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## IDEOLOGY AND EXISTENCE OF 50%-MAJORITY EQUILIBRIA IN MULTIDIMENSIONAL SPATIAL VOTING MODELS

Hervé Crès and M. Utku Ünver

### ABSTRACT

When aggregating individual preferences through the majority rule in an  $n$ -dimensional spatial voting model, the ‘worst-case’ scenario is a social choice configuration where no political equilibrium exists unless a super-majority rate as high as  $1 - 1/(n+1)$  is adopted. In this paper we assume that a lower  $d$ -dimensional ( $d < n$ ) linear map spans the possible candidates’ platforms. These  $d$  ‘ideological’ dimensions imply some linkages between the  $n$  political issues. We randomize over these linkages and show that there almost surely exists a 50%-majority equilibria in the above worst-case scenario, when  $n$  grows to infinity. Moreover, the equilibrium is the *mean voter*.

KEY WORDS • ideology • mean voter theorem • spatial voting • super majority

### 1. Introduction

It has been well known since Plott (1967) that a 50%-majority stable political equilibrium typically does not exist in a multidimensional voting setup. A way to restore the existence of a stable outcome is to require a super-majority rule to overrun the status quo, thus giving rise to the concept of  $\rho$ -majority equilibrium, where  $\rho \in [1/2, 1]$  is the proportion of the voting population a challenger must rally to take over. It is widely admitted that the smaller the rate of super majority needed to secure existence of an equilibrium (i.e. the less conservative the voting rule), the better.

There is a wide variety of literature on the level of super majority required for existence, both in deterministic or probabilistic setups (see, e.g. Ferejohn and Grether, 1974; Caplin and Nalebuff, 1988, 1991; Balasko and Crès, 1997). In a standard social choice setup where voters, endowed with continuous and convex preferences, have to choose among political alternatives in a non-empty, compact and convex subset of  $\mathbb{R}^n$ , Greenberg (1979) shows that a necessary and sufficient condition for the existence of a  $\rho$ -majority equilibrium is  $\rho \geq n/(n+1)$ .

To show that this bound is tight, Greenberg (1979) constructs a voting configuration where no incumbent is stable with respect to a super-majority rule with

rate smaller than  $n/(n+1)$ . The configuration is as follows. Take  $n+1$  independent points in  $\mathbb{R}^n$  and interpret them as the ideal political choices of  $n+1$  voters endowed with Euclidean preferences. Denote by  $S_n$  the  $n$ -dimensional simplex generated by the voters' ideal points. Fix an incumbent  $x \notin S_n$ ; then  $s(x) = \operatorname{argmin}\{\|x - s\|, s \in S_n\}$  is unanimously preferred to  $x$ , hence  $x$  is not stable under any  $\rho$ -majority rule with  $\rho < 1$ . Now, fix an incumbent  $x \in S_n$ ; then it is always possible to find a challenger preferred by  $n$  out of the  $n+1$  voters: indeed, denote by  $\bar{S}_n$  the  $(n-1)$ -dimensional simplex generated by the ideal points of these  $n$  voters ( $\bar{S}_n$  is a face of  $S_n$ ), then one can reconduct the previous argument, and show that  $\bar{s}(x) = \operatorname{argmin}\{\|x - \bar{s}\|, \bar{s} \in \bar{S}_n\}$  is preferred to  $x$  by all of these  $n$  voters.

This example is a 'worst-case' scenario. One easily sees that if the voters' ideal points are taken in a lower-dimensional subspace, then the upper bound decreases, but the gain remains small though and one gets the existence of political equilibria for not too conservative voting rules only when the number,  $n$ , of political issues is very low. This bound is:  $\rho = 1/2$  when  $n = 1$  (the so-called 'median voter theorem');  $\rho = 2/3$  when  $n = 2$ ; and for  $n \geq 3$ , then the required rate of super majority must be above  $3/4$  (and converges to the unanimity criterion when  $n$  goes to infinity), a level very rarely observed in practice. Indeed, constitutions or corporate charters build on super-majority rates which are very rarely above 70%,<sup>1</sup> although the number of political issues at stake in electoral processes is obviously often very large: it is not rare, when reading political platforms proposed by candidates in large elections, to denumerate several dozens of issues. Hence the question: why, if there are so many issues, do we observe such reasonable super-majority rates in practice?

A first answer might be that one should not believe in Greenberg's worst-case scenario. A second answer can be found in the Hinich–Ordeshook spatial voting model.<sup>2</sup> According to the latter, there are only a few political dimensions underlying the platforms proposed by the candidates. These few dimensions are claimed to be *ideological*. Ideologies imply linkages<sup>3</sup> between political issues and thus span a lower-dimensional linear space (dubbed the 'campaign space' in

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1. For decisions on issues which are delegated to the European Union, the rate was 72% in the Maastricht Treaty; it was decreased to a mix of 65% of the States and 55% of the population in the Constitutional Treaty.

2. This model was first proposed by Cahoon et al. (1976), Ordeshook (1976) and Hinich and Polard (1981) and then developed by Enelow and Hinich (1984) and Hinich and Munger (1997). See also Poole (2005) for an exhaustive survey of this approach.

3. These linkages formalize the fundamental insight of Converse (1964) according to which ideology (the Converseian 'belief system') interrelates and bundles the political issues: ideology is fundamentally the knowledge of what goes with what. As Converse (1964: 207) states it: ideology is 'a configuration of ideas and attitudes in which the elements are bound together by some form of constraint'.

the following) on which the original distribution of voters' ideal points is projected.

The assumption that political platforms are based on ideology stems from the belief that the cleavages between candidates separate along fewer, simpler lines than the  $n$ -dimensional policy space would imply. As Popkin (1994: 51) states it: 'Ideology is not the mark of sophistication and education, but of uncertainty and lack of ability to connect policies with benefits. . . Parties use ideologies to highlight critical differences between themselves, and to remind voters of their past successes'. This approach has some empirical relevance: Poole and Rosenthal (1991, 1996) show that in the USA, with the exception of the 32nd Congress, two dimensions are always capable of explaining more than 80% and up to 95% of the variation in the votes of elected officials on most issues. In the same vein, Poole and Rosenthal (1997) and McCarty et al. (1997) test the Hinich–Ordeshook spatial voting model on post-World War II Congressional roll-call voting and show that only two dimensions are required to account for most of the votes: the liberal–conservative continuum<sup>4</sup> and the dimension of conflict over race and civil rights. Equivalently, an analysis of French data yields, according to Rosenthal and Voeten (2004), that 'a stable two-dimensional spatial configuration explains deputies' vote choices in the Fourth Republic extraordinarily well'.

Another type of political debate often builds on more than two underlying dimensions: proxy fights in publicly traded corporations in the context of market failures.<sup>5</sup> The stakes are probably simpler to grasp than in ordinary political debates; moreover, shareholders usually have access to more measurable and precise information which is easier to aggregate. Yet corporate charters rarely choose rates of super majority beyond 65%.

The answer to the question 'why do we observe such a reasonable super-majority rate in practice?' seems to be: not only because the political competition articulates along fewer and simpler lines than the  $n$ -dimensional policy space would imply, but also one should not believe in Greenberg's worst-case scenario. The present paper goes one step further. Its main contribution is an aggregation theorem that links the two latter arguments: *one should not believe in Greenberg's worst-case scenario because the political competition happens in a lower-dimensional subspace* spanned by the underlying ideologies. Indeed, if we randomize on the linkages between issues imputed by ideologies, our main result (Theorem 1) states that *the Hinich–Ordeshook approach almost surely*

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4. A judgemental dimension that has been 'highly serviceable for simplifying and organizing events in most Western politics for the past century' (Converse, 1964: 214).

5. The heterogeneity of the shareholders' opinions can come from imperfect competition, the incompleteness of financial market structure or the presence of externalities. 'Ideological' dimensions in corporate politics can be the 'philosophy' with respect to debt versus equity, horizontal versus vertical integration, international diversification, expansion versus concentration, etc.

transforms Greenberg's worst-case scenario into the best-case scenario of a symmetric distribution of voting characteristics. As a consequence, we obtain (Theorem 2) a *mean-voter theorem*: the mean voter happens to almost always be the unique 50%-majority equilibrium, when the number of political issues grows large.

The paper is organized as follows: Section 2 introduces the definition of a political equilibrium in the classical Downsian spatial voting model; then Section 3 presents its Hinich–Ordeshook sophistication. Section 4 states and proves the aggregation theorems. Section 5 ends the paper with some concluding comments.

## 2. Voters, Platforms and the Majority Rule

The setup to model the electoral process and voting mechanism is the classical Downsian multidimensional spatial voting model (Downs, 1957). There are  $n$  measurable criteria of political activity, so that a *political platform* in the policy space can be represented as an  $n$ -dimensional vector:  $x \in \mathbb{R}^n$ . There are  $m$  voters in a set  $I$ . Each voter is endowed with an *Euclidean preference* relation on  $\mathbb{R}^n$ : agent  $i$ ,  $1 \leq i \leq m$ , has a *preferred choice* in the policy space,  $x_i \in \mathbb{R}^n$ , and his/her utility function over the space of political choices is decreasing with the Euclidean distance from his/her preferred choice:

$$\forall x \in \mathbb{R}^n \quad u_i(x) = - \|x_i - x\|.$$

A *society* is an  $m$ -tuple  $X = (x_i)_{i=1}^m$ .

We measure the stability of a political platform in a given society through the Simpson–Kramer approach. Given two political choices  $(a, b) \in \mathbb{R}^n \times \mathbb{R}^n$ ,  $\rho(b, a)$  measures the ratio of the electorate that strictly prefers  $b$  to  $a$ :

$$\rho(b, a) = \frac{\#\{i \in I \mid u_i(b) > u_i(a)\}}{m}.$$

The *score* of a political choice  $a \in \mathbb{R}^n$  is  $\rho(a) = \max_{b \in \mathbb{R}^n} \rho(b, a)$ . Clearly, the score of any political choice taken outside the closed convex hull,  $\langle X \rangle$ , of  $X$  will be 1: the challenger  $b$  that minimizes the distance between  $a$  and  $\langle X \rangle$  is unanimously preferred to  $a$ . Hence, looking for the ‘best’ status quo, i.e. the ones with lowest score, we can reduce our search to  $\langle X \rangle$ . The *min–max rate* of society  $X$  is  $\rho^* = \min_{a \in \mathbb{R}^n} \rho(a)$ . The *min–max set* of society  $X$  is  $S^*(X) = \{a \in \mathbb{R}^n \mid \rho(a) = \rho^*\}$ .

The *majority rule* with rate  $\rho \in [0, 1]$  states that candidate  $b$  is preferred by society  $X$  to (or defeats) candidate  $a$  if and only if  $\rho(b, a) > \rho$ . A candidate  $a$  is said to be  *$\rho$ -majority stable* in society  $X$  if and only if there is no alternative that defeats it, i.e. if and only if its score is not larger than  $\rho$ :  $\rho(a) \leq \rho$ . Such a candidate is a *political equilibrium* for the majority rule with rate  $\rho$ .

Since the seminal work of Plott (1967)<sup>6</sup> we know that 50%-majority stable equilibria generally do not exist when  $n \geq 2$ . To recover the existence of political equilibria, one has to impose a super-majority voting rule, i.e. a voting rule with rate  $\rho > 1/2$ . This paper deals with the existence of such political equilibrium based on super-majority voting. Along that search, political platforms in the min–max set have this appealing property that they are equilibria for the lowest rate of super majority, hence the *less conservative* voting rule.

The super-majority rate one has to impose in order to recover existence of equilibrium can be quite high: as extensively explained in the introduction, suppose that  $m = n + 1$  and the  $m$ -tuple  $X$  are the vertices of an  $n$ -dimensional simplex, then obviously any political choice in  $\langle X \rangle$  has a score of  $n/(n + 1)$  (it is enough to choose a challenger closer to any of the  $n + 1(n - 1)$ -dimensional faces of the simplex). Therefore, since any political choice outside  $\langle X \rangle$  has score 1, the min–max rate is  $n/(n + 1)$  and the min–max set is  $\langle X \rangle$ : one has to impose a super-majority rule of rate  $\rho \geq n/(n + 1)$  to get a stable political choice, and then all choices in  $\langle X \rangle$  are  $\rho$ -majority stable. Greenberg (1979) proves that the condition  $\rho \geq n/(n + 1)$ , to get the existence of a  $\rho$ -majority stable political equilibrium, is in fact *sufficient* as soon as the voter's preferences satisfy very mild properties of continuity and convexity. Hence, the latter case is a *worst-case scenario*, as far as getting not too conservative a min–max rate is concerned. The present paper can be read as an attempt at an argument against the relevance of this worst-case scenario.

Some convincing arguments along the same line are available in the social choice literature. One of the most important is given by Caplin and Nalebuff (1988, 1991). They give a dimension-free upper bound to the min–max rate under the conditions that preferred choices of agents are selected from a  $\sigma$ -concave distribution with compact and convex support. This upper bound (which, asymptotically, is lower than 64%) is given by the score of the *mean voter*, the voter whose preferred choice is the barycenter of all  $x_i$ . This literature can roughly be regarded as looking for multidimensional versions of the median voter theorem.

### 3. Ideology, Candidates and Political Campaigns

A central assumption of our model is that, although the number ( $n$ , here) of criteria for political activity can indeed be quite large, the political competition takes place in a subspace of lower dimension:  $d < n$ . In accordance with the Hinich–Ordeshook spatial voting model, this lower-dimensional space is considered to be the *ideological space*, assumed to be  $\mathbb{R}^d$  without loss of generality.

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6. More recent literature on the existence of the core in social choice problems is also related. See Banks (1995), McKelvey and Schofield (1986), Meirowitz (2004) and Saari (1997).

According to this approach, the ideologies are linked to the platforms by a linear map,  $L$ , from the ideological space to the policy space: a candidate,  $\pi^A \in \mathbb{R}^d$ , imputes a platform  $x^A \in \mathbb{R}^n$  such that  $x^A = x^0 + L\pi^A$ , where  $x^0$  is the platform of status quo policies. Finally, the  $d$ -dimensional affine subspace which is the image of  $\mathbb{R}^d$  by  $L$  translated by  $x^0$  is called the *campaign space*,  $C \subset \mathbb{R}^n$ , in the following. Before developing the strength of the model, let us illustrate through an example how the linear map  $L$  operates.

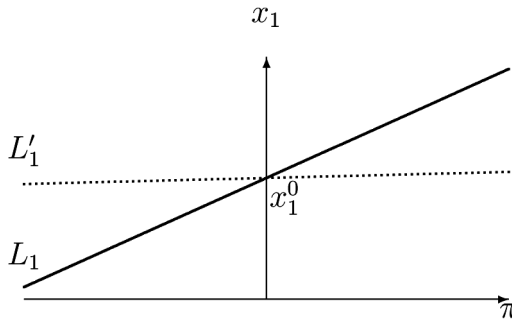
### 3.1 An Illustration

Issues of political activity are often precise and technical; consider two such classical issues such as (1) how much of the State's budget,  $x_1$ , must be allocated to buy helicopters, and (2) how much of the State's budget,  $x_2$ , must be allocated to create more slots in kindergartens. For the sake of simplicity, we limit the issues to these two, hence  $n = 2$ . The assumption is made that platforms proposed by candidates in this two-dimensional policy space can be explained through a (say) one-dimensional underlying linear subspace, e.g. the classical liberal-conservative (left-right) dimension; hence  $d = 1$ . Given a vector of status quo policies  $x^0$ , the sensitivity of  $x_j$ ,  $j = 1, 2$ , to the position  $\pi$  of the candidate in the ideological space is a fixed scalar  $l_j \in \mathbb{R}$ , therefore  $x_j$  is an *affine* function of  $\pi$ :

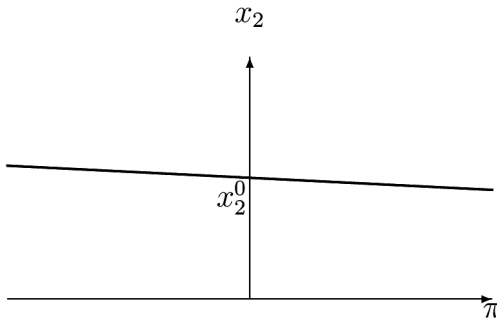
$$x_j = x_j^0 + l_j\pi, \quad j = 1, 2.$$

Figure 1 (Figure 2) plots the relation between ideology and helicopters (kindergartens, respectively). For example, the policy regarding kindergartens is almost not sensitive to ideology, and only slightly decreases ( $l_2$  is small and negative) with  $\pi$ : a leftist candidate wants to create slots in kindergartens because these structures are more used by low-class workers than by wealthy families; a rightist candidate uses slots' creation in kindergartens as an incentive to increase fertility. The policy regarding helicopters is more sensitive to ideology, and increasing ( $l_1$  is positive): rightist candidates are usually more hawkish, and spending on helicopters rises as ideology moves right, as shown by the plain line  $L_1$ .

The sensitivity of policies to ideology as depicted in Figures 1 and 2 implies a linkage between the two issues in the policy space: the induced campaign space,  $C$  (plain line on Figure 3), is going through the status quo  $x^0$  with slope  $l_2/l_1$ . The induced 'ideal candidate'  $\bar{x}_i$  of voter  $i$ , whose preferred platform is  $x_i$ , is obtained by orthogonal projection of  $x_i$  on the campaign space. Consequently, voter  $i$  votes for the candidate whose imputed platform is closest to his 'ideal candidate'  $\bar{x}_i$ . In the general  $(n, d)$  case, the Euclidean structure of the original voting configuration gives rise, through the orthogonal projection on  $C$ , to a social choice problem involving  $m$  voters with Euclidean preferences in  $\mathbb{R}^d$ . Hence, we are dealing with a  $d$ -dimensional spatial voting problem with  $m$



**Figure 1.** The relation between ideology and helicopters.



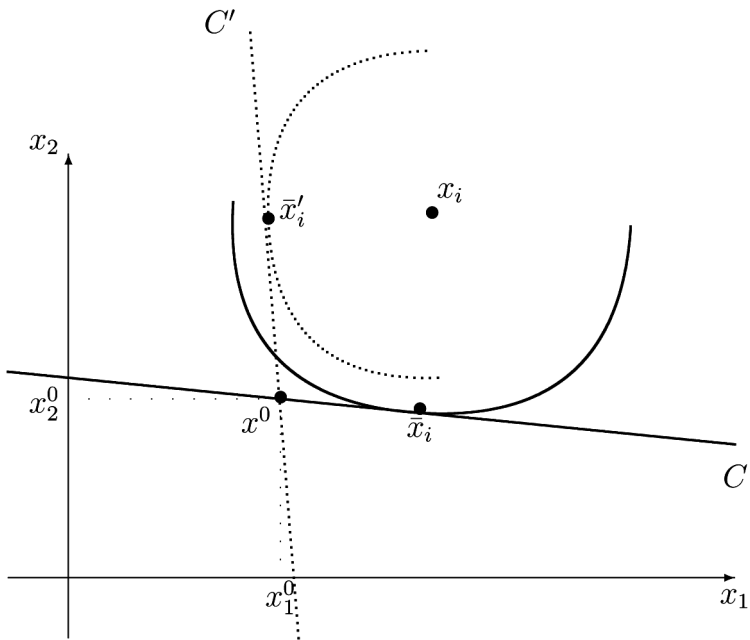
**Figure 2.** The relation between ideology and kindergartens.

voters and thus we are left with a combinatorial problem about  $m$ -tuples of points in  $\mathbb{R}^d$  rather than in  $\mathbb{R}^n$ .

If the Hinich–Ordeshook spatial voting model has the virtue of offering a more realistic view of electoral competition, we argue in the present paper that it moreover has *extremely nice properties as far as aggregation of individual preferences* is concerned. Indeed, we prove in the following that the worst-case configuration of the society (worst case as far as aggregation is concerned), i.e. when the point set  $X$  is an  $(n + 1)$ -tuple of points forming a  $n$ -dimensional simplex with equal voting rights on the vertices, transforms *almost surely* into a best-case configuration, i.e. the  $(n + 1)$ -tuples of projected points in  $\mathbb{R}^d X$  is symmetrically distributed.

The first step of our argument is to qualify what we mean by ‘almost surely’. Let us go back to the above illustration. Suppose now that an exogenous historical shock occurs, e.g. a terrorist attack. Most probably this event is going to have





**Figure 3.** The induced campaign space.

an impact on the sensitivity of the first issue (helicopters) to ideology: all candidates become hawkish and want to invest more into such a modern defense tool as helicopters, independently of his/her ideology. Hence, there is a new line  $L'_1$ , with a much smaller sensitivity rate:  $l'_1 < l_1$  (see the, almost flat, dotted line in Figure 1). It is probably going to be the case that everybody in the society, candidates *and voters*, are going to prefer an absolute increase  $\Delta x_1 > 0$  in the political platform; we assume that in such an event the perturbed status quo becomes  $x_1^0 = x_1^0 + \Delta x_1$ , and that for all  $i$ , the preferred platform's first component becomes  $x'_{i1} = x_{i1} + \Delta x_1$ . Hence, this general absolute increase  $\Delta x_1$  results in a *global (rightward) translation* of the spatial point-set configuration, and this translation has no impact on the geometric properties of our problem. Therefore, without loss of generality, we can consider  $\Delta x_1$  to be zero, and the only impact of this exogenous historical event is a drop in the sensitivity rate  $l_1$ . This results into a new campaign space  $C'$  (dotted in Figure 3) going through the status quo  $x^0$  with slope  $l_2/l'_1$ . The new induced 'ideal candidate'  $\bar{x}'_i$  of voter  $i$  is obtained by orthogonal projection of  $x_i$  onto the new campaign space  $C'$ .

The idea is that such *random shocks* always happen, although fortunately they are not all as dramatic as a terrorist attack, and that their 'media' treatment

and destiny can change the sensitivities of various issues to ideology. Then the central question is: how is this  $d$ -dimensional campaign space chosen? In the present paper, we take a purely Laplacian perspective and assume that  $C$  is selected *at random*, according to a ‘uniform’ distribution on the natural underlying space. We define  $C$  as an element in the Grassmanian  $G(n, d)$  of oriented  $d$ -subspaces in  $\mathbb{R}^n$ . Random historical and mediatic shocks generate a probability distribution over  $G(n, d)$ . Among the latter, one arises ‘naturally’: the unique rotation-invariant probability measure,  $\mu(n, d)$  (known as the Haar probability measure), on  $G(n, d)$ , which intuitively selects all  $d$ -dimensional campaign spaces ‘with equal probability’. Hence,  $\mu(n, d)$  will be dubbed *impartial* in the following. The idea behind impartiality is that the main themes at stake in a political campaign depend heavily on the exogenous shocks of recent history, and the exogenous treatment by the media of these shocks.

Finally, we define our political competition as a non-cooperative game as follows. Consider a three-stage game in which there are  $m$  voters, an incumbent, and a challenger. Voters’ Euclidean preference profiles are formed by ideological positions in a subset  $P \subseteq \mathbb{R}^m$  and are common knowledge. In the first stage the incumbent chooses an ideological position in the  $n$ -dimensional policy space. After observing this, the challenger chooses an ideological position in the policy space. Then, an impartial shock determines the  $d < n$ -dimensional campaign space. Each candidate’s campaign position is determined as the orthogonal projection of their ideological positions to the campaign space. After observing these, the  $m$  voters vote to maximize their utility for a candidate according to the two political platforms imputed on the campaign space. The incumbent is beaten by the challenger if and only if the latter gets more than 50% of the votes. Each candidate tries to maximize his/her chances of winning when choosing his ideological position in stage 1 or 2, respectively. A *50% political equilibrium* for given  $m$  is the ideological position chosen by the incumbent that is sustained by a subgame perfect equilibrium of this game for any configuration of voter preference profiles in  $P$ .

#### 4. Main Result

For any selected campaign space  $C$ , the original social choice problem characterized by the point set  $X$  in  $\mathbb{R}^n$  gives rise to a lower-dimensional social choice problem characterized by the (orthogonally projected) point set  $\bar{X}$  in  $\mathbb{R}^d$ . Suppose that  $X$  is a  $m - 1$ -dimensional simplex in  $\mathbb{R}^n$ , such that for each  $i$ ,  $x_i$  (a column vector in  $\mathbb{R}^n$ ) is the  $i$ th vertex of simplex  $X$ . We say that  $X$  is regular if  $\|x_i - x_j\| = \|x_i - x_k\|$  for any  $i, j, k$ . The simplex is  $O$ -centered if  $\sum_i x_i = 0$ . Note that for an  $O$ -centered regular simplex, we have  $x_i^T x_j = x_i^T x_k < 0$  for any  $i, j, k$ . In this section, we mute the translational part of the political shock as explained in the previous section and focus on the rotational shock to the

campaign space. First, we consider rotations pivoted at the center of gravity of the simplex  $X$ , that is, when the mean voter is the status quo (i.e. it is always on the campaign space). Center of gravity is normalized to  $O$ , the center of the coordinate system. As we will show, this has no cost in our approach, since the center of gravity is the unique 50% political equilibrium (Theorem 2).

**THEOREM 1.** *Let the campaign space  $C$  be impartially randomly selected. The point set  $\bar{X}$  coincides in distribution with a negatively correlated sample from a symmetric probability distribution in  $\mathbb{R}^d$  which becomes asymptotically independent as  $n \rightarrow \infty$  with the rate  $1/n$  when  $X$  is a regular  $O$ -centered simplex.*

*Proof.* Let  $n \geq m > d$ . Take a regular  $(m - 1)$ -dimensional  $O$ -centered simplex in  $\mathbb{R}^n$  whose vertices are column vectors of an  $n \times m$  matrix<sup>7</sup>  $X = [x_1 \ x_2 \ \dots \ x_m]$ . We have  $x_i^T x_j < 0$  for any  $i$  and  $j$ . Let  $\Pi$  be  $d \times n$  with  $\pi_{ii} = 1$  for  $i \in \{1, 2, \dots, n\}$  and all other entries of  $\Pi$  are  $0$ . We can denote the random  $d$ -subspace by  $\mathbf{C} = \Pi \mathbf{R}$  where  $\mathbf{R}$  is a random rotation matrix distributed with the Haar probability measure among the  $n \times n$  rotation matrix group denoted by  $R(n)$ , that is, every  $n \times n$  rotation matrix is chosen with equal probability as a draw of  $\mathbf{R}$ . Note that  $\mathbf{C}$  is distributed with probability measure  $\mu(n, d)$  in Grassmanian  $G(n, d)$ . Note that every rotation matrix is an orthogonal matrix. Let  $O(n)$  denote the  $n \times n$  orthogonal matrix group. First, we consider orthogonal matrices instead of rotation matrices. Let  $\mathbf{C}^* = \Pi \mathbf{A}$  be such that  $\mathbf{A}$  is a random orthogonal matrix distributed with the Haar probability measure in  $O(n)$ , that is, every  $n \times n$  orthogonal matrix is chosen with the same probability as a draw of  $\mathbf{A}$ . Note that orthogonal transformations include rotoinversions (where  $\det(A) = -1$ ) and rotations ( $\det(A) = 1$ ), we will rule out rotoinversions later. First note that every column of  $\mathbf{A}$  has a symmetric distribution and  $\mathbf{E}\mathbf{A} = 0$ , since if  $A$  is orthogonal then  $-A$  is orthogonal and both  $\mathbf{A} = A$  and  $\mathbf{A} = -A$  are equally likely events. Since orthogonal transformation preserves the inner-product of two vectors, we have  $\sum_i a_{ij} a_{ik} = 0$  for  $j \neq k$  and  $\sum_i a_{ij}^2 = 1$  for any orthogonal matrix  $A$ . Since every row permutation and column permutation of  $A$  is equally likely to occur,  $E(\mathbf{a}_{ij}^2)$  and  $E(\mathbf{a}_{ij} \mathbf{a}_{k\ell})$  is constant for every  $i, j, k$  and  $\ell$ . First,  $\sum_i E(\mathbf{a}_{ij}^2) = 1$  implies  $E(\mathbf{a}_{ij}^2) = \frac{1}{n}$ . Moreover,  $\sum_i E(\mathbf{a}_{ij} \mathbf{a}_{ik}) = 0$  implies  $E(\mathbf{a}_{ij} \mathbf{a}_{ik}) = 0$ , implying with the symmetry argument that  $E(\mathbf{a}_{ij} \mathbf{a}_{k\ell}) = 0$  for every  $i, j, k, \ell$  such that  $i \neq k$  or  $j \neq \ell$ . We are interested in the distribution of the columns of  $\mathbf{P} = \mathbf{C}^* X = \Pi \mathbf{A} X$ , the projection of  $X$  to the random subspace  $\mathbf{C}^*$ . We will show that each column vector has identical symmetric distribution and each pair of column vectors are negatively correlated.

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7. Let the  $(i, j)$ th entry of a matrix  $H$  be denoted by  $h_{ij}$  and  $j$ th column vector of  $H$  be denoted by  $h_j$ .

The two events  $\mathbf{P} = \Pi \mathbf{A} X$  and  $\mathbf{P} = \Pi(-\mathbf{A}) X$  are equally likely to occur, therefore each column of  $\mathbf{P}$  has a symmetric distribution. Since each  $x_i$  has the same length, each column vector  $\mathbf{A}x_i$  is an identical random vector. Let  $\Sigma^{ij} = \text{cov}(\mathbf{A}x_i, \mathbf{A}x_j)$  for  $i \neq j$ . The  $(g, h)$ th entry of  $\Sigma^{ij}$  is  $\sigma_{gh}^{ij} = E((\sum_k \mathbf{a}_{gk} x_{ki})(\sum_\ell \mathbf{a}_{h\ell} x_{\ell j}))$ . For  $g \neq h$ ,

$$\sigma_{gh}^{ij} = \sum_k \sum_\ell \underbrace{E(\mathbf{a}_{gk} \mathbf{a}_{h\ell})}_{=0} x_{ki} x_{\ell j} = 0.$$

For  $g = h$ ,

$$\begin{aligned} \sigma_{hh}^{ij} &= \sum_k \sum_\ell E(\mathbf{a}_{hk} \mathbf{a}_{h\ell}) x_{ki} x_{\ell j} = \sum_k \underbrace{E(\mathbf{a}_{hk}^2)}_{=\frac{1}{n}} x_{ki} x_{kj} + \sum_k \sum_{\ell \neq k} \underbrace{E(\mathbf{a}_{hk} \mathbf{a}_{h\ell})}_{=0} x_{ki} x_{\ell j} \\ &= \frac{1}{n} x_i^T x_j < 0. \end{aligned}$$

Since  $\mathbf{p}_i$  is the first  $d$  coordinates of  $\mathbf{A}x_i$  for each  $i$ , different coordinates of each pair of  $\mathbf{p}_i$  and  $\mathbf{p}_j$  are independently sampled and the same coordinates of each pair  $\mathbf{p}_i$  and  $\mathbf{p}_j$  are negatively correlated, where correlation goes to zero as  $n \rightarrow \infty$  (or  $m \rightarrow \infty$  since  $m \leq n$ ). We conclude our proof by observing that by Theorem 1 of Baryshnikov and Vitale (1994), the above matrix  $\mathbf{A}$  can be swapped with a random rotation matrix  $\mathbf{R}$ , and the point set  $\bar{X}$  consists of draws of columns of  $\mathbf{P} = \mathbf{C}X$ .

Our next result states that in the limit, there is a unique political equilibrium at the mean voter for 50%-majority rule as the number of voters goes to infinity for voter preference profiles each of which has most preferred positions forming a regular simplex (i.e.  $P$  is the set of all  $m - 1$ -dimensional regular simplices).

**THEOREM 2. (Mean Voter Theorem)** *Fix  $d$ . Take  $m \rightarrow \infty$ , then almost surely  $O$ , the mean voter, is the political equilibrium for the 50%-majority rule.*

*Proof.* Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$  be the underlying limiting marginal probability density function (pdf) for the columns of  $\lim_{m \rightarrow \infty} \bar{X}$ . Let  $E\rho(w)$  denote the expected value of the score of any point  $w \in \mathbb{R}^n$ .

First, we show that  $O(\in \mathbb{R}^n)$ , the mean voter, is a political equilibrium for the 50%-majority rule. Negatively correlated sampling of political positions has a smaller min-max rate than independent sampling of political positions from the same symmetric marginal distribution. Let  $\bar{Y}$  be a set of points independently sampled from pdf  $f$ . Under independent sampling, by Theorem 3 of Caplin and Nalebuff (1988) the min-max rate of  $O$  for  $\bar{Y}$  converges almost surely to the min-max rate of  $f$  at  $O \in \mathbb{R}^d$  when  $m \rightarrow \infty$ . The min-max rate of  $O$  for  $f$  is 0.5, since  $f$  is symmetric around  $O(\in \mathbb{R}^d)$  by Theorem 1. Since the min-max rate

cannot be smaller than 0.5, min–max rate of  $O$  for  $\bar{X}$  also almost surely converges to 0.5, that is,  $\lim_{m \rightarrow \infty} E\rho(O) = 0.5$ , implying  $O(\in \mathbb{R}^n)$ , the mean voter, is an equilibrium point for the 50.

Next, we prove that  $O(\in \mathbb{R}^n)$  is the unique equilibrium issue. Take any issue vector  $w \in \mathbb{R}^n \setminus \{O\}$  when  $m$  and  $n \rightarrow \infty$ . For finite  $n$ , let the first  $n$  entries of  $w$  be relevant. We will show that  $\lim_{m \rightarrow \infty} E\rho(w) > 0.5$ . Let  $\mathbf{C} = \mathbf{\Pi R}$  be the random campaign space spanned by the Haar measure, as in the proof of Theorem 1 where  $\mathbf{R}$  is the  $n \times n$  impartial random rotation matrix distributed with Haar measure on  $R(n)$  and  $\mathbf{\Pi}$  is the  $d \times n$  matrix with  $\pi_{ii} = 1$  for  $i \leq d$  and all other entries of  $\mathbf{\Pi}$  are zero. The random projection of the issue vector  $w$  on the campaign space is  $\mathbf{\Pi R}w$ . Note that  $\mathbf{\Pi R}w = O \in \mathbb{R}^d$  with probability 0 and  $\mathbf{\Pi R}w \neq O \in \mathbb{R}^d$  with probability 1, since  $w \neq O \in \mathbb{R}^n$ . Hence, the only relevant draws of  $\mathbf{R}$  for the calculation of  $E\rho(w)$  are all  $R \in R(n)$  such that  $\mathbf{\Pi R}w \neq O \in \mathbb{R}^d$ . Fix a rotation matrix  $R \in R(n)$  such that  $\mathbf{\Pi R}w \neq O$ . Since  $f$  is symmetric around  $O \in \mathbb{R}^d$  by Theorem 1, the min–max rate of the point  $\mathbf{\Pi R}w$  for  $f$  is greater than 0.5. Hence, expected min–max rate of the projection  $\mathbf{\Pi R}w$  is greater than 0.5 when  $m \rightarrow \infty$ , implying  $\lim_{m \rightarrow \infty} E\rho(w) > 0.5$  and concluding that  $w$  cannot be stable under the 50%-majority voting rule.

## 5. Conclusion

The present paper proposes a theorem of aggregation of individual preferences through the 50%-majority rule in a multidimensional spatial voting model. Of course, the result is obtained at a non-negligible cost in terms of assumptions: first, the regularity of the simplicial distribution of voters' ideal points; second, the 'uniform distribution' on the set of linkages between issues imputed by ideologies (the Haar probability measure on  $G(n, d)$ ). Of course, the robustness of the results when one relaxes these two assumptions should be studied. On the other hand, it is quite strong since it gives existence for the 50%-majority rule. Another important aspect is that it fingers the mean voter as the candidate most likely to be stable in the voting process. As underlined in Crès and Tvede (2005), mean voter theorems are very welcome in public economics because in many contexts the mean voter is the one who has the right incentives as far as making an economically efficient choice is concerned.

We would like to stress one last point. An important property of the approach chosen here is that it is compatible with the idea that politicians 'die in their ideological boots'. Poole (2003) shows a variety of evidence that members of the US Congress are ideologically consistent: they adopt an ideological position and maintain it over time. An interpretation of Theorem 2 is that *in the long run*, ignoring the historical, sociological and mediatic shocks which are going to shape the linkages between political issues, it might be a good strategy not to change one's mind. A strategic politician should choose an ideological position

that he/she believes will place, as frequently as possible over the years, his/her imputed platform at the center of gravity of the voters' ideal points. Different ideological positions come from different tastes, but also from different priors on the distribution of historical shocks (the so-called '*sens de l'histoire*') and therefore of linkages between issues. Maintaining that ideological position over time is essential for their credibility, and thus an important asset for future political successes. Now, what is a good strategy *in the short term*? A strategy here is neither the choice of an ideological position (basically chosen once for all at the beginning of one's career, although there might be more than one beginning) nor the choice of a political platform (automatically imputed by the ideological position), but an action that has an impact on the linkages between issues in a way that places the candidate at the center of gravity of the projected set of voters' ideal points. This is left for future work.

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