplitude and the length of the fluctuations are more pronounced than in the previous experiments.

5.1 Reversal of Fortune over the Transition

Consider two economies that are similar in all respects, except that in one (S-economy), the return on good 2 is lower than in the other (the F-economy). The returns satisfy $r_3 < r_2 < r_1$ in both economies. With the initial distribution of inventories, only good 1 is used in indirect trading in either economy. Over the transition, as the difference between $p_{3,1}$ and $p_{2,1}$ increases (top-left plot of Fig. (1)), good 3 becomes relatively more marketable than good 2. In the S-economy, where the good 2 commands a smaller return than in the F-economy, good 3 emerges as an asset exchanged in indirect trading,[14] whereas in the F-economy it does not. Said differently, in the F-economy, agents 1 choose to get a higher fraction of their income from hoarding capital. Conversely, in the S-economy, agents are willing to give up some capital income and to be more active in production.[15]

[14] The literature sometimes refers to this phenomenon as the emergence of commodity money.

[15] Specifically, the S-economy converges to the *speculative* equilibrium $\boldsymbol{\theta} = (1, 1, 0)$, $\mathbf{p} = \frac{1}{3}[\frac{1}{2}\sqrt{2}, \sqrt{2} - 1, 1]$, and the F-economy to the *fundamental* equilibrium $\boldsymbol{\theta} = (0, 1, 0)$, $\mathbf{p} = \frac{1}{3}[1, \frac{1}{2}, 1]$. The Appendix states the conditions for the existence of these equilibria.

Table 3: Reversal of Fortune

	Panel A: Fundamental Equilibrium		
	Bottom/Top	**Gini**	**Capital Share**
Initial Position	0.1840	0.1091	0.3685
Steady State	0	0.1131	0.5074
	Panel B: Speculative Equilibrium		
	Bottom/Top	**Gini**	**Capital Share**
Initial Position	0.1840	0.1235	0.3110
Steady State	0.5283	0.0595	0.3633

- Note: The first column of Panel A shows the ratio between the bottom and the top income level at the initial point of the transition and when the economy reaches the fundamental steady state (see Fig. (1)). For parameters' specification see Table (2). The second and third columns report the initial and final values of the Gini index, and of the average capital income share, respectively. In steady state, the bottom income group consists of type 2 individuals holding good 3. Their income is zero because they have no immediate prospect of obtaining a consumption good and have no capital income ($r_3 = 0$). Panel B shows similar data for the economy that converges to the speculative steady state with $r_2 = 0.1$. (In the initial phase of the transition agents 1 play also fundamental strategies).

Except for inequality, the two economies exhibit the similar macroeconomic and financial indicators until agents 1 of the S-economy change their strategies. From then on, the S-economy performs better. Aggregate production is larger, mostly thanks to an expansion in the production of good 3 (the production of good 1 is relatively smaller for a while in the S-economy because of a sudden drop in its marketability).

The middle-left graph of Fig. (1) contains the key insight of the experiment: The F-economy has an initial advantage in terms of income because of the higher returns in asset 2. Nevertheless, this initial advantage induces F-economy's agents 1 to maintain a rent-seeking behavior (i.e. to play fundamental strategies) whereas, their counterparts in the S-economy, engage in indirect trading. The reversal emerges gradually as the market thickness and the frequency of trade of good 3 become significantly large to compensate agents 1 for the loss of their rental income.

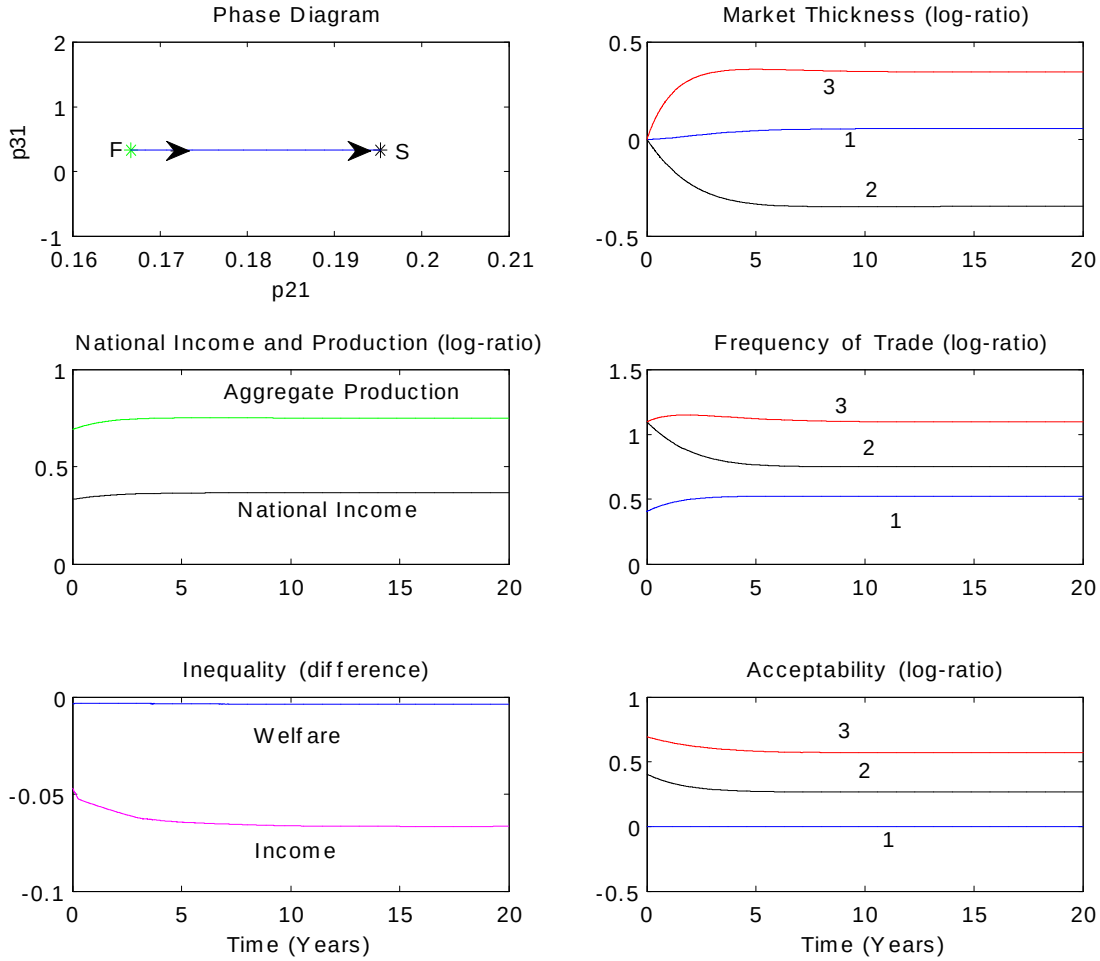
The crossing between the two economies occurs also with respect to income inequality (bottom-left graph of Fig. (1)). In the F-economy, the group of type 2 individuals holding good 3 converges to a zero income. These individuals do not earn any capital rent, and do not have any immediate prospect of trading their holding against a consumption good. Conversely, because in the S-economy good 3 is accepted by type 1 agents, the average income at the lower end is about half of that of the richest group (see Table (3)).

5.2 Market Frictions

The propagation mechanism stirred by an increase in the matching rate, α, is now considered. Assume that the economy is in a steady state. The immediate effect of the shock is a boost in production and national income, reflecting the more frequent trading activity. In the experiment depicted in Fig. (2) the frequency of trade in assets 2 and 3 doubles when the matching parameter goes up by 50%. Also the acceptability indices of these two assets increase substantially. But the shock also affects the distribution of assets, as it causes the economy to transit from a fundamental to a speculative steady state equilibrium. As a result, income inequality declines, because the holders of assets 2 and 3 earn a more meager capital income, relative to asset 1 holders. The top-right plot indicates that the volume of asset 2 shrinks, for this is partially replaced by asset 3. Since a larger stock of the wealth is invested in a low-return asset, the fraction of the national income generated by capital return goes down significantly, the bottom to top income ratio rises, and the level of inequality is significantly reduced (Table (4)).

In sum, a reduction of market frictions not only boosts average income up, but it also helps to reduce income disparity. Liquidity is, however, adversely affected.

Figure 2: Market Frictions



- Note: The pre-shock parameter values are in Table (2) with $r_2 = 0.1$ (fundamental equilibrium). The shock raises the matching parameter by 50%. The plots of ratios and differences are calculated with respect to the pre-shock state. The income capital share and inequality measures are reported in Table (4).

Table 4: Market Frictions

Steady State	**Bottom/Top**	**Gini**	**Capital Share**
F (pre-shock)	0	0.1131	0.5074
S (post-shock)	0.6274	0.0467	0.2757

- Note: The matching technology improves by 50%. The initial set of parameters are depicted in Table (2), with $r_2 = 0.2$. The economy transits from the fundamental to the speculative steady state, where trading is more frequent (Fig. (2)).

Table 5: Higher Return of Asset 2

Steady State	**Bottom/Top**	**Gini**	**Capital Share**
S (pre-shock)	0.5288	0.0594	0.3634
F (post-shock)	0	0.1131	0.5074

- Note: Inequality and capital share variations due to an increase in r_2 from 0.1 to 0.2. For remaining parameters see Table (2). The economy transits from the speculative to the fundamental equilibrium (Fig. (3)).

5.3 From the Speculative to the Fundamental Equilibrium

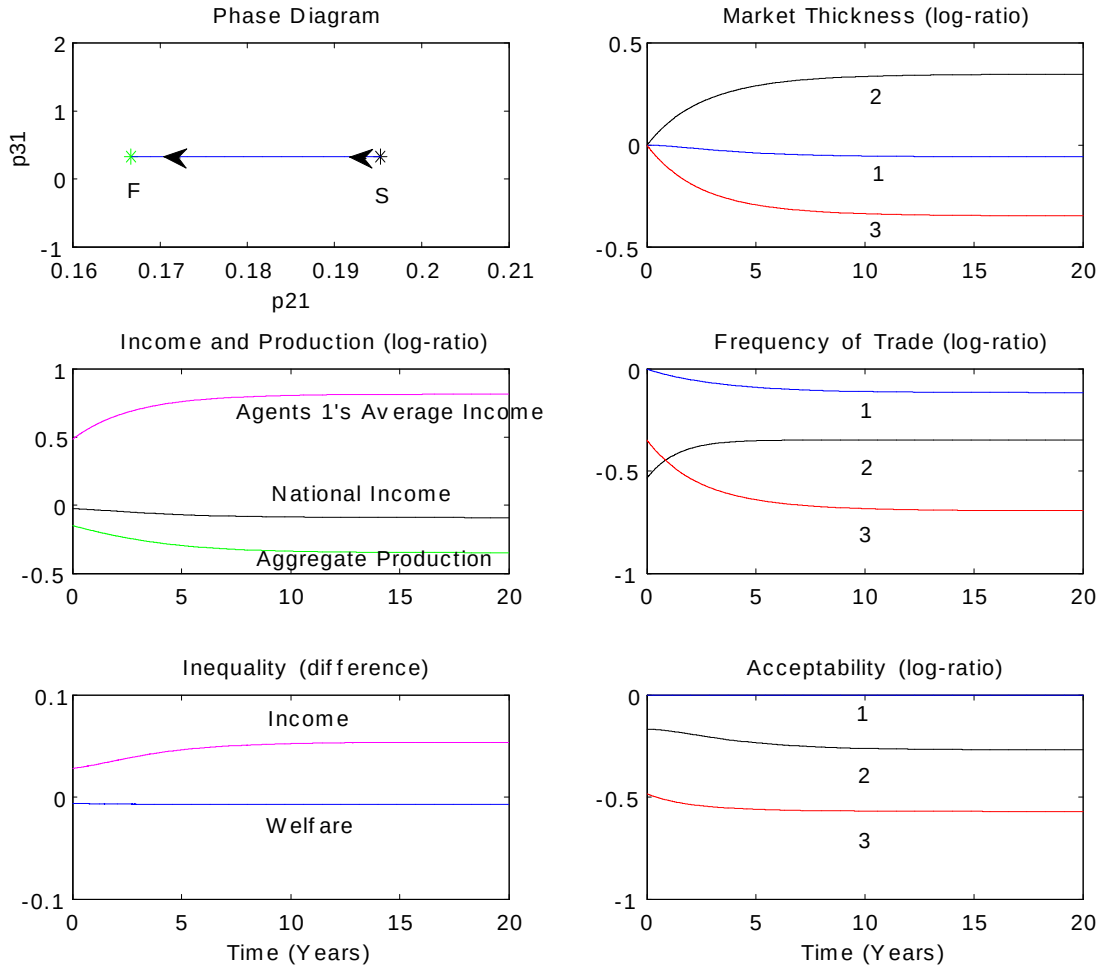
Consider now the adjustment process of an economy displaced from its current long-run equilibrium by a shock that affects the return of one of the three assets. Upon inspecting Panel A of Table (1), one realizes that equilibria in the configurations R2 and R3 can be obtained through an appropriate permutation of the R1 equilibria.[16]

This section proposes an experiment that generates a contemporaneous drop in liquidity and in national income,[17] as well as a rise in income inequality. Imagine that an R1 economy on a speculative steady state is hit by a shock that widens the return gap between assets 2 and 3. The shock is large enough to alter the long run equilibrium of the economy, but not the ranking of the rates of return, that remains of R1 type. More precisely, the shock induces agents 1 to give up indirect trading and to get a larger fraction of their income from hoarding capital. This more passive strategy has a number of negative consequences for economy's performance. Fig. (3) shows that the frequency of trade and the acceptability of assets decline. The liquidity drop is associated with a reduction of both national income and aggregate production. Welfare inequality declines slightly, but there is a substantial rise in income inequality. By renouncing to indirect trade, type 1 agents hoard asset 2 (see top-right plot), for longer periods. As asset 2 yields a better return than asset 3, a larger fraction of agents 1's income is now derived from rents. Therefore, the gap between income and production is larger after the shock (see left-middle graph of Fig. (3)). Because agents 2 face worse odds in trading away their holdings and they are also the ones that hold mostly the asset with the lowest return, there is a dramatic drop of the bottom/top income ratio. In fact, Table (5) indicates that income inequality doubles at the end of the adjustment process and that the capital income ratio goes up from 36 to 51 percent. The correlation is in line with Piketty's (2014) argument that a higher interest rate lead to greater capital income and

[16] The vectors of strategy and of assets for the R2 (R3) configuration can be obtained by shifting one position backward (forward) the elements of the corresponding R1 vectors.

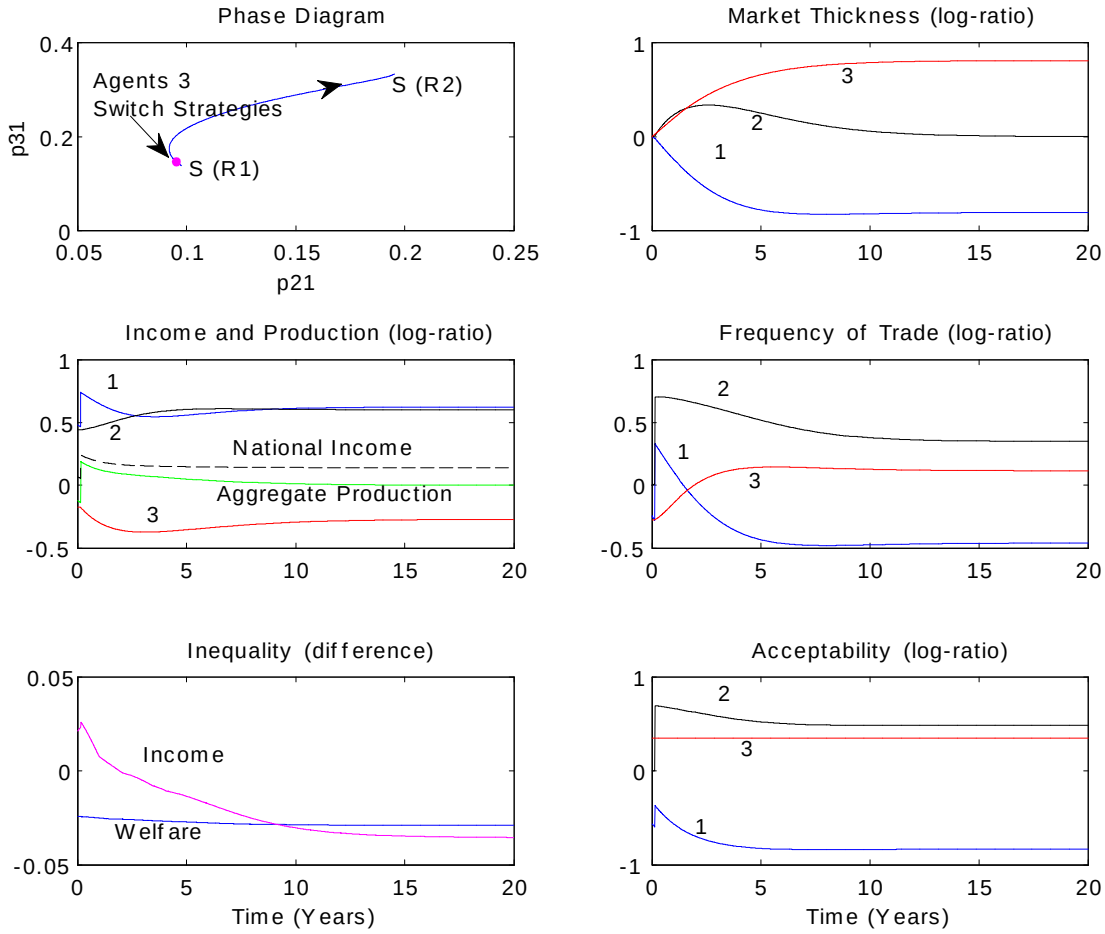
[17] Such a correlation has also been investigated in variations of Kiyotaki and Moore (2012). See, for instance, Aiello (2012), Shi (2012), and Del Negro et al. (2011).

Figure 3: Higher Return of Asset 2



- Note: The pre-shock set of parameters are displayed in Table (2), with $r_2 = 0.1$ (speculative equilibrium). The shock doubles the rate of return on asset 2. The ratios and differences are calculated with respect to an economy in its speculative Nash steady state.

Figure 4: Higher Return of Asset 3



- Note: The initial position is the speculative equilibrium, under R1 configuration. The shock raises the rate of return of asset 3 above that of the other two assets (r_3 goes from 0 to 0.22).

to more inequality. Notice that here the expansion in inequality is induced mostly by the decline in liquidity (i.e. by rent-seeking behavior). Despite the decline in national income and liquidity, the middle-left plot of Fig. (3) shows that the average income of type 1 agents goes substantially up, for it benefits from the higher returns of asset 2 – a second major factor contributing to the increased income inequality.

5.4 Transition between Speculative Equilibria: Multiple Switches

When the shock alters the order of the assets' returns there are broader effects on the agents' strategies. A sudden increase in r_3 above both r_1 and r_2, may induce two groups of agents to abandon their current strategies. Fig. (4) accounts for such a scenario. An R1 economy

Table 6: Higher Return of Asset 3

Steady State	**Bottom/Top**	**Gini**	**Capital Share**
S of R1 (pre-shock)	0.5288	0.0594	0.3634
S of R2 (post-shock	0.7358	0.0241	0.4453

- Note: The rate of return of asset 3, r_3, increases from 0 to 0.22. The remaining parameters are reported in Table (2), with $r_2 = 0.2$. The economy transits from the R1 speculative equilibrium to the R2 speculative equilibrium (Fig. (4)).

(see Table (1)), initially on the speculative steady state (1,1,0), moves, as a result of the shock, towards an R2 (1,0,1) equilibrium. In the new steady state, type 3 rather than type 1 agents play speculative strategies (although type 1 agents do not change their trading behavior, indirect trade in the R2 economy actually comes through fundamental strategies). Inequality has a non monotonic adjustment. First, it goes up, reflecting the greater capital income disparity across groups, and then declines, as the patterns of trade partially level the playing field. In particular, the shock and the responses that it triggers make type 1 and type 2 agents better off, whereas type 3 are worse off. Type 2 agents benefit as the return on the good they produce goes up. Type 1 agents gain from the fact that their production good is now more 'liquid', for it is accepted in indirect trading. Conversely, type 3 agents lose from the shock because their production good is no longer accepted in indirect trading. Overall, Table (6) shows that the shock rebalances substantially the income levels of the poorest and richest individuals and slashes the Gini index by half. Contrary to previous examples, however, this time there is a negative long-run correlation between inequality and capital share, because the capital windfall goes to type 2 agents that used to face the least favorable trading odds. Furthermore, liquidity of asset 1 and asset 3 move in opposite directions, a phenomenon that also contributes to the contraction of inequality.

6 Conclusion and Future Research

This article developed a simple algorithm for computing Dynamic Nash Equilibria for a class of models where agents with different skills trade goods in decentralized meetings. The method exploited a general feature of this class of models: the fact that a dynamical system describing the evolution of assets can be studied for any interesting (but not necessarily optimal) policy function, which here takes the form of a profile of trading strategies. The article then proposed an iterative procedure able to select a profile of strategies that satisfies the conditions of an open loop Nash equilibrium.

The algorithm proved flexible enough to replicate some of the most common applications

of dynamic general equilibrium theory with centralized markets, such as shocks to assets' returns, as well as to propose experiments geared towards economies with decentralized meetings, such as variations in the level of market frictions. The detailed connections between the policy functions and the behavior of aggregate variables that resulted in the numerical illustrations could not have been obtained using only analytical methods or by restricting the focus only to steady state analysis. The applications of the algorithm allowed for the uncovering of phenomena interesting on their own.

One was an example of reversal of fortune: The initial advantage of an economy endowed with an asset that fetches a higher return is dissipated over the transition, because of the limited participation into trading activity. As a result, in the long run it is the society with a low-return asset that becomes the leading economy. Furthermore, because in this economy agents trade more intensively, financial markets are more liquid, and income inequality declines. An improvement of market frictions can also induce an intensification of trade, and a reduction in income inequality. More in general, there is a positive correlation between liquidity and production, and a negative one between liquidity and inequality.

Because the features of the equilibria crucially depend on the ordering of the asset returns, the consequences are radically different depending on the magnitude of the shock and on the type of asset return hit by it. In one experiment, the shock favored the low-return asset, which also happened to have the lowest level of liquidity. Therefore, in the economy converged to an equilibrium with greater liquidity and more equal distribution of income.

The simulated economies showed a quite low level of the income Gini index: only between a quarter and a half of the one reported for the most egalitarian countries. One reason for such feature is that agents have the same level of productivity. In fact, the only source of inequality of the model economy is the marketability and the returns of assets. Yet, the model suggests that inequality originated from capital income is compressed substantially when assets becomes more liquid, because agents find it more profitable to earn income from production rather than from hoarding capital.

I conclude with two comments on future work. First, the algorithm generates a converging sequence of the profile of strategies, suggesting that a contraction is at work. Further research could investigate the algorithm's contraction property starting from some general features of the value functions and of the system representing the evolution of assets. Second, because the algorithm is not demanding from a computational point of view, it could be extended to environments where some of the special assumptions in KW, such as the unit-storage capacity or the fixed terms of trades, are relaxed along the lines, suggested, among others, by Molico (2006), Lagos and Rocheteau (2008), and Chu and Molico (2010).

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Appendix

Distribution of Inventories (Proposition 1)

For a given profile of strategies the system (5)-(7) converges globally to the unique steady state reported in the following table:

Strategies	Assets Distribution	*Strategies*	Assets Distribution
(0,1,0)	$[\mu_1, \frac{\mu_2\mu_1}{\mu_3+\mu_1}, \mu_3]$	*(1,0,1)*	$[p^{\#}_{1,2}, \mu_2, p^{\#}_{3,1}]$
(1,0,0)	$[\frac{\mu_1\mu_3}{\mu_2+\mu_3}, \mu_2, \mu_3]$	*(0,1,1)*	$[\mu_1, p^{\#}_{2,3}, p^{\#}_{3,2}]$
(0,0,1)	$[\mu_1, \mu_2, \frac{\mu_1\mu_3}{\mu_1+\mu_2}]$	*(0,0,0)*	$[\mu_1, \mu_2, \mu_3]$
(1,1,0)	$[p^*_{1,2}, p^*_{2,3}, \mu_3]$		

Case (0,1,0). Eq. (5) reduces to $\dot{p}_{1,2} = \alpha p_{1,3}\mu_3$, implying that the line $p_{1,2} = \mu_1$, is globally attractive (henceforth g.a.). Similarly because Eq. (7) $\dot{p}_{3,1} = \alpha p_{3,2}[p_{1,3} + \mu_2]$, $p_{3,1} = \mu_3$ is g.a.. Finally, along these lines the system collapses to

$$\dot{p}_{2,3} = (\mu_2 - p_{2,3})\mu_1 - p_{2,3}\mu_3,$$

which clearly converges globally to $\frac{\mu_2\mu_1}{\mu_3+\mu_1}$. In brief, under the profile of strategies (0,1,0) the distribution of inventories converges globally to the stationary distribution $[\mu_1, \frac{\mu_2\mu_1}{\mu_3+\mu_1}, \mu_3]$.

Case (0,0,1) and *Case (1,0,0).* One can verify that the stationary distribution converges to $[\mu_1, \mu_2, \frac{\mu_1\mu_3}{\mu_1+\mu_2}]$ and $[\frac{\mu_1\mu_3}{\mu_2+\mu_3}, \mu_2, \mu_3]$, respectively, using the same observations as in the previous case.

Case (1,1,0). Eq. (7) becomes $\dot{p}_{3,1} = \mu_2(\mu_3 - p_{3,1})$. Consequently, $\mu_3 = p_{3,1}$ is an invariant set. The Jacobian, J, of the system of the two remaining equations (5) and (6) along the line $\mu_3 = p_{3,1}$ is

$$J = \alpha \begin{bmatrix} -(\mu_3 + p_{2,3}) & -p_{1,2} \\ (\mu_2 - p_{2,3}) & -(\mu_3 + p_{1,2}) \end{bmatrix}.$$

The determinant is positive and the trace is negative; therefore, both eigenvalues are negative and the system is globally stable. To find the stationary distribution, set (5) and (6) to zero. They yield $p_{1,2} = \mu_1\mu_3/(\mu_3 + p_{2,3})$ and $p_{1,2} = \frac{\mu_3}{\mu_2/p_{2,3}-1}$, respectively. The two lines necessarily cross once and only once for $p_{2,3}$ in the interval $[0,\mu_3]$. The fixed point is $[p^*_{1,2}, p^*_{2,3}, \mu_3]$ where $p^*_{2,3} = \frac{1}{2}[-(\mu_1 + \mu_3) + \sqrt{(\mu_1 + \mu_3)^2 + 4\mu_1\mu_2}]$ and $p^*_{1,2} = \frac{\mu_1\mu_3}{\mu_3+p^*_{2,3}}$.

Cases (1,0,1) and (0,1,1). A Jacobian with similar properties can be obtained when the profiles of strategies are (1,0,1) or (0,1,1). The fixed point with (1,0,1) is $[p^{\#}_{1,2}, \mu_2, p^{\#}_{3,1}]$, where $p^{\#}_{1,2} = \frac{1}{2}[-(\mu_3 + \mu_2) + \sqrt{(\mu_3 + \mu_2)^2 + 4\mu_3\mu_1}]$ and $p^{\#}_{3,1} = \frac{\mu_1\mu_3}{p^{\#}_{1,2}+\mu_2}$. Similarly, under

(0,1,1) the fixed point is $[\mu_1, p^{\aleph}_{2,3}, p^{\aleph}_{3,2}]$ where $p^{\aleph}_{3,1} = \frac{1}{2}[-(\mu_2+\mu_1)+\sqrt{(\mu_2+\mu_1)^2+4\mu_2\mu_3}]$ and $p^{\aleph}_{2,3} = \frac{\mu_1\mu_2}{p^{\aleph}_{3,1}+\mu_1}$.

Case (0,0,0). The system converges globally $\mathbf{p} = [\mu_1,\mu_2,\mu]$. In this stationary state agents keep their production goods.

When the profile of strategies is (1,1,1) it is more difficult to characterize the properties of the Jacobian. This turns out not to be a Nash equilibrium, at least when the population is equally split across types.

Value Functions

Eq. (3) describes $V_{i,i+1}(t)$. The following does the same for $V_{i,i+2}(t)$

$$
\begin{aligned}
V_{i,i+2}(t) = \max_{\{\tau_i(s)\}_{s\geq t}} \int_t^{\infty} \alpha e^{-\alpha(s-t)} \{ & e^{-\rho(s-t)}([p_{i+1,i}(1-\theta_{i+1})+p_{i+2,i}](V_{i,i+1}+u_i)+ \\
& [p_{i,i+1}\theta_i + p_{i+2,i+1}](1-\tau_i)V_{i,i+1} \\
& [1-p_{i+1,i}(1-\theta_{i+1})-p_{i+2,i}-(p_{i,i+1}\theta_i+p_{i+2,i+1})(1-\tau_i)]V_{i,i+2}+ \\
& \frac{1-e^{-(s-t)\rho}}{\rho} r_{i+2}\} ds,
\end{aligned} \tag{9}
$$

Time derivatives of eqs. (3) and (9) give

$$
\begin{aligned}
\dot{V}_{i,i+1} = & -\alpha([p_{i,i+2}\tau_i(1-\theta_i)+p_{i+1,i+2}\tau_i]V_{i,i+2}+ \\
& p + [1-p_{i,i+2}\tau_i(1-\theta_i)-p_{i+1,i+2}\tau_i]V_{i,i+1}+ \\
& [p_{i+1,i}+p_{i+2,i}\theta_{i+2}]u_i) - r_{i+1} + (\alpha+\rho)V_{i,i+1}(s),
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
\dot{V}_{i,i+2} = & -\alpha([p_{i+1,i}(1-\theta_{i+1})+p_{i+2,i}](V_{i,i+1}+u_i) + [p_{i,i+1}\theta_i+p_{i+2,i+1}](1-\tau_i)V_{i,i+1} \\
& + [p_{i,i+1}\theta_i + p_{i+2,i+1}](1-\tau_i)V_{i,i+1}+ \\
& + [1-p_{i+1,i}(1-\theta_{i+1})-p_{i+2,i}-(p_{i,i+1}\theta_i+p_{i+2,i+1})(1-\tau_i)]V_{i,i+2})+ \\
& - r_{i+2} + (\alpha+\rho)V_{i,i+2},
\end{aligned}
$$

respectively.

The last two expressions can also be written as

$$
\begin{aligned}
\dot{V}_{i,i+1} = & -\alpha([p_{i,i+2}\tau_i(1-\theta_i)+p_{i+1,i+2}\tau_i](-\Delta_i) + V_{i,i+1}+ \\
& [p_{i+1,i}+p_{i+2,i}\theta_{i+2}]u_i) - r_{i+1} + (\alpha+\rho)V_{i,i+1}(s),
\end{aligned}
$$

$$\dot{V}_{i,i+2} = -\alpha([p_{i+1,i}(1-\theta_{i+1}) + p_{i+2,i} + [p_{i,i+1}\theta_i + p_{i+2,i+1}](1-\tau_i)]\Delta_i + [p_{i+1,i}(1-\theta_{i+1}) + p_{i+2,i}]u_i + V_{i,i+2}) - r_{i+2} + (\alpha+\rho)V_{i,i+2},$$

Subtracting side-by-side we obtain

$$\dot{\Delta}_i = -\alpha([p_{i,i+2}\tau_i(1-\theta_i) + p_{i+1,i+2}\tau_i + p_{i+1,i}(1-\theta_{i+1}) + p_{i+2,i} + (p_{i,i+1}\theta_i + p_{i+2,i+1})(1-\tau_i)](-\Delta_i) + + \Delta_i + [p_{i+1,i}\theta_{i+1} - p_{i+2,i}(1-\theta_{i+2})]u_i) - r_{i+1} + r_{i+2} + (\alpha+\rho)\Delta_i$$

Let

$$\chi_i \equiv p_{i,i+2}\tau_i(1-\theta_i) + p_{i+1,i+2}\tau_i + p_{i+1,i}(1-\theta_{i+1}) + p_{i+2,i} + (p_{i,i+1}\theta_i + p_{i+2,i+1})(1-\tau_i)$$

where $\phi_i \equiv [p_{i,i+2}\tau_i(1-\theta_i) + p_{i+1,i+2}\tau_i]$. Then the last two differential equations reduce to:

$$\dot{\Delta}_i = -\alpha((1-\chi_i)\Delta_i + [p_{i+1,i}\theta_{i+1} - p_{i+2,i}(1-\theta_{i+2})]u_i) - r_{i+1} + r_{i+2} + (\alpha+\rho)\Delta_i, \qquad (11)$$

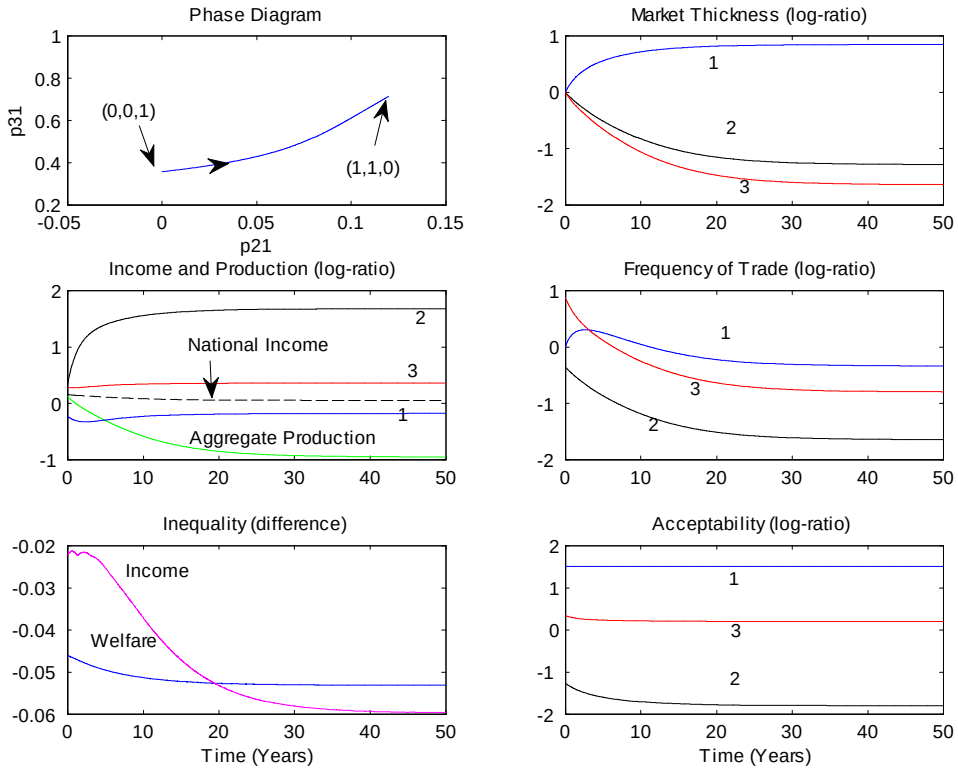
Since $(\alpha+\rho) > 0$, for any given pattern of asset distribution, the system is unstable. The Technical Appendix explains how this equation is used to verify whether a stationary distribution of assets is a Nash equilibrium.

An Experiment with Uneven Distribution of Skills

When the assumption that the population is equally split across the three groups is relaxed, multiple steady state equilibria may emerge (Wright (1995)). For instance, if μ_3 is sufficiently high, the (1,1,0) Nash equilibrium (high marketability of good 3) may coexist with (0,0,1).

Fig. (5) illustrates an economy that transits from a (0,0,1) to a (1,1,0) steady state. The movement is induced by a rise in the returns of stock 1. Although with the pre-shock set of parameters there are multiple equilibria, after the shock only the (1,1,0) steady state equilibrium exists. For a sufficiently high r_1, type 3 agents abandon indirect trading. If they switch, agents 1 and 2 flip their strategies as well because of the new marketability conditions. The switch of the three groups of agents actually occurs immediately after the shock, causing a swift drop in the acceptability of asset 2. This explains the income decline of type 1 agents. In the (0,0,1) equilibrium, type 2 and type 3 agents play speculative strategies, whereas type 1 play fundamental strategies. Interestingly, the largest income variation shows up not immediately after the shock but during the transition. Type 2 agents, by switching strategy, tend to accumulate more and more of the asset with the highest return. Overall

Figure 5: Higher Return of Asset 1



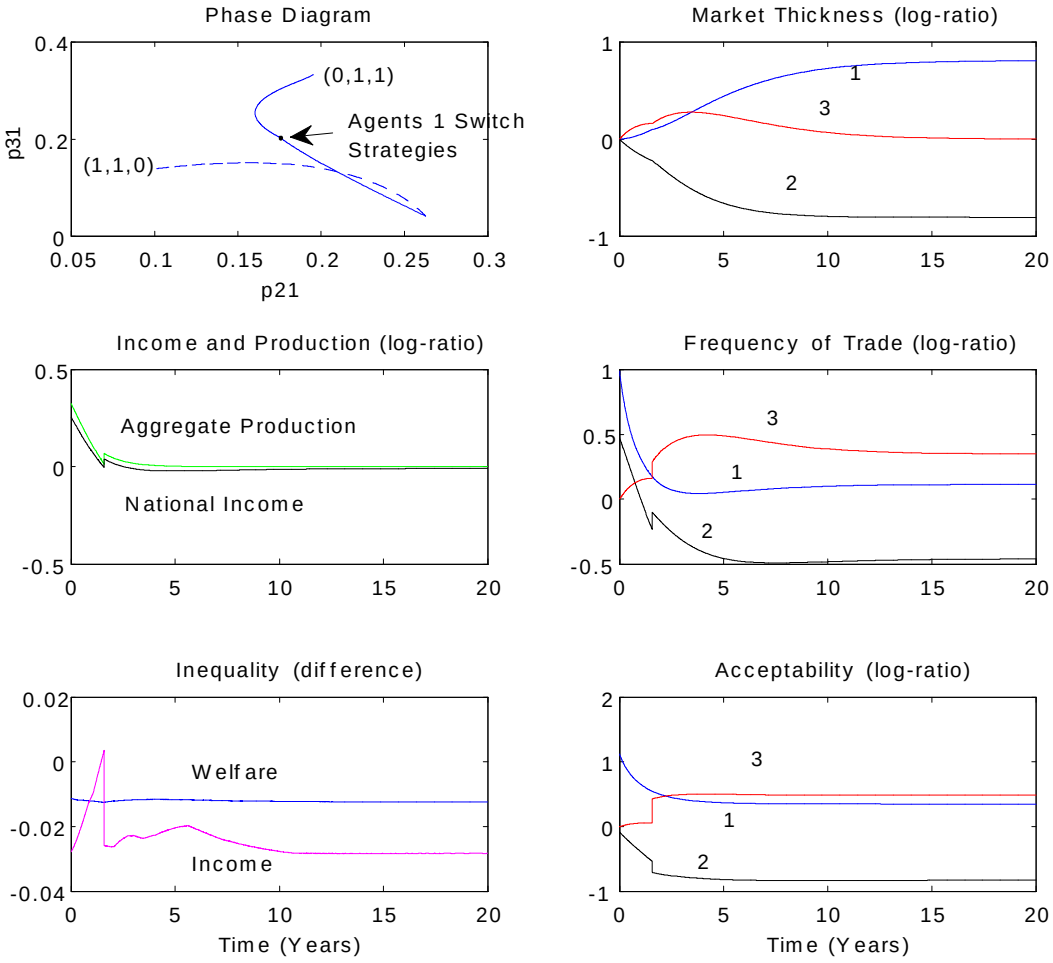
Note - A shock raises the returns of asset 1 by 10%. As a result, the economy moves from the (0,0,1) (multiple) equilibrium to a (1,1,0) (unique) equilibrium. The population is distributed as follows: $\mu_1 = \mu_2 = 1/7$ and $\mu_3 = 5/7$. The rest or the parameters are as in Table (2) with $r_2 = 0.2$.

production tends to decline because with the new strategies there is less trade. National income goes up after the shock thanks to the windfall of capital income in favor of asset 1 holders, but then also slides down as a result of the weaker trade in goods 2 and 3.

An Experiment with Model B

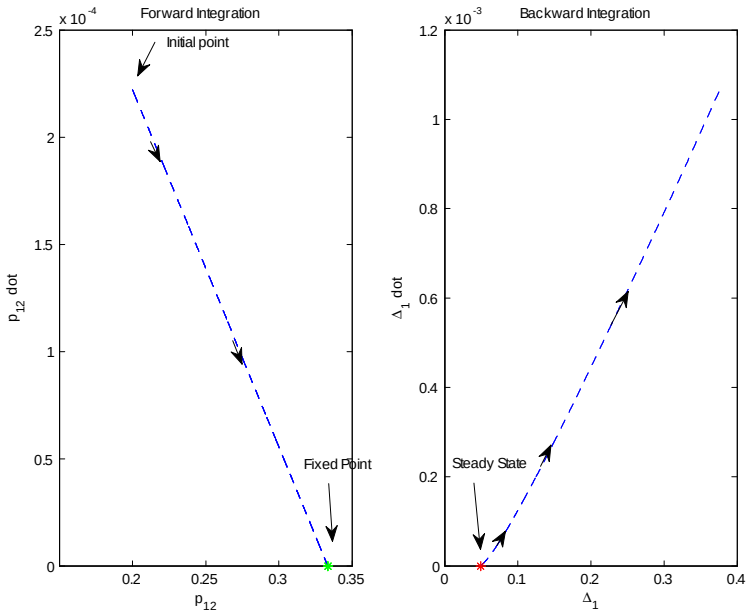
Multiple equilibria also emerge when the population is equally split across the three groups, for the three combinations of returns listed in row R4-R6, in Table (1). When $r_3 < r_1 < r_2$, the fundamental equilibrium is (0,1,1), whereas the speculative one is (1,1,0) (type 1 and type 3 agents follow a speculative behavior). The top-left graph of Fig. (6) shows two alternative paths for the distribution of assets: If agents coordinate on the (1,1,0) steady state equilibrium, they follow the (1,1,0) profile all along the transition. If, instead, they coordinate on the (0,1,1) equilibrium, type 1 agents switch strategies along the transition. The plots in the figure show a substantial difference in macroeconomic time series associated with the two paths. The 'speculative path' is associated with lower inequality in income and welfare. It also exhibits higher national income in the early phase of the transition because all individuals engage in indirect trade. The three liquidity plots show the greater (smaller) role of asset 3 (2) in the economy converging to the speculative equilibrium, reflecting the greater (smaller) role in indirect trading.

Figure 6: Dynamics in Model B



Note - The rate of return on asset 2 and 3 are 8 and 9 percent, respectively. The remaining parameters are reported in Table (2). The ratios and differences are taken with respect to the economy converging to the (1,1,0) fundamental steady state equilibrium.

Figure 7: Combining Forward and Backward Integration



Note - For parameters see Table (2), with $r_2 = 0.1$. The initial value of $p_{1,2}$ is 60 percent of its steady state value. The slope of the right plot is $\alpha\chi_1(t) + \delta$ with $\tilde{\tau}_1(t) = 0$ and $\boldsymbol{\theta}(t) = [0, 1, 0]$. The path of $\mathbf{p}(t)$ in $\chi_1(t)$ is obtained by integrating the system (5)-(7) forward in time (left plot).

Technical Appendix [Not intended for publication]

Forward and Backward Integration

The right plot of Fig. (7) illustrates a solution of (8) for $i = 1$ for a particular pattern of χ_1. This pattern is obtained by integrating forward in time (5)-(7) starting from some arbitrary initial $\mathbf{p}(0)$, under an exogenous profile of strategies.

Stationary Nash Equilibria (Proposition 2)

The stationary distribution of inventories are derived from (3.1) under the assumption that $\mu_1 = \mu_2 = \mu_3 = \frac{1}{3}$. The key condition to determine whether a stationary distribution is a NE is the sign of Δ_i. From (11) it follows that $\Delta_i > 0$ if

$$p_{i+1,i}\theta_{i+1} - p_{i+2,i}(1 - \theta_{i+2}) > \frac{r_{i+2} - r_{i+1}}{\alpha u_i} \tag{12}$$

Consistency requires that $\theta_i = 0$ (1) with $\Delta_i > 0$ (< 0). This section reviews the consistency conditions (12) for the six rankings listed in Table (1).

R1 ($r_3 < r_2 < r_1$). There are two unique NE: (0,1,0) and (1,1,0). The (0,1,0) equilibrium requires that

$$p_{2,1} - p_{3,1} > \frac{r_1 - r_3}{\alpha u_1} \tag{13}$$

$$-p_{1,2} < \frac{r_2 - r_1}{\alpha u_2}, \tag{14}$$

and

$$0 > \frac{r_2 - r_1}{\alpha u_3}, \tag{15}$$

with $p_{2,1} = \frac{1}{3} - p_{2,3}$, $p_{1,2} = \frac{1}{3}$, $p_{2,3} = \frac{1}{6}$, and $p_{3,1} = \frac{1}{3}$. Conditions (14) and (15) are clearly verified. From (13) it follows that the (0,1,0) exists if $\frac{1}{6} > \frac{r_1 - r_3}{\alpha u_1}$. For the (1,1,0) equilibrium the stationary distribution is $\mathbf{p} = \frac{1}{3}[a, b, 1]$. The above three conditions are replaced by

$$p_{2,1} - p_{3,1} < \frac{r_3 - r_2}{\alpha u_1},$$

$$0 < \frac{r_1 - r_3}{\alpha u_2},$$

and

$$p_{1,3} > \frac{r_2 - r_1}{\alpha u_3},$$

respectively, with $p_{2,1} = \frac{1}{3} - p_{2,3}$, $p_{1,2} = \frac{a}{3}$, $p_{2,3} = \frac{b}{6}$, and $p_{3,1} = \frac{1}{3}$. Again, the last two conditions are obviously satisfied. The first condition says that in a NE agents 1 play speculative if

$$-\frac{b}{6} < \frac{r_3 - r_2}{\alpha u_1}.$$

There are no other NE. A similar proof applies for the combination of cost R2 and R3. A summary of the outcome follows.

R2 ($r_2 < r_1 < r_3$). There are two unique NE: (1,0,0) and (1,0,1). The inequalities $\Delta_1 < 0$ and $\Delta_2 > 0$ are always verified. If $p_{1,3} - p_{2,3} = -\frac{1}{6} > \frac{r_2 - r_1}{\alpha u_3}$ agents 3 play fundamental strategies and the NE is (1,0,0). If $p_{1,3} - p_{2,3} = -\frac{b}{3} < \frac{r_2 - r_1}{\alpha u_3}$ agents 3 play speculative strategies and the NE is (1,0,1).

R3 ($r_1 < r_3 < r_2$). There are two unique NE: (0,0,1) and (0,1,1). The conditions $\Delta_1 > 0$ and $\Delta_3 < 0$ are always verified. If $p_{3,2} - p_{1,2} > \frac{r_1 - r_3}{\alpha u_2}$ agents 2 play fundamental strategies and the NE is (0,0,1). The condition is $-\frac{1}{6} > \frac{r_1 - r_3}{\alpha u_2}$. Otherwise, if $-\frac{b}{6} < \frac{r_1 - r_3}{\alpha u_2}$ agents two play speculative strategies and the equilibrium is (0,1,1).

In the combinations of returns that follow, one fundamental NE always exist. This could coexist with another equilibrium in which two types of agents play speculative strategies (multiple equilibria).

R4 ($r_2 < r_3 < r_1$). The conditions for the NE (1,1,0) are $\Delta_1 < 0$, $\Delta_2 < 0$, and $\Delta_3 > 0$, that is

$$p_{2,1} - p_{3,1} < \frac{r_3 - r_2}{\alpha u_1}$$

$$0 < \frac{r_1 - r_3}{\alpha u_2}$$

$$p_{1,3} > \frac{r_2 - r_1}{\alpha u_3}$$

with $p_{2,1} = \frac{1}{3}(1-b)$, $p_{3,1} = \frac{1}{3}$, and $p_{1,3} = \frac{1}{3}(1-a)$. Because $p_{2,1} - p_{3,1} < 0$, the first condition from the top is satisfied. The following one clearly always holds, as $r_1 > r_3$. Similarly $r_2 - r_1 < 0$ ensures that the last inequality is true. Therefore the (1,1,0) NE exists for any set of parameters – as long as $r_2 < r_3 < r_1$. Next, the (1,0,1) equilibrium is verified. The three conditions implied by (12) are now

$$0 < \frac{r_3 - r_2}{\alpha u_1}$$

$$p_{3,2} > \frac{r_1 - r_3}{\alpha u_2}$$

$$p_{1,3} - p_{2,3} < \frac{r_2 - r_1}{\alpha u_3}$$

with $p_{3,2} = \frac{1}{3}(1-a)$, $p_{1,3} = \frac{1}{3}(1-b)$, and $p_{2,3} = \frac{1}{3}$. The top inequality is always verified. The middle one requires $\frac{1}{3}(1-a) > \frac{r_1 - r_3}{\alpha u_2}$. The bottom one requires $-\frac{b}{3} < \frac{r_2 - r_1}{\alpha u_3}$. In sum, if $\frac{1}{3}(1-a) > \frac{r_1 - r_3}{\alpha u_2}$ and $-\frac{b}{3} < \frac{r_2 - r_1}{\alpha u_3}$ agents 2 and agents 3 play speculative strategies. The (1,0,1) NE exists along with the (1,1,0) NE.

Because the following two cases are qualitatively similar to the one just discussed, the existence conditions are simply summarized.

R5 ($r_3 < r_1 < r_2$). The (0,1,1) is the fundamental NE. It exists for any set of parameters (for which the value functions are non negative). Under the following two conditions it also exists the (1,1,0) NE:

$$p_{2,1} - p_{3,1} > \frac{r_3 - r_2}{\alpha u_1}$$

$$p_{1,3} > \frac{r_2 - r_1}{\alpha u_3}$$

with $p_{2,1} = \frac{1}{3}(1-b)$, $p_{3,1} = \frac{1}{3}$ and $p_{1,3} = \frac{1}{3}(1-a)$. The top one requires $-\frac{b}{3} > \frac{r_3 - r_2}{\alpha u_1}$ and the bottom one that $\frac{1}{3}(1-a) > \frac{r_2 - r_1}{\alpha u_3}$.

R6 ($r_1 < r_2 < r_3$). The (1,0,1) is the fundamental NE. It exists for any set of parameters (for which the value functions are non negative). The two conditions for the existence of (0,1,1) NE (speculative) are:

$$p_{2,1} > \frac{r_3 - r_2}{\alpha u_1}$$

$$p_{3,2} - p_{1,2} < \frac{r_1 - r_3}{\alpha u_2}$$

with $p_{2,1} = \frac{1}{3}(1-a)$, $p_{3,2} = \frac{1}{3}(1-b)$. With these values the two inequalities become $\frac{1}{3}(1-a) > \frac{r_3-r_2}{\alpha u_1}$ and $-\frac{b}{3} < \frac{r_1-r_3}{\alpha u_2}$.

Liquidity

This section constructs the indices of market thickness, frequency of trade, and acceptability. The market thickness captured by the stock of commodity i on the market at a given time is

$$x_i(s) = p_{i+2} + (\mu_{i+1} - p_{i+1}),$$

for all i. Let $o_i(s)ds$ be the probability that good i is offered (but not necessarily traded) on the market between time s and $s + ds$. Then

$$\begin{aligned} o_i(s) = \alpha p_{i+2}[p_{i+1} + (\mu_i - p_i) + (p_i + \mu_{i+2} - p_{i+2})\theta_{i+2}] + \\ \alpha(\mu_{i+1} - p_{i+1})[p_i + (\mu_{i+2} - p_{i+2}) + (p_{i+1} + \mu_i - p_i)(1 - \theta_{i+1})]; \end{aligned}$$

for $i = 1, 2, 3$.

Let $t_i(s)ds$ the probability that good i is traded on the market between time s and $s+ds$. Then

$$\begin{aligned} t_i(s) = \alpha\{p_{i+2}(\mu_i - p_i(1 - \theta_{i+2}) + \theta_{i+1}p_{i+1}) + (\mu_{i+1} - p_{i+1})[p_i + \\ (\mu_i - p_i)(1 - \theta_{i+1}) + (\mu_{i+2} - p_{i+2})(1 - \theta_{i+2})]\}. \end{aligned}$$

The 'velocity' of circulation of good i is $v_i(s) = \frac{t_i(s)}{x_i(s)}$. This quantity is not a good indicator of the moneyness of an object, because, paradoxically, the velocity can be very high even if it is rarely traded.

Finally, the acceptability of commodity i is

$$a_i(s) = \frac{t_i(s)}{o_i(s)}.$$

This indicates how willing people are to accept commodity i, once it is being offered.

Numerical Methods

All the programming is done in Matlab. The main running file, called 'dynamics_KW', sets the parameters, specifies the initial distribution of inventories and the initial guess of the strategies, launches the iteration solution procedure, and generates the plots. The iteration procedure is based on the interaction between two files, triggered within the file 'dynamics_KW': 'backward_ode_kw89' and 'forward_ode_kw89'. The 'backward' file gives instructions to integrate the distribution of inventories forward. The resulting distribution

of inventories (reversed with respect to time) is saved and used as an input for in the 'backward' file. This integrates the value functions using, as initial condition, their steady state values. The 'backward' file delivers the three the value functions (in differences) of the representative agents $i = 1, 2, 3$ as well as a profile of strategies. The 'dynamics_KW' file saves this profile and uses it as new initial guess for the overall population. After each iteration, the 'dynamics_KW' file computes the differences between vector θ_i $(t)$and of $\tau_i(t)$. When no discrepancy is noticed, the iteration is stopped, the resulting trajectory of strategies and of the assets distribution is recorded as an equilibrium, and all remaining variables (acceptability, consumption, etc...) are computed along such a trajectory. In both in the backward and in the forward files the ordinary differential equations are computed at a fixed 0.0001 time-step of a year. The Runge-Kutta approximation method with adjustable steps readily available in Matlab, is not used because it would require an additional layer of coding to synchronize the timing in the 'backward_ode_kw89' and 'forward_ode_kw89' file.