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# Proxy fights in incomplete markets: when majority voting and sidepayments are equivalent\*

Hervé Crès<sup>†</sup> and Mich Tvede<sup>‡</sup>

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## Abstract

This article provides a study of corporate control in a general equilibrium framework for production economies. When markets are incomplete, trading assets does not allow agents to fully resolve their conflict of interest: at the market equilibrium, shareholders disagree on the way to evaluate production plans which ly outside the market span, and the objective function of the firm is not well defined. Two ways of resolving these conflicts are compared here. The first one (see, e.g., Drèze (1974) and Grossman & Hart (1979)) relies on sidepayments between shareholders. The second one (see, e.g., Drèze (1985) and DeMarzo (1993)) relies on majority voting in the assembly of shareholders: a *stable* production plan is one which cannot be overruled by a majority of shareholders. Since voting occurs in a multi-dimensional setup, super majority rules are needed to ensure existence of such ‘political’ equilibria. The most interesting equilibria are those which are stable with respect to the super majority rule with smallest rate. The present paper provides a framework where these two approaches yield the same equilibria.

**Keywords:** Incomplete markets, shareholders’ voting, sidepayments.

**JEL-classification:** D21, D52, D71, G39.

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# 1 Introduction

When shareholders disagree on the investment policies, control rights over the firm are very important, and become the stakes of ‘proxy fights’. This is what happens in general equilibrium models of production economies where markets are incomplete: agents (consumers/shareholders) trade assets but, at market equilibria, they typically disagree on the way to evaluate income streams which lie outside the market span; indeed they compute them using different systems of shadow prices. Hence profit maximization is not a well defined objective function for the firm<sup>1</sup>.

Among the ways that have been proposed in the literature to resolve these conflicts arising at equilibrium, two are compared in the present paper. The first way, proposed by Diamond (1967), Drèze (1974) and Grossman & Hart (1979), consists in allowing sidepayment between shareholders. At equilibrium, there is no alternative investment policy that makes everybody better off, even if one allows for sidepayments and transfers between shareholders<sup>2</sup>. Drèze (1974) bases its argument on efficiency grounds and Grossman & Hart (1979) on competitive behavior. But in all cases, similar criteria are proposed as an objective for the firm: profit should be maximized with respect to a system of shadow prices that *averages* the idiosyncratic shadow prices of all shareholders. The difference is that Diamond (1967) and Drèze (1974) use the *after* trade shareholdings to average the shadow prices, whereas Grossman & Hart (1979) use the *initial* shareholdings; thus it depends on whether the manager is acting in the interest of final or initial shareholders.

A second way to resolve these disputes between shareholders is based on majority voting in assemblies of shareholders. Among others, Drèze (1985) and DeMarzo (1993) propose the same concept of *stable* production equilibria: they are such that, at equilibrium, within each firm, the production plans of other firms remaining fixed, no alternative production plan can rally a majority of shareholders/shares (depending on whether the governance is of the ‘one person–one vote’ or ‘one share–one vote’ type) against the status quo. As Gevers (1974) already noted, the first problem into which this approach runs is existence: the seminal work of Plott (1967)<sup>3</sup> shows that in multi-dimensional voting models, a simple majority political equilibrium typically does not exist. A way to restore

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<sup>1</sup>For details on standard general equilibrium models of production with incomplete markets and the role of the firms, see, e.g., Duffie & Shafer (1988), Geanakoplos, Magill, Quinzii & Drèze (1990), Magill & Quinzii (1996) and Kelsey & Milne (1997) as well as more recent studies: Citanna & Villanacci (1997), Dierker, Dierker & Grodal (1999) and Bettzüge & Hens (2000).

<sup>2</sup>Grossman & Hart (1979) describes a takeover scheme where a ‘new’ manager, proposing an alternative investment policy, solicits payments from the shareholders who perceive they derive benefits from the change to buy proxies from the shareholders who perceive that the change impairs their welfares.

<sup>3</sup>Developped, among others, by McKelvey (1976, 1979), Schoffield (1978) and Cohen (1979).

existence is to instore a super majority rule<sup>4</sup>: to overturn the status quo, a challenger should rally a proportion bigger than the simple majority of the voting population. The question then arises of a ‘suitable’ level a super majority,  $\rho \in [1/2, 1]$ : it should be high enough to ensure existence, and low enough so that there are not too many equilibria. The standard way to proceed<sup>5</sup> is to associate to each proposal its *score*. The score of a proposal (the incumbent, or status quo) is the fraction of the voting population supporting, against this proposal, its most dangerous challenger, i.e., the alternative proposal that rallies the maximal fraction of voters against the incumbent. The most stable proposals are the ones with lowest score.

The main contribution of the present paper is to set a framework in which the equilibrium concepts provided by the two above mentioned approaches – the one based on sidepayments, and the one based on super majority voting – coincide. The model is that of a finance economy with two periods,  $S$  states of nature in period 1, one good in each state and a continuum of agents. A generic agent is endowed with unrestricted initial endowments and characterized by a linear-quadratic utility function, as in a standard version of the CAPM. It happens to be that, under certain condition on the distribution of primitive characteristics of the agents<sup>6</sup>, the Drèze-Grossman-Hart equilibria based on sidepayments are stable for the smallest possible rate of super majority, a rate that is shown to be smaller than 64%.

The paper is organized as follows. Section 2 introduces the model and three equilibrium concepts. Section 2.1 provides the setup of the model. In Section 2.2, the central equilibrium concept is defined and computed; it is the concept of *stock market equilibrium with fixed production plans* (Definition 1): firms initially announce production plans, then agents trade the corresponding assets, and finally an equilibrium price system occurs for the assets, defining equilibrium portfolios and allocations. The remaining of Section 2 is devoted to identify, among the stock market equilibrium with fixed production plans, which fullfill the Drèze-Grossman-Hart criterion, and which are stable with respect to super majority voting. In Section 2.3, ownership structures are defined as distributions of power among shareholders in the corporate control mechanism<sup>7</sup>; henceforth a first subset,  $\mathcal{S}_1$ , of the set of stock market equilibria with fixed production plans is described

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<sup>4</sup>As proved by, e.g., Ferejohn and Grether (1974), Greenberg (1979), Caplin & Nalebuff (1988, 1991).

<sup>5</sup>Which dates back to Simpson (1969) and Kramer (1973), often called the ‘min-max rule’.

<sup>6</sup>The fundamental result of the literature on social choice that is used here is that of Caplin & Nalebuff (1988, 1991), Balasko & Crès (1997).

<sup>7</sup>E.g., it can be the uniform distribution — as for the ‘one person–one vote’ governance — or the distribution of shareholdings — as for the ‘one share–one vote’ governance — where the considered shareholdings are the pre-trade ones — for a governance à la Grossman & Hart — or the post-trade ones — for a governance à la Diamond-Drèze.

(Definition 2): those for which the production plan proposed by each firm fulfills the Drèze-Grossman-Hart criterion, i.e., maximizes profit with respect to the *average shadow prices* of the shareholders —where the weights to average shadow prices are given by the ownership structure. In Section 2.4, a second subset,  $\mathcal{S}_2$ , of the set of stock market equilibria with fixed production plans is described (Definition 3): those which are the stable with respect to the super majority rule with smallest rate according to the Simpson-Kramer min-max approach.

Section 3 contains the results of the paper. Section 3.1 provides the main result: the set,  $\mathcal{S}_1$ , of equilibria that are stable with respect to sidepayments is a subset of the set,  $\mathcal{S}_2$ , of equilibria that are stable with respect to super majority rules with rates smaller than 64% (Theorem 1), and  $\mathcal{S}_1$ , and thus  $\mathcal{S}_2$ , are not empty (Theorem 2). Section 3.2 yields conditions under which similar results hold for the simple majority rule (Proposition 1). Section 3.3 provides some concluding remarks and comments.

## 2 The Model

### 2.1 The setup

Consider a finance economy with two periods,  $t \in \{0, 1\}$ , and  $S$  states of nature in period 1, indexed by  $s$ ,  $s \in \{1, \dots, S\}$  ( $s = 0$  denotes the initial period). There is one good in each state, and a continuum of agents. A generic agent is endowed with a vector of initial endowments  $\omega = (\omega^0, \omega^1, \dots, \omega^S) \in \mathbf{R}^{S+1}$  and has a utility function of the linear-quadratic type:

$$u(x) = \lambda x^0 + \sum_{s=1}^S \pi^s \left( \gamma x^s - \frac{1}{2} (x^s)^2 \right),$$

where  $x^s$  is the agent's consumption in state  $s$ ;  $\pi = (\pi^1, \dots, \pi^S)$  is the (common) vector of objective probabilities over the states of nature at date 1; the vector  $(\lambda, \gamma) \in \mathbf{R}_{++}^2$  is the idiosyncratic characteristics of this generic agent's preferences. Utility functions are thus of the mean-variance type and moreover satisfy the assumption of linear risk tolerance with an identical coefficient of linear risk tolerance.

A generic agent is thus indexed by a vector of primitive characteristics  $\mu = (\omega, \lambda, \gamma) \in \mathbf{R}^{S+1} \times \mathbf{R}_{++}^2$ . Let  $\mathcal{M} \subset \mathbf{R}^{S+1} \times \mathbf{R}_{++}^2$  be the support of the distribution of the agents. We assume that  $\mathcal{M}$  is compact and convex and that agents are distributed over  $\mathcal{M}$  according to a continuous density function  $f$ . Define  $M = (\Omega, \Lambda, \Gamma) \in \mathbf{R}^{S+3}$ , the vector of aggregate characteristics, i.e., such that

$$\forall s : \int_{\mathcal{M}} \omega^s f(\mu) d\mu = \Omega^s; \quad \int_{\mathcal{M}} \lambda f(\mu) d\mu = \Lambda; \quad \int_{\mathcal{M}} \gamma f(\mu) d\mu = \Gamma.$$

We assume that  $\lambda > 0$  and  $\gamma > \max\{\Omega^1, \dots, \Omega^S\}$  for all  $\mu \in \mathcal{M}$  in order to have monotone preferences on the relevant domain.

There are  $J$  firms indexed by  $j$ ,  $j \in \{1, \dots, J\}$ . Firm  $j$  is characterized by its production set  $\mathcal{Y}_j \subset \mathbf{R}^{S+1}$ . These production sets are supposed to be closed and convex, and moreover the set of efficient production plans (denoted  $\mathcal{Z}_j = \{y_j \in \mathcal{Y}_j | (\{y_j\} + \mathbf{R}_+^{S+1}) \cap \mathcal{Y}_j = \{y_j\}\}$ ) is supposed to be bounded. Firms are owned by agents. The initial distribution of shares within firm  $j$  is described by a continuous real function  $\delta_j$  over  $\mathcal{M}$ , satisfying

$$\int_{\mathcal{M}} f(\mu) \delta_j(\mu) d\mu = 1, \text{ for all } j.$$

## 2.2 Stock market equilibria with fixed production plans

Given announced production plans,  $y = (y_j)_{j=1}^J \in \prod_{j=1}^J \mathcal{Z}_j$  (it is assumed that firms only announce efficient production plans, and that the  $y_j$ 's are taken in general position), a market span  $\langle Y \rangle$ , of dimension  $J$ , is available for agents to trade in, where  $Y$  denotes the payoffs matrix:

$$Y = \begin{pmatrix} y_1^1 & \dots & y_J^1 \\ \vdots & & \vdots \\ y_1^S & \dots & y_J^S \end{pmatrix}.$$

Denote  $y^s \in \mathbf{R}^J$  the  $s$ -th row of the matrix  $Y$ ,  $s \in \{1, \dots, S\}$ , and  $y^0$  the vector of period 0 inputs. Agents trade the  $J$  equity contracts,  $(y_j)_{j=1}^J$ , at the market prices,  $q = (q_j)_{j=1}^J \in \mathbf{R}^J$ . The budget constraint of a generic agent  $\mu$  who buys the portfolio  $\theta(\mu) \in \mathbf{R}^J$  is<sup>8</sup>:

$$\begin{aligned} x^0(\mu) &= \omega^0 + q \cdot [\delta(\mu) - \theta(\mu)] + \delta(\mu) \cdot y^0 \\ x^s(\mu) &= \omega^s + \theta(\mu) \cdot y^s \end{aligned} \quad \text{for all } s \neq 0 \quad (1)$$

Each agent maximizes his utility by choosing an optimal portfolio  $\theta(\mu)$  and the corresponding optimal consumption plan  $x(\mu)$  under the constraints (1).

**Definition 1** *A stock market equilibrium with fixed production plans ( $SME(y)$ ) is a vector  $\mathcal{E} = (y, q, x, \theta)$  such that*

- given  $(y, q)$ , for all  $\mu \in \mathcal{M}$ ,

$$x(\mu) = \arg \max \{u_\mu(x) \mid \exists \theta : (x, \theta) \text{ satisfies (1)}\}$$

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<sup>8</sup>Whether it is the the initial or final shareholders who pay for the inputs,  $y^0$ , i.e., whether they are paid according to the initial or final portfolio,  $\delta$  or  $\theta$ , does not make any difference in the present paper, as far as the equilibrium concept and results are concerned.

- the associated optimal portfolio map  $\theta : \mathcal{M} \rightarrow \mathbf{R}^J$  satisfies

$$\int_{\mathcal{M}} f(\mu)\theta_j(\mu)d\mu = 1,$$

for all  $j$ .

Fix the following notation: the subscript  $\cdot_1$  under a vector means that only the last  $S$  coordinates (corresponding to period one) are considered. Let  $\Pi$  be the  $S$ -dimensional diagonal matrix with  $\pi = (\pi_1, \dots, \pi_S)$  in the diagonal.

**Claim 1** *At equilibrium,*

$$x_1(\mu) = \omega_1 + YB(\gamma\mathbf{1}_S - \omega_1 - \lambda v) \quad (2)$$

$$\theta(\mu) = B(\gamma\mathbf{1}_S - \omega_1 - \lambda v) \quad (3)$$

$$Du_\mu(x(\mu)) = \lambda \begin{pmatrix} 1 \\ \Pi Y B v \end{pmatrix} + \begin{pmatrix} 0 \\ \Pi(I - YB)(\gamma\mathbf{1}_S - \omega_1) \end{pmatrix} \quad (4)$$

$$q = v^t Y B \quad (5)$$

where  $B = (Y^t \Pi Y)^{-1} Y^t \Pi$  and  $v = \frac{1}{\Lambda}(\Gamma\mathbf{1}_S - \Omega_1 - Y\mathbf{1}_J)$ .<sup>9</sup>

*Proof:* Standard. See, e.g., Magill & Quinzii (1996), Section 17.

*Q.E.D.*

### 2.3 Stock market equilibria

Diamond (1967), Drèze (1974) and Grossman & Hart (1979) are important contributions to the problem of resolving the conflicts arising between shareholders in the context of incomplete financial markets. These papers define a concept of stock market equilibrium based on allowing sidepayments between disagreeing shareholders. Within each firm, an alternative production plan can be proposed to shareholders instead of the status quo; the fundamental scheme is that of a ‘takeover’ where the ‘new’ manager, proposing the alternative investment policy, solicits payments from the shareholders who (perceive they) derive benefits from the change to buy proxies from the shareholders who (perceive they) derive losses from the change. An equilibrium is such that there does not exist alternative production plans and a set of transfers between shareholders that makes them all better off. Drèze (1974) argues on grounds of efficiency while Grossman & Hart (1979) argues on

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<sup>9</sup>The matrix  $YB$  is associated with the  $\pi$ -projection on the market span  $\langle Y \rangle$ . Of course, for all  $w \in \langle Y \rangle$ ,  $YBw = w$ .

grounds of competitive behavior. But in both cases they show that, at these equilibria, the production plans chosen by firms are those which maximize profits with respect to the *mean* shadow prices of the shareholders. The weights to average shadow prices are the shares of the final shareholders for Drèze (1974) and initial shareholders for Grossman & Hart (1979).

This naturally leads to the definition, in the present setup, of stock market equilibria where the production plans chosen by firms are those that maximize profits with respect to the mean shadow prices of the shareholders for a given ownership structure. The latter concept consists of a map,  $\eta : \mathcal{M} \rightarrow \mathbf{R}^J$ , whose  $j$ 's component,  $\eta_j$ , is a continuous density on  $\mathcal{M}$  characterizing the ownership structure within firm  $j$ . An intuitive interpretation of the function is that  $\eta_j(\mu)$  is the ‘weight’ given to agent  $\mu$  in the selection procedure of the production plan. Thus, it characterizes a mode of governance. Hence this last definition corresponds to the Drèze criterion for  $\eta \equiv \theta$ , and to the Grossman & Hart criterion for  $\eta \equiv \delta$ .

**Definition 2** *Given an ownership structure  $\eta$ , at a stock market equilibrium with fixed production plans:  $\mathcal{E} = (y, q, x, \theta)$ , the **mean shadow price vector** for firm  $j$  is  $\bar{p}_j(\eta) \in \mathbf{R}_{++}^{S+1}$  defined by*

$$\bar{p}_j(\eta) = \frac{1}{\int_{\mathcal{M}} f(\mu)\eta_j(\mu)d\mu} \int_{\mathcal{M}} f(\mu)\eta_j(\mu)Du_{\mu}(x(\mu))d\mu.$$

**A stock market equilibrium** for the ownership structure  $\eta$  ( $SME(\eta)$ ) is a stock market equilibrium with fixed production plans,  $\mathcal{E} = (y, q, x, \theta)$ , which satisfies:

$$\forall j, y_j = \arg \max\{\bar{p}_j(\eta) \cdot z_j \mid z_j \in \mathcal{Y}_j\}. \quad (6)$$

*Remarks:*

1. Obviously, thanks to Claim 1 (and the linearity of the optimal consumption), given an ownership structure  $\eta$ , at a  $SME(y)$ ,  $\mathcal{E}$ , the mean shadow price vector is the shadow price vector of the **mean shareholder** defined by the vector of primitive characteristics  $\bar{\mu}_j(\eta) \in \mathbf{R}^{S+3}$ :

$$\bar{\mu}_j(\eta) = \frac{1}{\int_{\mathcal{M}} f(\mu)\eta_j(\mu)d\mu} \int_{\mathcal{M}} \mu f(\mu)\eta_j(\mu)d\mu$$

i.e.,

$$\bar{p}_j(\eta) = Du_{\bar{\mu}_j(\eta)}(x(\bar{\mu}_j(\eta))).$$

2. The mean shareholder  $\bar{\mu}_j(\eta)$  might not be in  $\mathcal{M}$ .



## 2.4 Majority stable equilibria

The aim of the present paper is to relate the latter concept of  $\text{SME}(\eta)$  to the one of majority stable equilibrium (see, e.g., Drèze (1985), DeMarzo (1993), Crès (2000)). Informally, a  $\text{SME}(y)$ ,  $\mathcal{E}$ , is a  $\rho$ -majority stable equilibrium provided that, within each firm, there is no alternative production plan which is preferred by more than  $\rho \times 100$  percent of the shareholders. A more formal description of the latter concept follows.

Fix a  $\text{SME}(y)$ ,  $\mathcal{E} = (y, q, x, \theta)$ , and an ownership structure,  $\eta$ , with  $\eta_j(\mu) \geq 0$  for all  $j$  and  $\mu$ . Agent  $\mu$  prefers the challenger  $z_j \in \mathcal{Y}_j$  over the status quo,  $y_j$ , all other production plans,  $y_{-j} = (y_k)_{k \neq j}$ , being fixed, if and only if

$$u_\mu(x(\mu) + \theta_j(\mu)(z_j - y_j)) > u_\mu(x(\mu)). \quad (7)$$

Let  $\mathcal{M}_\mathcal{E}(z_j, y_j) \subset \mathcal{M}$  be the set of agents satisfying<sup>10</sup> equation (7). The relative weight of agents preferring the challenger over the incumbent is the following:

$$P_{\mathcal{E}, \eta}(z_j, y_j) = \frac{\int_{\mathcal{M}_\mathcal{E}(z_j, y_j)} f(\mu) \eta_j(\mu) d\mu}{\int_{\mathcal{M}} f(\mu) \eta_j(\mu) d\mu}. \quad (8)$$

Then define the ‘score’ (in a Simpson-Kramer perspective, as explained in the introduction) of the status quo  $y_j$  as the relative weight of agents who prefer its most dangerous challenger:

$$P_{\mathcal{E}, \eta}(y_j) = \sup_{z_j \in \mathcal{Y}_j} P_{\mathcal{E}, \eta}(z_j, y_j). \quad (9)$$

**Definition 3** *A  $\rho$ -majority stable equilibrium for the ownership  $\eta$  ( $\rho$ -MSE( $\eta$ )) is a stock market equilibrium with fixed production plans,  $\mathcal{E} = (y, q, x, \theta)$ , which satisfies:*

$$\forall j : P_{\mathcal{E}, \eta}(y_j) \leq \rho. \quad (10)$$

The last part of this section is devoted to prove that the most dangerous challengers for the status quo are infinitesimally close to it.

For  $y_j \in \mathcal{Z}_j$  there exists  $p_j \in \mathbf{R}_+^{S+1} \setminus \{0\}$  such that  $y_j$  maximizes profits for firm  $j$  with respect to  $p_j$ . It is a **supporting price** of  $y_j$ . Suppose that shareholders can propose only infinitesimal changes,  $\varepsilon t_j \in \langle p_j \rangle^\perp$ , of the production plan  $y_j$  (where  $\langle p_j \rangle^\perp$  is the hyperplane orthogonal to  $p_j$ ); agent  $\mu$  supports the change (for  $\varepsilon$  sufficiently small) if and only if:

$$\theta_j(\mu) Du_\mu(x(\mu)) \cdot t_j > 0. \quad (11)$$

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<sup>10</sup>For governances à la Grossman & Hart, where the production decision incurs to the initial shareholders, we assume as in Grossman & Hart (1979) that agent have *competitive price perceptions*.

Forgetting about the measure zero agents (in  $\mathcal{M}$ ) having, at equilibrium, no shares of firm  $j$ , denote  $\mathcal{L}_\varepsilon(t_j) \subset \mathcal{M}$  the set of agents which are *indifferent* to the change,  $\varepsilon t_j$ . It is described by the equation:

$$Du_\mu(x(\mu)) \cdot t_j = \lambda t_j^0 + \Pi((I - YB)(\gamma \mathbf{1}_S - \omega_1) + \lambda YBv) \cdot t_{j1} = 0. \quad (12)$$

Equation (12) is *linear* in  $\mu$ . Therefore, at a SME( $y$ ),  $\mathcal{E}$ , for any direction of infinitesimal change,  $t_j$ , the set  $\mathcal{L}_\varepsilon(t_j)$  is a *hyperplane* in  $\mathcal{M}$  which separates the set of agents who support the change — denoted  $\mathcal{L}_\varepsilon^+(t_j) \subset \mathcal{M}$  — from the set of agents who oppose to it —  $\mathcal{L}_\varepsilon^-(t_j)$ . The following lemma (whose proof follows immediately from quasi-concavity of utility functions) states that the most dangerous challengers, for the status quo  $y_j$ , are proposals  $z_j$  which are infinitesimally close to  $y_j$ .

**Lemma 1** *For a given ownership structure  $\eta$  and a fixed SME( $y$ ),  $\mathcal{E}$ ,*

$$\forall j, P_{\mathcal{E}, \eta}(y_j) = \sup_{t_j \in \langle p_j \rangle^\perp} \frac{\int_{\mathcal{L}_\varepsilon^+(t_j)} f(\mu) \eta_j(\mu) d\mu}{\int_{\mathcal{M}} f(\mu) \eta_j(\mu) d\mu}.$$

### 3 Results

A restricted family of governances is considered here. The governances considered in this section are of the form:

$$\eta_j(\mu) = \begin{cases} (\theta_j(\mu))^a & \text{for } \theta_j(\mu) > 0 \\ 0 & \text{for } \theta_j(\mu) \leq 0 \end{cases} \quad \text{or} \quad \eta_j(\mu) = \begin{cases} (\delta_j(\mu))^a & \text{for } \delta_j(\mu) > 0 \\ 0 & \text{for } \delta_j(\mu) \leq 0. \end{cases}$$

where  $a$  is a real number, depending on whether it is based on post-trade or pre-trade shares. Hence, the classical ‘one shareholder–one vote’, resp. ‘one share–one vote’, governances are members of the considered family for  $a = 0$ , resp.  $a = 1$ . Governances of the first, resp. second, form are denoted  $\theta_+^a$  and  $\delta_+^a$ .

#### 3.1 SME( $\eta$ ) exist and are $\rho$ -MSE( $\eta$ ) for $\rho \leq 1 - 1/e$ .

We restrict the analysis to the case of  $\nu$ -concave distributions of primitive characteristics. A density function,  $g$ , is  $\nu$ -concave over  $\mathcal{M}$  if for all  $\mu, \mu' \in \mathcal{M}$ , for all  $\lambda \in [0, 1]$ ,

$$g((1 - \lambda)\mu + \lambda\mu') \geq ((1 - \lambda)g(\mu)^\nu + \lambda g(\mu')^\nu)^{1/\nu}.$$

This assumption is regarded in the literature (see Caplin & Nalebuff (1988, 1991)) as imposing some degree of homogeneity in the agents’ primitive characteristics. Notice

$\nu = \infty$  yields the uniform distribution. Define<sup>11</sup> the rate of super majority:

$$r(\alpha) = 1 - \left( \frac{S - 1 + \alpha}{S + \alpha} \right)^{S-1+\alpha}.$$

**Theorem 1** *Suppose  $f : \mathcal{M} \rightarrow \mathbf{R}_+$  is  $\sigma$ -concave, then a  $SME(\eta)$  is a  $\rho$ - $MSE(\eta)$  for  $\eta \equiv \theta_+^a$  provided that  $\rho \geq r(a + 1/\sigma)$ . Moreover, suppose that  $\{\mu | \delta_j(\mu) > 0\}$  is convex and that  $\delta_j$  is  $\tau$ -concave for all  $j$  on this set, then a  $SME(\eta)$  is a  $\rho$ - $MSE(\eta)$  for  $\eta \equiv \delta_+^a$  provided that  $\rho \geq r(1/\sigma + a/\tau)$ .*

*Proof:* Take a  $SME(\eta)$ . Any proposed direction of infinitesimal change  $t_j \in \langle \bar{p}_j(\eta) \rangle^\perp$  is, by construction, orthogonal to the price vector with respect to which profit is maximized, i.e., orthogonal to  $Du_{\bar{\mu}_j(\eta)}(x(\bar{\mu}_j(\eta)))$ . Therefore equation (11) is satisfied for  $\mu = \bar{\mu}_j(\eta)$ : hence, the mean shareholder's primitive characteristics,  $\bar{\mu}_j(\eta)$ , is on  $\mathcal{L}_\mathcal{E}(t_j)$  for any proposed direction of infinitesimal change  $t_j$ . As a consequence, the hyperplane  $\mathcal{L}_\mathcal{E}(u_j)$  always goes through the center of gravity of the distribution of primitive characteristics. Theorem 1 in Caplin & Nalebuff (1991) allows to conclude<sup>12</sup> as soon as the relevant distribution of voting weights is  $\nu$ -concave over a convex support. It is the case by assumption on the primitive characteristics for  $\eta$  based on pre-trade shares,  $\delta$ , because the product of a  $\sigma$ -concave and a  $\tau$ -concave distribution is  $\sigma\tau/(\sigma + \tau)$ -concave — see Lemma 2 in the appendix. For  $\eta$  based on post-trade shares,  $\theta$ , it follows (i) from the property of linearity of the portfolio mapping  $\theta$  which ensures that  $\{\mu | \theta_j(\mu) > 0\}$  is convex for all  $j$  (it is the truncation by a hyperplane of the convex support  $\mathcal{M}$ ), and; (ii) from the property of linearity of the portfolio mapping  $\theta$ , i.e.,  $\theta$  is 1-concave — again see Lemma 2 in the appendix allows to conclude.

*Q.E.D.*

Hence, existence of  $\rho$ - $MSE(\eta)$  follows immediately from existence of  $SME(\eta)$ , which is stated in the following theorem (adapted from Drèze (1974) for continuous distributions of agents).

**Theorem 2** *For all considered governances, a  $SME(\eta)$  exists.*

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<sup>11</sup>The ratio  $r(\alpha)$  is increasing in  $\alpha$  and bounded above by  $1 - 1/e \approx 0.632$  when  $\alpha \geq 2 - S$ .

<sup>12</sup>In loose terms, the theorem states that there is no way to cut, by a hyperplane, a compact and convex support endowed with a  $\nu$ -concave distribution through its centroid so that one of the two resulting pieces is larger than  $100r(1/\nu)$  percent of the weight. The game played is very simple: two players have to share a 'cake' — the support  $\mathcal{M}$  endowed with a  $\nu$ -concave distribution  $g$  — the first player indicates a point in the cake, the second one has to cut the cake by a hyperplane, through the indicated point, as unevenly as possible and takes the biggest piece; what is left to the first player is the Simpson-Kramer score of the point he indicated. The point with highest score (the min-max) is the center of gravity of the cake; its score is  $1 - r(1/\nu)$ .

*Proof:* Four consecutive maps are considered: a map from production plans to equilibrium prices; a map from production plans to equilibrium portfolios and consumption bundles; a map from production plans to mean gradients of shareholders of firms, and; a map from production plans to production plans that maximize profits with respect to the shadow price vectors of mean gradients of shareholders. A fixed point of the map from production plans to production plans is then shown to exist and to induce an equilibrium with the desired properties.

“From production plans to equilibrium prices”: Let the map  $q : \prod_{j=1}^J \tilde{\mathcal{Z}}_j \rightarrow \mathbf{R}^J$  be defined by

$$q(y) = \frac{1}{\Lambda} (\Gamma \mathbf{1}_S - \Omega_1 - Y \mathbf{1}_J)^T \Pi Y$$

where  $\tilde{\mathcal{Z}}_j \subset \mathcal{Y}_j$  is the closure of the convex hull of the set of efficient production plans, i.e.  $\tilde{\mathcal{Z}}_j = \text{cl co } \mathcal{Z}_j$ . Then  $q : \prod_{j=1}^J \tilde{\mathcal{Z}}_j \rightarrow \mathbf{R}^J$  is continuous. Moreover,  $q(y)$  is the vector of equilibrium prices as stated in Claim 1.

“From production plans to equilibrium portfolios and consumption bundles”: Let the map  $\eta : \prod_{j=1}^J \tilde{\mathcal{Z}}_j \times \mathcal{M} \rightarrow \mathbf{R}^J$  be defined according to the considered voting governance, i.e.

$$\eta_j(y, \mu) = \begin{cases} \delta_{+j}^a(\mu)_j & \text{for governances based on pre-trade shares} \\ \theta_{+j}^a(y, \mu) & \text{for governances based on post-trade shares} \end{cases}$$

where

$$\theta(y, \mu) = (Y^t \Pi Y)^{-1} (Y^t \Pi (\gamma \mathbf{1}_S - \omega_1) - \lambda q(y)^t)$$

and let the map  $x : \prod_{j=1}^J \tilde{\mathcal{Z}}_j \times \mathcal{M} \rightarrow \mathbf{R}^J$  be defined by

$$x^s(y, \mu) = \begin{cases} \omega^0 + q \cdot (\delta - \theta(y, \mu)) + \delta \cdot y^0 & \text{for } s = 0 \\ \omega^s + \theta(y, \mu) \cdot y^s & \text{for } s \in \{1, \dots, S\} \end{cases}$$

Then  $\eta : \prod_{j=1}^J \tilde{\mathcal{Z}}_j \times \mathcal{M} \rightarrow \mathbf{R}^J$  and  $x : \prod_{j=1}^J \tilde{\mathcal{Z}}_j \times \mathcal{M} \rightarrow \mathbf{R}^J$  are continuous. Moreover,  $\theta(y, \mu)$ , resp.  $x(y, \mu)$ , is the equilibrium portfolio, resp. consumption bundle, as stated in Claim 1.

“From production plans to mean gradients”: Let the map  $p : \prod_{j=1}^J \tilde{\mathcal{Z}}_j \rightarrow \mathbf{R}_+^{(S+1)J}$  be defined by

$$p_j(y) = \frac{1}{\int_{\mathcal{M}} f(\mu) \eta_j(y, \mu) d\mu} \int_{\mathcal{M}} Du_\mu(x(y, \mu)) f(\mu) \eta_j(y, \mu) d\mu.$$

Then  $p : \prod_{j=1}^J \tilde{\mathcal{Z}}_j \rightarrow \mathbf{R}_+^{(S+1)J}$  is continuous.

“From production plans to production plans”: Let the correspondence  $\zeta : \prod_{j=1}^J \tilde{\mathcal{Z}}_j \rightarrow \prod_{j=1}^J \tilde{\mathcal{Z}}_j$  be defined by

$$\zeta_j(y) = \{y_j \in \tilde{\mathcal{Z}}_j \mid y_j \in \arg \max_{z_j \in \tilde{\mathcal{Z}}_j} p_j(y) \cdot z_j\}.$$

Clearly,  $\zeta$  is upper hemi-continuous and  $\zeta(y)$  is non-empty and convex for all  $y \in \prod_{j=1}^J \tilde{\mathcal{Z}}_j$ .

Hence, there exists  $y^* \in \prod_{j=1}^J \tilde{\mathcal{Z}}_j$  such that  $y^* \in \zeta(y^*)$  according to Kakutani's fixed point theorem.

*Q.E.D.*

### 3.2 Existence of $\rho$ -MSE( $\eta$ ) for $\rho \leq 1/2$ .

Consider first the case where financial markets are complete ( $J = S$ ); then there is unanimity of shareholders<sup>13</sup> and there trivially exists  $\rho$ -MSE( $\eta$ ) for all  $\rho \geq 0$  and any  $\eta$  such that agents short in the stock of a firm do not have the right to vote. Indeed, in that case, all gradients  $(Du_\mu)_{\mu \in \mathcal{M}}$  and supporting price vectors  $(p_j)_{j=1}^J$  are positively colinear to a common vector  $p$ ; and the technologically possible changes  $t_j$  are such that  $p \cdot t_j \leq 0$ :  $t_j \in \langle p_j \rangle^\perp$  is unanimously rejected by all shareholder positively endowed with shares of firm  $j$ .

The case where either the degree of market incompleteness, or the dimension of the sets of efficient production plans is one. Then the Drèze-Grossman-Hart criterion is not the most fruitful criterion to extract the equilibria that are stable for the super majority rules with smallest rate  $\rho$ . Indeed, in these cases, a concept of *median* shareholder naturally comes to mind and it is possible find equilibria that are stable with respect to the simple majority rule.

**Proposition 1** *Suppose that either  $S - J = 1$  or  $\dim \tilde{\mathcal{Z}}_j = 1$  for all  $j$  then a  $\rho$ -MSE( $\eta$ ) exists for all  $\rho = 1/2$ .*

*Proof:* “Case  $S - J = 1$ ”: If  $S - J = 1$  then the normalized gradients, i.e. the  $(1/\lambda)Du_\mu^s(x(\mu))$ 's, are in a 1-dimensional affine subset of  $\mathbf{R}_{++}^{S+1}$ . Therefore, the median gradients of the normalized gradients are well defined and clearly they are upper hemi-continuous correspondences of production plans. Hence, the proof of Theorem 2 carries over, with median gradients replacing mean gradients.

“Case  $\dim \tilde{\mathcal{Z}}_j = 1$  for all  $j$ ”: If  $\dim \tilde{\mathcal{Z}}_j = 1$  for all  $j$  then the set of alternatives is 1-dimensional and induced preferences on the  $\tilde{\mathcal{Z}}_j$ 's are convex for all  $j$ . Therefore, the median shareholders are well defined and clearly they are upper hemi-continuous correspondences of production plans. Hence, the proof of Theorem 2 carries over, with median shareholders replacing mean gradients and utility maximization of median shareholders replacing profit maximization.

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<sup>13</sup>This absence of disputes between shareholders is also the consequence of the *partial spanning* condition developed by Ekern & Wilson (1974) and Leland (1974): firms cannot innovate outside the market span; it is also observed in some restricted versions of the CAPM —see Magill & Quinzii (1996), Theorem 17.3 (iv).

### 3.3 Concluding comments and remarks.

It is not possible, within this simple framework, to compare the stability properties of the Drèze versus the Grossman & Hart criteria. Indeed, although Theorem 1 asks for stronger conditions for a governance à la Grossman & Hart, i.e., that all initial distributions of shares be  $\tau$ -concave, in a sequential version of the model, the initial distribution of shares of one period is the final distribution of shares of the preceding period, and thus is also linear, i.e., 1-concave. Hence the relevant rate of super majority rule should be  $r(a + 1/\sigma)$  for both types of governances. On the other hand,  $r(\alpha)$  being increasing in  $\alpha$ , it is the case that governances of the ‘one person-one vote’ type yield stable equilibria for smaller rates of super majority rule:  $r(1/\sigma)$  instead of  $r(a + 1/\sigma)$ , with  $a = 1$  for governances of the ‘one share-one vote’ type, although the gain is modest. Hence the temptation to propose a negative  $a$ : this means giving more power to shareholders with fewer shares! The difficulty is that, if some shareholders do not want to be long then the considered equity,  $a$ , should stay strictly bigger than -1, otherwise the weight, in the voting process, of a shareholder endowed with a shares tending toward zero might not be anymore negligible.

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## Appendix

### Product of a $\phi$ -concave and a $\psi$ -concave distribution

**Definition 4** A map,  $F : K \rightarrow \mathbf{R}_+$  where  $K \in \mathbf{R}^k$  is compact and convex, is  $\nu$ -concave provided that

$$F((1-t)a + tb) \geq ((1-t)F(a)^\nu + tF(b)^\nu)^{1/\nu}$$

for all  $a, b \in K$  and  $t \in [0, 1]$ .

**Lemma 2** If  $G : K \rightarrow \mathbf{R}_+$  is  $\phi$ -concave and  $H : K \rightarrow \mathbf{R}_+$  is  $\psi$ -concave. Then  $F : K \rightarrow \mathbf{R}_+$  defined by  $F(a) = G(a)H(a)$  for all  $a \in K$  is  $\nu$ -concave for all

$$\nu \leq \frac{\phi\psi}{\phi + \psi}.$$



*Proof:* It follows from the definition of  $\nu$ -concavity that if

$$\begin{aligned} & (((1-t)G(a)^\phi + tG(b)^\phi)^{1/\phi}((1-t)H(a)^\psi + tH(b)^\psi)^{1/\psi})^\nu \\ & \geq (1-t)(G(a)H(a))^\nu + t(G(b)H(b))^\nu \end{aligned}$$

for all  $a, b \in K$  and  $t \in [0, 1]$  then  $F : K \rightarrow \mathbf{R}_+$  is  $\nu$ -concave.

Let  $g, h : [0, 1] \rightarrow \mathbf{R}_+$  be defined by

$$g(t) = ((1-t)G(a)^\phi + tG(b)^\phi)^{1/\phi}$$

$$h(t) = ((1-t)H(a)^\psi + tH(b)^\psi)^{1/\psi}$$

then  $g$  is  $\phi$ -concave and  $h$  is  $\psi$ -concave. Let  $f : [0, 1] \rightarrow \mathbf{R}_+$  be defined by  $f(t) = (g(t)h(t))^\nu$  then the second-order derivative is

$$\begin{aligned} D^2 f &= \nu(gh)^{\nu-2}((\nu-1)((gDf)^2 + (fDg)^2) + 2\nu(gDf)(fDg) \\ &\quad + fg(gD^2 f + fD^2 g)) \\ &\leq \nu(gh)^{\nu-2}((\nu-\phi)(gDf)^2 + 2\nu(gDf)(fDg) + (\nu-\psi)(fDg)^2). \end{aligned}$$

The “ $\leq$ ” follows from the fact that  $g$  being  $\phi$ -concave is equivalent to  $g^\phi$  being concave so  $D^2 g^\phi = \phi g^{\phi-2}((\phi-1)(Dg)^2 + gD^2 g) \leq 0$  implying  $gD^2 g \leq (1-\phi)(Dg)^2$  – similarly for  $h$  and  $\psi$ .

Finally  $(\nu-\phi)(gDf)^2 + 2\nu(gDf)(fDg) + (\nu-\psi)(fDg)^2 \leq 0$  for all values of  $gDf$  and  $fDg$  if and only if  $\nu \leq \phi\psi/(\phi+\psi)$ . Hence,  $F(a) : K \rightarrow \mathbf{R}_+$  is  $\nu$ -concave for all  $\nu \leq \phi\psi/(\phi+\psi)$ .

*Q.E.D.*