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Abstract

We provide several characterizations of unanimity decision rules, in a public choice model where preferences are constrained by attributes possessed by the alternatives (Nehring and Puppe, 2007a,b). Solidarity conditions require that when some parameters of the economy change, the agents whose parameters are kept fixed either all weakly lose or they all weakly win. *Population-monotonicity* (Thomson, 1983a,b) applies to the arrival and departure of agents, while *replacement-domination* (Moulin, 1987) applies to changes in preferences. We show that either solidarity property is compatible with *voter-sovereignty* and *strategy-proofness* if and only if the attribute space is quasi-median (Nehring, 2004), and with *Pareto-efficiency* if and only if the attribute space is a tree. Each of these combinations characterizes unanimity.

JEL classification codes: D63, D71, H41.

Keywords: Solidarity, Population-monotonicity, Replacement-domination, Unanimity, Strategy-proofness, Attribute-based Domains, Generalized Single-Peaked Domains.

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1 Introduction

Unanimity decision rules are used in many political institutions, such as the United Nations Security Council, the ratification procedure for treaties in the European Union, and criminal law juries. Surprisingly few justifications for such rules are available (Berga, Bergantiños, Massó and Neme, 2004; Ju, 2005). We provide several characterizations of unanimity in a general public choice problem. Decision rules prescribe an outcome as a function of the preferences submitted by individuals. The set of admissible preferences is constrained by a set of objective attributes possessed by the alternatives: we consider the class of attribute-based domains, introduced by Nehring and Puppe (2007a, 2007b, 2005, 2010). Solidarity conditions are the key element in all of our characterizations.

Solidarity is a principle of justice with respect to changes in circumstances. It says that all agents not responsible for these changes should be affected in the same direction. Possible changes include the arrival or departure of individuals, as well as changes in their preferences. "Replacement-domination" is introduced by Moulin (1987) in the context of quasi-linear binary public decision. It applies to a model with a fixed population of agents and requires that the replacement of the preferences of one agent causes the other agents to either all weakly win or all weakly lose. "Population-monotonicity" is introduced by Thomson (1983a, 1983b) in the context of bargaining. It applies to a model with a variable population of agents and requires that when one agent joins the population, the other agents whose preferences are kept fixed either all weakly win or they all weakly lose.[1]

In the context of public choice, solidarity conditions are first studied in location models. Thomson (1993) considers a continuous line over which agents have single-peaked preferences. For any preference profile, a target rule selects the Pareto-efficient alternative that is closest to some exogenously fixed alternative on the line. Thomson (1993) shows that the target rules are the only Pareto-efficient rules that satisfy replacement-domination.

[1] For surveys on these two conditions, see Thomson (1995, 1999).

Ching and Thomson (1997) show that these rules are also the only Pareto-efficient rules that satisfy population-monotonicity. Ching and Thomson (1997), Vohra (1999) and Klaus (1999, 2001) extend these results for single-peaked preferences on a tree network. Klaus (1999, 2001) further shows that the target rules are the only ones that satisfy unanimity (if all agents' preferred alternative is the same, it should be selected), strategy-proofness (reporting true preferences is a weakly dominant strategy for all agents) and either condition of solidarity, on a tree network. Finally Klaus (1999, 2001) extends this second result to Euclidean spaces, when agents' preferences are separable across dimensions and quadratic, and characterizes coordinatewise target rules on this domain. Gordon (2007b) obtains an impossibility result for single-peaked preferences on a circle, except on small discrete domains (less than five alternatives) of symmetric preferences.[2]

In this paper, we extend the analysis to a larger class of models in which the set of admissible preferences is constrained by a set of objective attributes possessed by the alternatives. This class of domains generalizes discrete versions of all of the locations models listed in the last paragraph. One difference is that these models assume a continuum of alternatives, while we assume a discrete set. This difference is however not fundamental. The real novelty of our work is that the class of attribute-based domains is larger than the class of domains structured around a location model. For example, the unrestricted domain, the domain of separable preferences over sets of objects (Barberà, Sonnenschein and Zhou 1991) and models of voting under constraints (Barberà, Massó and Neme, 1997 and 2005) can be viewed as attribute-based domains.

Our starting point, in section 3, is a characterization by Nehring and Puppe (2007b), which mirrors results by Barberà, Massó and Neme (1997, 2005) in the closely related model of voting under constraints These authors show that the rules that satisfy voter-sovereignty (any alternative is selected for some profile) and strategy-proofness in any attribute-based domain form a class of "voting by issues" rules which make separate decisions between

[2]Solidarity conditions were also studied in the problem of locating multiple public goods by Miyagawa (1998, 2001), Ehlers (2002, 2003), and Umezawa (2012) and in the problem of selecting a probabilitic location by Ehlers and Klaus (2001). These models differ from ours in that they include some alternatives which are not considered best by any preference in the domain.

each attribute and its complement.

In section 4, we show that in any attribute-based domain, the rules satisfying voter-sovereignty, strategy-proofness and solidarity, when they exist, are unanimity rules, in which each of the attributes of some prespecified "target" alternative can only be defeated by an unanimous vote. This result can be viewed as a generalization of a discrete counterpart of the similar characterizations by Klaus (1999, 2001) on trees and Euclidean spaces. Nehring and Puppe (2007b) have characterized the class of attribute spaces in which unanimity voting by issues rules exist, the "quasi-median" spaces (Nehring 2004). It follows that the quasi-median domains are exactly the ones where these three conditions are compatible. We provide examples to illustrate how this class of domains extends the discrete counterparts of the domains studied by Klaus (1999, 2001).

In section 5, we study the compatibility of Pareto-efficiency and solidarity in attribute-based domains. Here, we do not assume strategy-proofness and hence cannot restrict attention to "voting by issues" rules from the outset, but this last condition turns out to be implied by the other two.[3] Unfortunately, our result is negative. "Trees", which are precisely the discrete counterpart of the domains studied by Ching and Thomson (1997), Vohra (1999) and Klaus (1999, 2001) in continuous location models are the only attribute-based domains where the two conditions are compatible. Finally, we provide a characterization on discrete trees, that mirrors the results obtained by these authors in the continuum case. Our proof differs from theirs, in that we rely on the theory of voting by issues.

2 The model

In this section, we present the class of attribute-based domains (Nehring and Puppe 2007a,b). Then we present the fixed and variable population models and the conditions we are interested in.

Let A be a nonempty finite set of alternatives. Let $\mathcal{H} \subseteq 2^A$ be a non-empty family of

[3] In a companion paper, Gordon (2007a) studies Pareto-efficiency and solidarity in a general public choice problem and establishes several of these conditions' general implications. We defer the discussion of the relation between the two papers to the end of section 5.

subsets of A, with typical element $H \subseteq A$. The elements of $\mathcal{H}$ are called *attributes.* We can think of each of them as a descriptive characteristic relevant for the choice and defined by the set of alternatives that possess it. For example, if the set A is a set of possible constitutions for a nation, the attributes could be "federal", "non-federal", "presidential", "parliamentary" and the attribute "presidential" is defined as the subset of constitutions in A that are presidential. Suppose that $\mathcal{H}$ satisfies the following three conditions. Non-triviality: for all $H \in \mathcal{H}$, $H \neq \emptyset$. Closedness under negation: for all $H \in \mathcal{H}$, $H^c \in \mathcal{H}$. Separation: for all $x \neq y \in A$, there is $H \in \mathcal{H}$ such that $x \in H$ and $y \notin H$. A family $\mathcal{H}$ that satisfies these three conditions is called an *attribute space.*[4] Following Nehring (1999), an attribute space enables us to define a notion of betweenness of alternatives as follows. Let $a, b, c \in A$. We say that b *is between* a *and* c, denoted by $b \in [a, c]$ if b possesses all attributes that are common to a and c. For all $a, b, c \in A$,

$$b \in [a, c] :\Leftrightarrow \text{for all } H \in \mathcal{H},\ \{a, c\} \subseteq H \Rightarrow b \in H.$$

Here are some simple examples. It is important to realize that an attribute space could be much more intricate than the ones shown here.

Example 1 (Lines and Trees) *Suppose that the alternatives can be ordered from left to right by some linear ordering $\leq$ on X. The family $\mathcal{H}$ of all sets of the form $H_{\leq x} :=$ $\{a \in A : a \leq x\}$ or $H_{\geq x} := \{a \in A : a \geq x\}$ for all $x \in A$ define an attribute space. Each attribute is thus of the form "greater than or equal to x" or "lesser than or equal to x". The induced line betweenness is given, for all $a, b, c \in A$, by $b \in [a, c] :\Leftrightarrow [a \leq b \leq c$ or $c \leq b \leq a]$. More generally, an attribute space can be defined on a tree, defined as a graph (set of undirected edges) on A without cycles. For each $x \in A$, the set $A \setminus \{x\}$ can be uniquely represented as the union of two connected components H_x^+ and H_x^- whose intersection is the singleton $\{x\}$. The family $\mathcal{H}$ of all sets of the form H_x^+ and H_x^- for all $x \in A$ define an attribute space.*

Example 2 (The hypercube) *Let $A = \{0, 1\}^K$. An alternative is represented by a sequence $a = \left(a^1, ..., a^K\right)$ with $a^k \in \{0, 1\}$. For all k, let $H_0^k := \{a : a_k = 0\}$ and $H_1^k :=$*

[4] Note that the set A can be recovered from an attribute space $\mathcal{H}$ since $A = \bigcup_{H \in \mathcal{H}} H$.

$\{a : a_k = 1\}$ *and consider the family* $\mathcal{H}$ *of all such subsets. The induced hypercube betweenness is given, for all* $a, b, c \in A$, *by* $b \in [a, c] :\Leftrightarrow \left[\text{for all } k : a^k = c^k \Rightarrow b^k = a^k = c^k\right]$. *Geometrically, b is between a and c if and only if b is contained in the subcube spanned by a and c.*

Example 3 (Cycles) *Let* $A = \{a_1, ..., a_k\}$ *and consider the k-cycle on* A, *i.e. the graph with the edges* (a_l, a_{l+1}) *with the convention* $a_{l+k} = a_l$. *If k is odd, define* $\mathcal{H}$ *as the family of sets of the form* $\left\{a_l, ..., a_{l-1+\frac{k\pm1}{2}}\right\}$. *If k is even, define* $\mathcal{H}$ *as the family of sets of the form* $\left\{a_l, ..., a_{l-1+\frac{k}{2}}\right\}$.

A subset $S \subseteq A$ is *convex* if it is the intersection of attributes. By convention $\cap_{\emptyset} = A$, hence A is also convex. For all $B \subseteq A$, the *convex hull* of B, denoted by $Co(B)$ is the smallest convex set that contains it. For all $a, b \in A$, the *segment* $[a, b]$ is the convex hull of $\{a, b\}$. The elements of $[a, b]$ are exactly the alternatives that are between a and b.

A binary relation R_i is called a *preference* if it is reflexive, transitive and complete, i.e. a weak ordering.[5] Let P_i and I_i be the associated strict preference and indifference relation. Let $\mathcal{H}$ be an attribute space. A preference R_i is adapted to $\mathcal{H}$ if there exists $p(R_i) \in A$ such that for all $a \neq b \in A$, $a \in [p(R_i), b] \Rightarrow a \ P_i \ b$. This means that $p(R_i)$ is the "single peak" of preference R_i, i.e. its most preferred alternative. Moreover if a has more attributes (in an inclusion sense) in common with $p(R_i)$ than b, then preference R_i prefers a to b.

Let $\mathcal{R}$ be the set of preferences, which are adapted to $\mathcal{H}$. In this paper, we focus on domains of preferences, which are adapted to some attribute space.

The line attribute space in Example 1 generates a discrete version of the classic single-peaked domain on a line studied by Moulin (1980), Thomson (1993) and Ching and Thomson (1997). The tree attribute space in Example 1 generates a discrete version of the domain of single-peaked preferences on a tree studied by Ching and Thomson (1997), Vohra (1999), Klaus (1999, 2001) and Schummer and Vohra (2002). The attribute space in

[5] Nehring and Puppe work with linear orderings (2007a,b). Here we consider weak orderings. Either domain is rich enough for our results to hold.

Example 2 generates the separable domain over subsets of objects, introduced by Barberà, Sonnenschein and Zhou (1991). The attribute space in Example 3 generates a discrete version of the domain of single-peaked preferences on a circle studied by Gordon (2007a) and Schummer and Vohra (2002). The unrestricted domain, the domain of separable and single-peaked domains on product of lines are also special cases.

We consider problems with a fixed population and problems with a variable population. For each, we define a relevant solidarity condition.

Fixed population. Let N be a fixed nonempty finite set of agents with generic agent denoted by i. A *fixed population problem* is defined by an attribute space $\mathcal{H}$ and a fixed population N. Each agent i is equipped with a preference $R_i \in \mathcal{R}$. A preference profile is a list $R_N = (R_i)_{i \in N} \in \mathcal{R}^N$. A rule is a mapping $f : \mathcal{R}^N \to A$. We are interested in solutions such that when the preference of one agent changes, the other agents are all affected in the same direction: either they all weakly win or the all weakly lose.

Replacement-domination. For all $R_N \in \mathcal{R}^N$, all $i \in N$, all $R_i' \in \mathcal{R}$, either for all $j \in N \setminus \{i\}$, $f\left(R_i', R_{N\setminus\{i\}}\right) \; R_j \; f\left(R_N\right)$, or for all $j \in N \setminus \{i\}$, $f\left(R_N\right) \; R_j \; f\left(R_i', R_{N\setminus\{i\}}\right)$.

We assume that $|N| \geq 3$, which is the minimal cardinality for which replacement-domination has bite.

Variable population. Let $\mathcal{N}$ be a finite set of potential agents with generic agent denoted by i. A *variable population problem* is defined by an attribute space $\mathcal{H}$ and a set of potential agents $\mathcal{N}$. It consists of the collection of problems with fixed population $(\mathcal{H}, N)$ for all nonempty $N \subseteq \mathcal{N}$. A preference profile is given by a set N and a list $R_N \in \mathcal{R}^N$. A (variable population) rule is a mapping $f : \cup_N \mathcal{R}^N \to A$. For all $N \subset \mathcal{N}$ finite, the restriction of f to $\mathcal{R}^N$ is denoted by f_N. We are interested in solutions such that when one agent joins the economy and the preferences of the agents already present in the economy are kept fixed, these agents who were present before the change are all affected in the same direction: either they all weakly win or they all weakly lose.

Population-monotonicity. For all $N \subset \mathcal{N}$, $R_N \in \mathcal{R}^N$, all $i \in \mathcal{N} \setminus N$, all $R_i' \in \mathcal{R}$, either for all $j \in N,\ f(R'_i, R_N)\ R_j\ f(R_N)$, or for all $j \in N,\ f(R_N)\ R_j\ f(R'_i, R_N)$.

We assume that $|\mathcal{N}| \geq 3$, which is the minimal cardinality for which population-monotonicity has bite.

In each of these models, we study the compatibility of these conditions with the following additional requirements. A fixed population rule satisfies *voter-sovereignty* if it is onto, i.e. if every alternative is selected by the rule for some profile in its domain. Next, a rule is *strategy-proof* if revealing their true preferences is a weakly dominant strategy for all agents: for all $R_N \in \mathcal{R}^N$, all $i \in N$ and all $R'_i \in \mathcal{R}$, we have $f(R_N)\ R_i\ f\left(R'_i, R_{N\setminus\{i\}}\right)$. Last, a rule satisfies *Pareto-efficiency*, if at any profile, there is no alternative that is weakly preferred by all agents in the economy and strictly preferred by at least one agent to the alternative selected by the choice function: for all $R_N \in \mathcal{R}^N$, there exists no $a \in A$ such that for all $i \in N,\ a\ R_i\ f(R_N)$ and for some $j \in N,\ a\ P_j\ f(R_N)$. A variable population rule f satisfies either of these three properties if for all N, the restriction f_N satisfies it. We are interested in rules that satisfy solidarity and either voter-sovereignty and strategy-proofness, or Pareto-efficiency.

3 Voting by issues

In this Section, we present a classic result in the literature, on which our results are based. It characterizes the class of rules that satisfy voter-sovereignty and strategy-proofness as "voting by issues." The results in this section were obtained by Nehring and Puppe (2007b) in the attribute-based framework. Similar results were previously obtained by Barberà, Massó and Neme (1997, 2005) in the related model of voting under constraints.

An issue is defined as the pair formed by an attribute and its complement. Under "voting by issues," the agents vote separately on each issue, over the two corresponding competing attributes, using an issue-specific voting rule for each of the issues. The alternative that is selected is the one that possesses all of the adopted attributes. Of course, such a procedure is well defined only if the issue-specific voting rules satisfy certain joint restrictions.

More precisely, consider a fixed population problem $(\mathcal{H}, N)$. A *family of winning coalitions* is a non-empty family $\mathcal{W}$ of subsets of N satisfying $[W \in \mathcal{W}$ and $W \subseteq W'] \Rightarrow W' \in \mathcal{W}$. For example, for any $q \in (0,1)$, the *quota q family* consists of the coalitions $W \subseteq N$ such that $|W| > q\,|N|$. A *structure of winning coalitions* is a list $(\mathcal{W}_H)_{H\in\mathcal{H}}$ of families of winning coalitions indexed by the attributes of $\mathcal{H}$ such that for each $W \subseteq N$ and each attribute H, we have

$$W \in \mathcal{W}_H \Leftrightarrow N \setminus W \notin \mathcal{W}_{H^c}. \tag{1}$$

Let $\mathcal{W} = (\mathcal{W}_H)_{H\in\mathcal{H}}$ be a structure of winning coalitions. *Voting by issues* associated with $\mathcal{W}$ is the rule f: $\mathcal{R}^N \to A$ such that for all $R_N \in \mathcal{R}^N$,

$$a = f(R_N) :\Leftrightarrow \text{for all } H \in \mathcal{H} \text{ such that } a \in H \text{ we have } \{i : p(R_i) \in H\} \in \mathcal{W}_H.$$

In general, the rule f need not be well-defined.[6] Nehring and Puppe (2007b) provide the following important characterization.

Proposition 1 (Nehring and Puppe, 2007b) *A rule f satisfies voter-sovereignty and strategy-proofness if and only if it is voting by issues and well-defined.*

Obviously, this result extends to variable population problems in the following way.

Corollary 1 *A variable population rule satisfies voter-sovereignty and strategy-proofness if for each nonempty $N \subseteq \mathcal{N}$, the rule f_N is voting by issues and well-defined.*

An important feature of voting by issues is that the rules in this class always selects an alternative that lies in the convex hull of the peaks of the agents: for all $R_N \in \mathcal{R}^N$, $f(R_N) \in Co(\{p(R_i) : i \in N\})$. In particular, for all preference profile with exactly two distinct peaks, the choice function selects an alternative that lies between the two peaks: for all R_N such that there is $a \neq b \in A$ such that $\{p(R_i)\}_{i\in N} = \{a, b\}$, we have $f(R_N) \in [a, b]$.

[6] Nehring and Puppe (2007b) provide a necessary and sufficient condition on the structure of winning coalitions under which voting by issues is well-defined: the "intersection property." Barberà, Massó and Neme (1997) provide a different necessary and sufficient condition, also labelled "intersection property" under which separable voting under constraints is well-defined.

4 Strategy-proofness and Solidarity

In this section, we study the compatibility of voter-sovereignty, strategy-proofness and either solidarity conditions.

In the last section, we introduced *voting by issues.* A rule in this class is a *unanimity rule* if there exists a "target" $\widehat{a} \in A$ whose attributes can only be defeated by unanimity, i.e. such that $\mathcal{W}_H = 2^N \setminus \{\emptyset\}$ for all H such that $\widehat{a} \in H$ and $\mathcal{W}_H = \{N\}$ for all H such that $\widehat{a} \notin H$. Such a rule is denoted by $f^{\widehat{a}}$. As Nehring and Puppe (2005) note, it follows directly from this definition that for any $R_N \in \mathcal{R}^N$, $f^{\widehat{a}}(R_N)$ is well-defined if and only if there exists an alternative that is both in $Co(\{p(R_1), ..., p(R_n)\})$ and in $[\widehat{a}, p(R_i)]$ for all $i \in N$. If there is such an alternative, it is necessarily unique and it is precisely $f^{\widehat{a}}(R_N)$. The condition $f^{\widehat{a}}(R_N) \in Co(\{p(R_1), ..., p(R_n)\})$ expresses the fact that any attribute that is unanimously supported gets approved. The condition $f^{\widehat{a}}(R_N) \in [\widehat{a}, p(R_i)]$ expresses the fact that any attribute possessed by $\widehat{a}$ and supported by at least one agent gets approved. Similarly, in a variable population model, a rule f is a variable-population unanimity rule if there exists $\widehat{a} \in A$, such that for all N, the restriction f_N is the unanimity rule $f^{\widehat{a}}$.

We now show that unanimity rules are the only rules that satisfy voter-sovereignty, strategy-proofness and the relevant solidarity condition both with a fixed and a variable population.

Proposition 2 (i) *A rule f satisfies voter-sovereignty, strategy-proofness and replacement-domination if and only if it is a unanimity rule.* (ii) *A variable-population rule f satisfies voter-sovereignty, strategy-proofness and population-monotonicity if and only if it is a unanimity rule.*

Proof. It is clear that a unanimity rule satisfies the conditions. We prove the converse implications.

(i) From Proposition 1, we know that f is voting by issues and well-defined, characterized by a winning coalition structure $(\mathcal{W}_H)_H$.

First, we show that for all $H \in \mathcal{H}$, either $\mathcal{W}_H = 2^N \setminus \{\emptyset\}$ or $\mathcal{W}_H = \{N\}$. Suppose by contradiction that this is not the case. Then let $H \in \mathcal{H}$, $W \in \mathcal{W}_H$ and $i \in W$, such that $W \neq N$ and $W \setminus \{i\} \neq \emptyset$ and $W \setminus \{i\} \notin \mathcal{W}_H$. Let $a \in H$ and $b \notin H$. Let $R^a \in \mathcal{R}$ be a preference such that $p(R^a) = a$ and for all $x \in H$ and all $y \notin H$, $x \; P^a \; y$. Similarly, let $R^b \in \mathcal{R}$ be such that $p\left(R^b\right) = b$ and for all $x \notin H$ and all $y \in H$, $x \; P^b \; y$. Let $R_N \in \mathcal{R}^N$ such that for all $j \in W$, $R_j := R^a$ and for all $j \in N \setminus W$, $R_j := R^b$. Next, let $R_i' := R^b$. By definition of voting by issues, this implies that $f(R_N) \in H$ but $f\left(R_i', R_{N\setminus\{i\}}\right) \notin H$. Therefore the replacement of R_i by R_i' hurts all agents in $W \setminus \{i\}$ and benefits all agents in $N \setminus W$, which contradicts replacement-domination. Thus, for all $H \in \mathcal{H}$, either $\mathcal{W}_H = 2^N \setminus \{\emptyset\}$ or $\mathcal{W}_H = \{N\}$.

Next, let $\mathcal{H}_f := \left\{H \in \mathcal{H} : \mathcal{W}_H = 2^N \setminus \{\emptyset\}\right\}$. We now show that $\bigcap_{H \in \mathcal{H}_f} H \neq \emptyset$. Suppose by contradiction that this set is empty. Then there is $H_1, ..., H_l \in \mathcal{H}_f$, where $l \geqslant 2$, such that $\bigcap_{k=1}^{l-1} H_k \neq \emptyset$ and $\bigcap_{k=1}^{l} H_k = \emptyset$. Let Let $R_a, R_b \in \mathcal{R}$ be such that $p(R_a) \in \bigcap_{k=1}^{l-1} H_k$ and $p(R_b) \in H_l$. Let R be the profile $(R_a, R_b, ..., R_b)$ where R_b is repeated $|N| - 1$ times. Since all the H_k for $k = 1, ..., l$ are in $\mathcal{H}_f$, we have $f(R) \in \bigcap_{k=1}^{l} H_k = \emptyset$, which is a contradiction. Therefore $\bigcap_{H \in \mathcal{H}_f} H \neq \emptyset$.

Last, we show that this intersection has exactly one element $\widehat{a}$. Suppose that $a \neq b \in \bigcap_{H \in \mathcal{H}_f} H$. Let $H \in \mathcal{H}$ such that $a \in H$ and $b \notin H$. If $H \in \mathcal{H}_f$, then $b \in \bigcap_{H \in \mathcal{H}_f} H$ is contradicted. If instead $H \notin \mathcal{H}_f$, then by (1), $H^c \in \mathcal{H}_f$ and $a \in \bigcap_{H \in \mathcal{H}_f} H$ is contradicted. Therefore there is $\widehat{a} \in A$ such that $\widehat{a} = \bigcap_{H \in \mathcal{H}_f} H$. Moreover, for all $R_N \in \mathcal{R}^N$, we have $f(R_N) \in Co(p(R_i) : i \in N)$ and $f(R_N) \in [f(R_i), \widehat{a}]$ for all $i \in N$. Therefore f is a unanimity rule with parameter $\widehat{a}$.

(ii) First, we show that for each $H \in \mathcal{H}$, each population $N \subseteq \mathcal{N}$ with $|N| \geq 3$, each $W \subsetneq N$ such that $|W| \geq 2$, and each $i \in W$, we have $W \in \mathcal{W}_{H,N} \implies W \setminus \{i\} \in \mathcal{W}_{H,N\setminus\{i\}}$. Let H, N, W and i satisfy these conditions and suppose that $W \in \mathcal{W}_{H,N}$. Let $a \in H$ and $b \notin H$. Let $R^a \in \mathcal{R}$ be a preference such that $p(R^a) = a$ and for all $x \in H$ and all $y \notin H$, $x \; P^a \; y$. Similarly, let $R^b \in \mathcal{R}$ be such that $p\left(R^b\right) = b$ and for all $x \notin H$ and all $y \in H$, $x \; P^b \; y$. Let $R_N \in \mathcal{R}^N$ such that for all $j \in W$, $R_j := R^a$ and for all $j \in N \setminus W$, $R_j := R^b$. By definition of voting by issues, since $W \in \mathcal{W}_{H,N}$, we have $f(R_N) \in H$. Next, consider the profile $R_{N\setminus\{i\}}$, where agent i has left. By population-monotonicity, we have $f\left(R_{N\setminus\{i\}}\right) \in H$. By definition of voting by issues, this implies that $W \setminus \{i\} \in \mathcal{W}_{H,N\setminus\{i\}}$.

Second, we show that for each $H \in \mathcal{H}$, each population $N \subsetneq \mathcal{N}$ with $|N| \geq 2$, each $W \subsetneq N$ such that $|W| \geq 1$, and each $i \notin N$, we have $W \in \mathcal{W}_{H,N} \implies W \in \mathcal{W}_{H,N\cup\{i\}}$. Let H, N, W and i satisfy these conditions and suppose that $W \in \mathcal{W}_{H,N}$. Let $a \in H$ and $b \notin H$. Let $R^a \in \mathcal{R}$ be a preference such that $p(R^a) = a$ and for all $x \in H$ and all $y \notin H$, $x \; P^a \; y$. Similarly, let $R^b \in \mathcal{R}$ be such that $p\left(R^b\right) = b$ and for all $x \notin H$ and all $y \in H$, $x \; P^b \; y$. Let $R_N \in \mathcal{R}^N$ such that for all $j \in W$, $R_j := R^a$ and for all $j \in N \setminus W$, $R_j := R^b$. By definition of voting by issues, we have $f(R_N) \in H$. Next, consider the profile (R'_i, R_N), where agent i is added, with the preference $R'_i := R^b$. By population-monotonicity, we have $f(R'_i, R_N) \in H$. By definition of voting by issues, this implies that $W \in \mathcal{W}_{H,N\cup\{i\}}$.

The implication proved in last paragraph, together with the implication for all H, N, W, $[W \in \mathcal{W}_{N,H}$ and $W \subseteq W'] \Rightarrow W' \in \mathcal{W}_{N,H}$ implies that for all H, either $\mathcal{W}_{H,N} = 2^N \setminus \{\emptyset\}$ for all N or $\mathcal{W}_{H,N} = \{N\}$ for all N.

Last, we define the family $\mathcal{H}_f := \left\{H \in \mathcal{H} : \mathcal{W}_H = 2^N \setminus \{\emptyset\}\right\}$, which as we saw does not depend on N, and $\widehat{a} = \bigcap_{H\in\mathcal{H}_f} H$ exactly like in (i). Thus for all N, the rule f_N is the unanimity rule with parameter $\widehat{a}$.■

For all $(a, b, c) \in A$, if the set $[a, b] \cap [b, c] \cap [a, c]$ is not empty, it necessarily contains a single element called the median of (a, b, c), denoted by $med\,(a, b, c)$. An alternative $a \in A$ is called a *median alternative* for $\mathcal{H}$ if and only if for all $b, c \in A$ the triple (a, b, c) has a median. Nehring and Puppe (2005, 2010) obtained the following result.

Proposition 3 (Nehring and Puppe 2005, 2010) *For all $\widehat{a} \in A$, and all N such that $|N| \geq 2$, the unanimity rule $f^{\widehat{a}}$ is well-defined on $\mathcal{R}^N$, if and only if $\widehat{a}$ is a median alternative for $\mathcal{H}$.*

It is easy to see that the direct implication is true. If $f^{\widehat{a}}$ is well-defined at some profile with peaks located at b and c, it is necessarily a median of $\widehat{a}$, b and c. The fact that when $\widehat{a}$ is a median point, the rule $f^{\widehat{a}}$ is well-defined is less clear. Nehring and Puppe (2005, 2010) provide an abstract proof of this fact. For completeness, we provide here a constructive proof of this result, which we believe is more transparent.

Proof. First, suppose that the rule $f^{\widehat{a}}$ is well-defined on $\mathcal{R}^N$, with $|N| \geq 2$. Let b, c be arbitrary alternatives in A. Consider a profile R_N such that all agents have their peaks in $\{b, c\}$ and not all agents have the same peak. Then by definition, $f^{\widehat{a}}(R_N)$ is an element of $[\widehat{a}, b] \cap [\widehat{a}, c] \cap [b, c]$. Therefore this set is not empty. Since this is true for all $b, c \in A$, then $\widehat{a}$ is a median alternative.

Conversely, suppose that $\widehat{a}$ is a median alternative. Let $|N| \geq 2$. Without loss of generality, let $1, ..., n$ be the agents in N. Define the sequence $x_1 := p_1$ and for all $i \in \{2, ..., n\}$, let $x_i := med(\widehat{a}, x_{i-1}, p_i)$. Since $\widehat{a}$ is a median point, the sequence is well-defined. By construction, $x_n \in Co(\{x_{n-1}, p_n\}) \subseteq Co(\{x_{n-2}, p_{n-1}, p_n\}) \subseteq ... \subseteq Co(\{x_1, ..., p_k\})$. Again by construction, we know that $x_n \in [\widehat{a}, p_n]$; $x_n \in [\widehat{a}, x_{n-1}] \subseteq [\widehat{a}, p_{n-1}]$; ... and $x_n \in [\widehat{a}, x_{n-1}] \subseteq ... \subseteq [\widehat{a}, x_2] \subseteq [\widehat{a}, p_1]$. Therefore x_n is an element in $Co(\{x_1, ..., p_k\})$ that is in the interval $[\widehat{a}, p_i]$ for all $i = 1, ..., n$. Therefore $f^{\widehat{a}}(R_N)$ is well-defined and $f^{\widehat{a}}(R_N) = x_n$. Since this construction is feasible for all $R_N \in \mathcal{R}^N$, the rule $f^{\widehat{a}}$ is well-defined.■

An attribute space is called *quasi-median* (Nehring 2004) if it contains at least one median alternative and it is called *median* if all alternatives in A are median. From these definitions and Proposition 3, it follows that the attribute-structures that admit a unanimity rule are exactly the quasi-median spaces. From this observation and Proposition 2, we obtain the following characterization.

Corollary 2 *The following three statements are equivalent.*

(*i*) *The attribute space $\mathcal{H}$ admits a rule that satisfies voter-sovereignty, strategy-proofness and replacement-domination.*

(*ii*) *The attribute space $\mathcal{H}$ admits a variable population rule that satisfies voter-sovereignty, strategy-proofness and population-monotonicity.*

(*iii*) *$\mathcal{H}$ is quasi-median.*

Nehring and Puppe (2005) provide several interesting characterizations of quasi-median spaces. We end this section by two examples of such spaces, whose associated domains are not discrete counterparts of the domains considered by Klaus (1999, 2001).

Example 4 *A subset of at most* L *out of* K *public projects, with* $1 \leq L < K$*, has to be selected. An alternative specifies which of the projects will be carried out. Attributes are the sets of alternatives of the form "yes to project* k*" and "no to project* k*" for all* $k = 1, ..., K$*. This attribute space has exactly* $K + 1$ *median alternatives. These are all the alternatives of the form "only project* k *is carried out" for all* $k = 1, ..., K$ *and "no project is carried out". Consequently, exactly* $K + 1$ *rules satisfy the conditions of Proposition 2. In any rule in this class, each of the projects, except perhaps one, require unanimous support for approval. If a project does not require unanimous support for approval, a single vote in its favour suffices to get it approved.*

Example 5 *Let* $A := \{a, b, c, d, e\}$*. The attributes are the sets* $\{a, b, c\}$*,* $\{b, c, d\}$*,* $\{c, d, e\}$*,* $\{a, e\}$*,* $\{a, b\}$ *and* $\{d, e\}$*. This attribute space has two median points* b *and* d*. Therefore the two rules that satisfy the conditions of Proposition 2 are the unanimity rules with parameter either* b *or* d*.*

5 Pareto-efficiency and Solidarity

In this section, we study the compatibility of Pareto-efficiency and either solidarity conditions. Ching and Thomson (1997), Vohra (1999) and Klaus (1999, 2001) have shown that Pareto-efficiency is compatible with either replacement-domination or population-monotonicity on domains of single-peaked preferences defined on trees.[7] Both the set of alternatives and the preference domain these authors consider are continua. Their discrete counterpart in our setting is the class of domains of single-peaked preferences on discrete trees, such as the ones presented in Example 1.

In the light of the results by Ching and Thomson (1997), Vohra (1999) and Klaus (1999, 2001), it is natural to ask the following question: Are these properties compatible in other attribute-based preferences domains? Unfortunately, we find that the answer to this question is negative: tree structures are the only attribute-based domains on which

[7] Other properties are studied on these domains by Demange (1982) and Danilov (1994) and Schummer and Vohra (2002).

Pareto-efficiency and solidarity properties are compatible.

To establish this result, we first provide an abstract characterization of the attribute spaces that generate domains in the class presented in Example 1. It is well known and easily verified that the attribute spaces $\mathcal{H}$ constructed in Example 1 are median and in addition satisfy the following condition.

Condition (T) : *For all* $H, H' \in \mathcal{H}$, *at least one of the sets* $H \cap H'$, $H \cap H'^c$, $H^c \cap H'$, $H^c \cap H'^c$ *is empty.*

Conversely, we establish that these two conditions, the median condition and (T), characterize tree structures, a result we believe is of independent interest.[8] From any median attribute space $\mathcal{H}$ that satisfies (T), one can recover a discrete graph-theoretic tree, that is a graph (a finite set of vertices and edges) that is connected (any two vertices are connected through some path) and has no cycles.

Proposition 4 *Let* $\mathcal{H}$ *be a median attribute space that satisfies* (T). *Then the graph on* $A = \bigcup_{H \in \mathcal{H}} H$ *whose edges are the pairs* (a, b) *such that there exists a unique attribute* $H_a \in \mathcal{H}$ *such that* $a \in H_a$ *and* $b \in H_a^c$ *is a tree, i.e. it is connected and has no cycles.*

Proof. Throughout the proof, let $\mathcal{H}$ be a tree attribute space.

Step 1: The graph defined in the Proposition is connected.

For any two alternatives a, b let $a \sim b$ if $[a, b] = \{a, b\}$. We will show that (a, b) is an edge if and only if $a \sim b$. Let a, b be alternatives such that $a \sim b$. Let H and H' be any two attributes such that $a \in H \cap H'$ and $b \in H^c \cap H'^c$. We need to prove that $H = H'$. By contradiction, suppose that this is not the case. For example $H' \subsetneq H$. Then there exists $c \in H' \setminus H$. Then by condition (T) it must be that either $H \subset H'$. Moreover b and c are in H^c and b is in H'^c. Then the median of a, b and c is an element of $[a, b]$, but it is neither a

[8] Buneman (1971) and Bandelt and Dress (1986) provide related constructions, without assuming a median space. Their construction, however, allows introducing "latent alternatives", i.e. additional alternatives outisde of the set A in order to construct a tree consistent with an attribute structure that satisfies (T). Our result shows that when the space is median, latent alternatives are not needed. In addition, our construction relies on elementary arguments.

nor b, since it is an element of H^c (ruling out a) and it is also an element of H (ruling out b). This contradicts that $[a,b]=\{a,b\}$. Therefore necessarily $H=H'$, which proves that (a,b) is an edge.

It follows immediately from the previous paragraph that the graph defined in the Proposition is connected.

<u>Step 2:</u> *Monotonicity of attributes along a path.*

Let a and b be arbitrary alternatives. Consider a path $a_0,...,a_n$ with $a_0=a$ and $a_n=b$ connecting a and b. For all i, since (a_i,a_{i+1}) is an edge, there exists exactly one attribute, let it be denoted H_i, such that $a_i\in H_i$ and $a_{i+1}\notin H_i$. We will show that for all i, we have $H_{i-i}\subset H_i$. Since (a_{i-1},a_i) form an edge, and a_{i-1} and a_i are already separated by H_{i-1} and H_{i-1}^c, then they are not separated by H_i and H_i^c. Since $a_i\in H_i$, then $a_{i-1}\in H_i$.

Similarly, since (a_i,a_{i+1}) form an edge, and a_i and a_{i+1} are already separated by H_i and H_i^c, then they are not separated by H_{i-1} and H_{i-1}^c. Since $a_i\in H_{i-1}^c$, then $a_{i+1}\in H_{i-1}^c$.

Thus $H_i\cap H_{i-1}$ contains a_{i-1}, $H_i\cap H_{i-1}^c$ contains a_i and $H_i^c\cap H_{i-1}^c$ contains a_{i+1}, therefore (T) implies that $H_{i-1}\subseteq H_i$. The inclusion is strict because $a_i\in H_i\cap H_{i-1}^c$.

<u>Step 3:</u> *The graph does not contain any cycles.* Suppose by contradiction that there exists a path $a_0,...,a_n$ with $a_n=a_0$. By Step 2, we find that $H_0\subset ...\subset H_n\subset H_0$, a contradiction. We conclude that the graph does not contain any cycles, i.e. it is a tree in a graph theoretic sense.■

From now on, a **tree attribute space** is a median attribute space that satisfies (T). We are now ready to show that Pareto-efficiency and solidarity are only compatible on tree attribute spaces. Note that we do not assume strategy-proofness, thus we cannot restrict attention to voting by issues from the outset, as we did in the previous section.

Proposition 5 *Let $\mathcal{H}$ be an attribute space. (i) Let $|N|\geq 3$. Let f be a rule. If f satisfies Pareto-efficiency and replacement-domination, then $\mathcal{H}$ is a tree attribute space. (ii) Let $|\mathcal{N}|\geq 3$. Let f be a variable-population rule. If f satisfies Pareto-efficiency and population-monotonicity, then $\mathcal{H}$ is a tree attribute space.*

The idea of proof is the following. We associate to any rule satisfying the conditions a two agents rule, which turns out to be a Pareto-efficient unanimity rule (Steps 1 and 2). Then in Steps 3 and 4, we show that this implies that $\mathcal{H}$ is a tree.

Proof. Throughout the proof, let $\mathcal{H}$ be an attribute space.

(i). Let $|N| \geq 3$. Let f be a rule that satisfies Pareto-efficiency and replacement-domination.

Let $M := \{1,2\}$ and $g : \mathcal{R}^M \rightarrow A$, such that for all $(R_1, R_2) \in \mathcal{R}^M$, we have $g(R_1, R_2) := f(R'_N)$, where $R'_N \in \mathcal{R}^N$ is such that $R'_1 = R_1$ and $R'_i = R_2$, for all $i \geq 2$.

<u>Step 1:</u> *The rule g satisfies anonymity, strategy-proofness and Pareto-efficiency.*

First, it is obvious that the rule g satisfies Pareto-efficiency.

Next, we prove that for all (R_1, R_2), we have $g(R_1, R_2)\ I_i\ g(R_2, R_1)$ for all $i \in \{1,2\}$. Let $(R_1, R_2) \in \mathcal{R}^M$. Then $g(R_1, R_2) = f(R'_N)$, where $R'_N \in \mathcal{R}^N$ is such that $R'_1 = R_1$ and $R'_i = R_2$, for all $i \geq 2$.. Consider the transformation of $R'_N = (R_1, R_2, ..., R_2)$ in $R''_N = (R_2, R_1, ..., R_1)$ where for each agent $i \in \{3, ..., n\}$, the preference $R'_i = R_2$ is replaced by the preference $R''_i = R_1$ in decreasing index order. Then the preference $R'_1 = R_1$ is replaced by the preference $R''_1 = R_2$ and last the preference $R'_2 = R_2$ is replaced by the preference $R''_2 = R_1$. At each step in the transformation, the Pareto-set for the preference subprofile of the agents whose preferences are kept fixed is the same at the Pareto-set for the entire profiles (of any profile along the path). By Pareto-efficiency and replacement-domination, this implies that the preferences R_1 and R_2 remain indifferent between the images by f along the path. Therefore $g(R_1, R_2) = f(R_1, R_2, ..., R_2)\ I_i\ f(R_2, R_1, ..., R_1) = g(R_2, R_1)$, for all $i \in \{1,2\}$.

Last, we prove that g satisfies strategy-proofness. By anonymity, it suffices to prove that for all $(R_1, R_2) \in \mathcal{R}^M$ and all $R'_2 \in \mathcal{R}$, we have $g(R_1, R_2)\ R_2\ g(R_1, R'_2)$. We have $g(R_1, R_2) = f(R_1, R_2, ..., R_2)$. Consider the sequence of transformations where the preference $R_i = R_2$ is replace by the preference $R'_i = R'_2$, for each $i \in \{2, ..., n\}$ in increasing order. When replacing the preference of an agent of indices in $\{3, ..., n-1\}$, the image by f remains unchanged by the same argument as in the last paragraph. In the first replacement, agent 2 cannot strictly benefit, since this would imply agent 3 strictly benefits, which would imply that $f\left(R'_2, R_{N \setminus \{2\}}\right)$ Pareto-dominates $f(R_1, R_2, ..., R_2)$ for (R_1, R_2), i.e. for

R_N. Similarly, in the last replacement, agent n cannot strictly benefit. If this where the case, since all agents $1, ..., n-1$ are affected in the same direction, these agents must weakly lose (at least one of them strictly), otherwise $f(R_1, R_2', ..., R_2', R_2)$ is not Pareto-efficient for $(R_1, R_2', ..., R_2', R_2)$. But this in turns contradicts the Pareto-efficiency of $f(R_1, R_2', ..., R_2')$ for $(R_1, R_2', ..., R_2')$. Therefore agent n weakly loses. Last $f(R_1, R_2', ..., R_2') = g(R_1, R_2')$. In summary, we obtain that $g(R_1, R_2)\ R_2\ g(R_1, R_2')$, i.e. g satisfies strategy-proofness.

Last, since g satisfies strategy-proofness and voter-sovereignty, it only depends on the agents' peaks. Therefore, the property established in the second paragraph of Step 1 implies that g satisfies anonymity.

Step 2: The rule g is a unanimity rule. Let $\widehat{a}$ be the unique alternative such that for all (R_1, R_2), $g(R_1, R_2) = med(\widehat{a}, p(R_1), p(R_2))$.

Step 3: The attribute space $\mathcal{H}$ satisfies condition (T).[9]

Suppose by contradiction that $\mathcal{H}$ violates condition (T). If this is the case, then there are alternatives $a_1, ..., a_4 \in A$ and attributes H and H', such that $a_1 \in H \cap H'$, $a_2 \in H \cap H'^c$, $a_3 \in H^c \cap H'^c$, and $a_4 \in H^c \cap H'$. Then $\widehat{a}$ is an element of exactly one of these four sets. Suppose for example that $\widehat{a} \in H \cap H'$. Consider now a profile (R_1, R_2) such that $p(R_1) = a_2$ and $p(R_2) = a_4$, and moreover for all $a \in H \cap H'$, we have $a_3\ P_1\ a$ and $a_3\ P_2\ a$. Therefore none of the alternatives in $H \cap H'$ is Pareto-efficient. However, $g(R_1, R_2) = med(\widehat{a}, p(R_1), p(R_2))$, which is an element of $H \cap H'$, contradicting Pareto-efficiency. Therefore the attribute space $\mathcal{H}$ satisfies condition (T).

Step 4: The space $\mathcal{H}$ is a median space.

Suppose, by contradiction that $\mathcal{H}$ is not a median space. Then, we know from Nehring and Puppe (2007b, Proposition 4.1) that there is a family of attributes $H_1, ..., H_k$, with $k \geq 3$, such that $\bigcap_{l=1}^{k} H_l = \emptyset$ and for each $h \in \{1, ..., k\}$, $\bigcap_{l \neq h} H_l \neq \emptyset$.For each $h = 1, ..., k$, let $a_j \in \bigcap_{l \neq h} H_l$. For each pair l, l', $l \neq l'$, the sets $H_l \cap H_{l'}$, $H_l^c \cap H_{l'}$ and $H_l \cap H_{l'}^c$ are non-empty. Therefore, by Step 3, it must be that, for each pair l, l', $l \neq l'$, $H_l^c \cap H_{l'}^c = \emptyset$.

[9] Steps 3 and 4 in the proof can be deduced from the main result in Nehring and Puppe (2007a), more precisely from Claim (a) in their Theorem. For the sake of completeness, we provide a direct proof that exploits the special structure of this model to avoid the complexities of their analysis.

Therefore any alternative $a \in A$ is an element of exactly $k-1$ sets H_l and one set $H_{l'}$. Without loss of generality, let's suppose that $\widehat{a} \in H_1 \cap ... \cap H_{k-1} \cap H_k^c$. Consider now a profile (R_1, R_2) such that $p(R_1) = a_1$ and $p(R_2) = a_2$. Then $g(R_1, R_2) = med(\widehat{a}, p(R_1), p(R_2))$, which is an element of $[p(R_1), p(R_2)] \subseteq \bigcap_{l=3}^{k} H_l$, of $[\widehat{a}, p(R_1)] \subseteq H_2$ and of $[\widehat{a}, p(R_2)] \subseteq H_1$. Since $\bigcap_{l=1}^{k} H_l = \emptyset$, this is a contradiction. Therefore $\mathcal{H}$ is a median space.

Therefore $\mathcal{H}$ is a tree attribute space.

(ii). The proof follows exactly the same steps, therefore we only provide a sketch. Let $|\mathcal{N}| \geq 3$. Let f be a variable-population rule that satisfies Pareto-efficiency and population-monotonicity. Let g be the restriction of f to $\mathcal{R}^M$ with $M := \{1, 2\}$. Each preference replacement in the proof of (i) is achieved in two steps by first withdrawing the agent whose preference is replaced and then adding him back with the new preference. At the end, all agents with labels other than 1 and 2 are removed. The remaining steps are identical to those in (i).■

To end this section, we provide the discrete counterparts of the characterizations obtained by Ching and Thomson (1997), Vohra (1999) and Klaus (1999, 2001) on trees.[10] Our proof differs from theirs, as it relies on the theory of voting by issues and does not require an infinite set of alternatives.

Proposition 6 *Let $\mathcal{H}$ be a tree attribute space. (i) Let $|N| \geq 3$. Let f be a rule. Then f satisfies Pareto-efficiency and replacement-domination if and only if f is a unanimity rule. (ii) Let $|\mathcal{N}| \geq 3$. Let f be a variable population rule. Then f satisfies Pareto-efficiency and population-monotonicity if and only if f is a unanimity rule.*

Proof. It is clear that a unanimity rules satisfies the conditions. We prove the converse implication. Throughout the proof, let $\mathcal{H}$ be a tree.

(i). Let $|N| \geq 3$. Let f be a rule that satisfies Pareto-efficiency and replacement-domination. Let $M := \{1, 2\}$ and $g : \mathcal{R}^M \to A$, such that for all $(R_1, R_2) \in \mathcal{R}^M$, we have $g(R_1, R_2) := f(R'_N)$, where $R'_N \in \mathcal{R}^N$ is such that $R'_1 = R_1$ and $R'_i = R_2$, for all

[10] The proofs in Vohra (1999) and Klaus (1999, 2001) require infinitely many alternatives. This assumption is implicitly used, for example, in Vohra's Lemma 3.

$i \geq 2$. From Step 1 in the proof of Proposition 5, we know that g satisfies Pareto-efficiency and strategy-proofness and that it is a unanimity rule with status quo $\widehat{a}$. Let $f^{\widehat{a}}$ be the unanimity rule with status quo $\widehat{a}$ on $\mathcal{R}^N$. We will prove here that $f = f^{\widehat{a}}$.

Let $R_N \in \mathcal{R}^N$. Let $a^* := f^{\widehat{a}}(R_N)$ and $b^* := f(R_N)$. Since a^* and b^* are in the Pareto-set for R_N, there are two distinct agents $i, j \in N$ such that $a^* \in [p(R_i), p(R_j)]$ and $b^* \in [p(R_i), p(R_j)]$. We will transform the profile R_N into the profile R'_N such that $R'_1 = R_i$ and $R'_2 = ... = R'_n = R_j$.

Step 1. First, let $L := N \setminus \{i, j\}$. Let $R'_l := R_j$. Replace one by one the preference R_l of each agent $l \in L$ by $R'_l := R_2$. Once this is done, we obtain the profile R'_N such that $R'_i = R_i$ and $R'_l = R_j$ for all $l \neq i$. If $i = 1$, the transformation ends here. If not, then $1 \neq i$, so that $R'_1 = R_j$. In this case, let $R''_1 := R_i$ and $R''_i := R_j$. Replace first the preference $R'_1 = R_j$ by $R''_1 = R_i$ and then the preference $R'_i = R_1$ by the preference $R''_i = R_2$. The transformation ends and we obtain the profile R''_N such that $R''_1 = R_i$ and $R''_2 = ... = R''_n = R_j$. At each elementary step in this transformation, there are always at least two agents whose preferences are kept fixed and are R_i and R_j. We will show that the image by f remains b^* along the path. Along the path, the set of preferences represented in the profile decreases or remains constant at each step. By Pareto-efficiency and replacement-domination, this implies that the change weakly benefits preferences R_i and R_j. Since this is true at each step, and by transitivity, it must be that $f(R''_N)\ R_i\ f(R_N)$ and $f(R''_N)\ R_j\ f(R_N)$. But since $f(R_N) = b^* \in [p(R_i), p(R_j)]$, this implies that $f(R''_N) = b^*$.

But $f(R''_N) = g(R''_1, R''_2) = g(R_i, R_j) = med(\widehat{a}, p(R_i), p(R_j))$. By definition of a^*, we know that $a^* \in [a^*, p(R_j)] \cap [a^*, p(R_i)]$. By definition of i and j, we know that $a^* \in [p(R_i), p(R_j)]$. Therefore $med(\widehat{a}, p(R_i), p(R_j)) = a^*$, i.e. $f(R_N) = f^{\widehat{a}}(R_N)$. Since this is true for all $R_N \in \mathcal{R}^N$, we conclude that f is the unanimity rule with status quo $\widehat{a}$.

(ii) The proof follows exactly the same steps, therefore we only provide a sketch. In this case the rule g is the restriction of the variable-population rule f to $\mathcal{R}^{\{1,2\}}$. Each preference replacement in the proof of (i) is achieved in two steps by first withdrawing the agent whose preference is replaced and then adding him back with the new preference. Last all agents with labels other than 1 and 2 are removed.■

Vohra (1999) and Klaus (2001) both point out that their characterization on trees is valid even in the subdomain of symmetric preferences. This is not the case in the discrete setting, as shown in the following example.

Example 6 *Let* $A := \{0, 1, x\}$, *with* $x \geq 2$. *Consider the domain of preferences represented by the utility functions* $u_0(a) = -|a|$, $u_1(a) = -|a-1|$ *and* $u_x(a) = -|a-x|$ *for all* $a \in A$. *Then the rule, which for any preference profile maximizes the linear ordering* $x \succ 0 \succ 1$ *on the set of Pareto-efficient alternatives satisfies the conditions of Proposition 6 and is not the restriction to this domain of any voting by issues rule.*

We end this section by discussing the relation between the results by Gordon (2007a) and the ones in this paper. Gordon (2007a) considers a general variable population public choice problem, without assumptions on the preference domain, other than symmetry (all agents have the same set of possible preferences) and that the set of alternatives does not depend on the population. In this very general framework, Gordon (2007a) does not provide a characterization, but establishes that under Pareto-efficiency, both population-monotonicity the one hand, and replacement-domination together with replication-indifference (the decision change that follows the cloning replication of the entire population leaves all agents indifferent) have the following strong implications. First, either combination implies strategy-proofness and even the stronger requirement of group-strategy-proofness (no group of agents can jointly benefit from misrepresenting the preferences of its members). Second, they imply anonymity (up to Pareto-indifference). Third, they imply that there is a "status-quo alternative" that is always Pareto-dominated by the choice of the rule. These results are obtained under the assumption that the population is variable and that the set $\mathcal{N}$ is infinite.[11]

In contrast, we do not assume that such an infinite variable population is available and focus on a more restricted class of models. Our results confirm that, in this more specific

[11] In a similarly general framework, Bu (2013) establishes a general equivalence between *false-name-proofness,* which requires non-manipulability via the creation of fictitious identities, and strategy-proofness, anonymity and population-monotonicity. It would be interesting to study the implications of false-name-proofness in the class of attribute-based preference domains.

context, the above implications remain true as long as the fixed population has at least three agents in the fixed population model, or as long as there are at least three potential agents in the variable population model, and without the assumption of replication-indifference. The status quo alternative of a unanimity rule on a tree is its target $\widehat{a}$. Moreover, both anonymity and the existence of a status quo alternative are also shown to be implications of the weaker conditions of voter-sovereignty, strategy-proofness and solidarity.

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