Production externalities: internalization by voting

Hervé Crès, Mich Tvede

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Abstract

We study internalization of production externalities in perfectly competitive markets where production plans are decided by majority voting. Since shareholders want firms to maximize dividends of portfolios rather than profits, they are interested in some internalization. Two governances, namely the shareholder governance (one share, one vote) and the stakeholder democracy (one stakeholder, one vote), are compared. We argue that perfect internalization is more likely to be the outcome of the stakeholder democracy than the shareholder governance.

Keywords: general equilibrium, majority voting, production externalities, shareholder governance vs. stakeholder democracy, social choice.

JEL-classification: D21, D51, D72, G39, L21.

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1 Introduction

Decisions on some activities, such as consumption, saving and work, are made individually while decisions on other activities, such as production and policy, are made collectively. In case of perfectly competitive and complete markets, the outcome of each type of decision has a positive impact on the outcome of the other type: (1) utility maximization results in equal marginal rates of substitution across consumers so shareholders agree unanimously on profit maximization as the right objective for collective decision making; and, (2) profit maximization results in the maximal dividends to shareholders making the outcome Pareto optimal.

However, in case of perfectly competitive markets and direct externalities between firms none of these two properties hold: (1) equalization of marginal rates of substitution does not make the shareholders agree on profit maximization; and on top of that, (2) the outcome of firm by firm profit maximization is typically not Pareto optimal. In order to understand why shareholders disagree with profit maximization, consider at one extreme a consumer with shares in only one firm: she wants that firm to maximize its profit irrespectively of the impact on profits of other firms, corresponding to no internalization. At the other extreme, consider a consumer with a fraction of the market portfolio, where the market portfolio is the sum of portfolios of all shareholders: she wants every firm to maximize aggregate profit[1], corresponding to perfect internalization.

The argument is illustrated in Hansen & Lott (1996). Indeed they emphasize that, besides the traditional benefits of risk reduction, portfolio diversification offers additional benefits for shareholders through helping internalize externalities. Investigating cross-ownership of stocks by institutions and domestic mutual funds in the computer and automobile industries, they provide evidence that externalities exist and shareholders are well diversified.

[1]In models with representative consumers such as most macroeconomic models as well as all models with agents having some fraction of the market portfolio such as the CAPM model of finance, all shareholders agree that the aggregate profit should be maximized.

Hence for perfectly competitive markets and direct externalities between firms there is a genuine social choice problem in every firm as well as a genuine problem with efficiency in the economy. We study how these problems can be solved through majority voting in firms. In equilibrium consumers maximize utilities subject to budget constraints, production plans in firms are stable with respect to majority voting and markets clear. Our main findings are the following: (1) in terms of behavior of firms, voting is equivalent to maximizing weigted sums of profits with weights being in the intersection of the convex hulls of portfolios of majorities of voters; (2) equilibria exist provided the rate of majority is at least $\max\{q/(q+1), (n-1)/n\}$, where n is the number of firms and q is the dimension of the production set; and, (3) perfect internalization of externalities can be the outcome of voting in case the market portfolio is in the intersection of the convex hulls of portfolios of majorities of voters.

Two governances are considered: the shareholder governance (one share, one vote); and, the stakeholder democracy (one stakeholder, one vote), where all consumers are allowed to vote in all firms. The performance of the two governances is compared with respect to aggregation of preferences and market efficiency.

In general, for the shareholder governance a majority of shareholders in some firm tends to have more shares in that firm than in some other firm, so they put "too much" weight on the profit of that firm when voting over production plans in that firm. Therefore perfect internalization is typically not the outcome of voting for the shareholder governance. In the stakeholder democracy, majorities in one firm are also majorities all other firms. Therefore perfect internalization can be the outcome of voting for the stakeholder democracy. At first sight the stakeholder democracy appears unrealistic. However, public regulation in democracies can be seen as a proxy of the stakeholder democracy.

The decision making in firms in case of market failures has received some attention for quite many years. One strand of research has focused on the link between equilibrium and (constrained) Pareto optimality. Contributions include Drèze (1974) and Grossman & Hart (1979), who emphasize the role of the mean shareholder, or the mean gradient of the shareholders to be more precise, in the context of incomplete financial markets. In Drèze (1985) and Dierker & Dierker (2010) the role of the control group is emphasized. In Dierker & Grodal (1999), Bejan (2008), Bejan & Bidian (2010) and Magill, Quinzii & Rochet (2010) among others imperfect competition is considered. In particular Bejan (2008) and Bejan & Bidian (2010) consider properties of equilibria where firms maximize the wealth of shareholders.

Another strand of research has focused on majority voting. Contributions include DeMarzo (1993), who emphasizes the role of the dominant shareholder, Gevers (1974), Kelsey & Milne (1996), Ritzberger (2005), Tvede & Crès (2005), Crès (2008) and Demichelis & Ritzberger (2011). In Tvede & Crès (2005) multidimensional median voter/shareholder theorems à la Greenberg (1979) are obtained. In the present paper we bridge the two strands of research: equilibria are based on majority voting; and, the link between equilibrium and optimality is studied.

The problems arising in the governance of firms because of incomplete markets or imperfect competition are ultimately problems of (indirect) externalities. In case of imperfect competition, decisions on production plans impact shareholders through income and prices and consumers through prices. In case of incomplete markets, decisions on production plans impact shareholders through income and spanning and consumers through spanning. In contrast with the cases of incomplete markets and imperfect competition, the case of direct production externalities makes it possible to fully characterize the relation between the behavior of firms and the distribution of portfolios. Moreover the link between equilibrium and optimality becomes transparent and directly related to the distribution of portfolios. Characterizations parametrized by the distribution of portfolios rather than gradients of consumers have the advantage of being based on available information as can be seen in Hansen & Lott (1996).

The paper is organized as follows: in Section 2, the set-up including definitions of equilibria and assumptions is presented; in Section 3 the results are stated; and, in Section 4 some concluding remarks are offered.

2 The model

In the present section the economy is described and the notion of equilibrium is introduced.

2.1 Set-up

Consider an economy with ℓ goods, m consumers and n firms. Markets are perfectly competitive and there are direct externalities between firms. In general perfect competition makes sense in case all agents are small relative to the rest of the economy. In case of direct externalities between firms, both production plans and externalities should be small for firms to be small. Indeed actions in one firm should influence at most a few other firms a lot or a lot of other firms a little.

Let $p = (p_1, \ldots, p_\ell)$, where $p_k \geq 0$ for all k, be a price vector. Price vectors are normalized such that prices sum to one. Let $S = \{p \in \mathbb{R}^\ell_+ \mid \sum_k p_k = 1\}$ be the set of normalized prices.

Consumers are characterized by their identical consumption sets $X = \mathbb{R}^\ell$, endowment vectors $\omega_i \in \mathbb{R}^\ell$, utility functions $u_i : X \to \mathbb{R}$ and portfolios $\delta_i = (\delta_{i1}, \ldots, \delta_{in})$ where $\delta_{ij} \geq 0$ for all i and j and $\sum_i \delta_{ij} = 1$ for all j. Let $e^n = \sum_i \delta_i$ be the market portfolio, so $e^n_j = 1$ for all j. Consumption sets are assumed to be unbounded from below to ensure that for all lists of individual production plans, consumers are able to finance consumption plans in their consumption sets. Since firms are not necessarily maximizing their profits, the value of a firm can be negative.

There are direct externalities between firms: in every firm an action is taken and the production plan of every firm depends on the actions taken in all firms. Firms are described by their sets of action $A_j \subset \mathbb{R}^q$ and production functions $f_j : A \to \mathbb{R}^\ell$, where $A = \prod_{j'} A_{j'}$, such that $y_j = f_j(a)$, where $a = (a_1, \dots, a_n)$, is the production plan of firm j. Actions could include choice of some inputs or outputs. As an example suppose firms choose inputs while output in every firm depends on aggregate inputs: if $K_j \geq 0$ is the input of firm j, then the output of firm j is $(\sum_{j' \neq j} K_{j'})^\alpha K_j^\beta$ as in Romer (1986). Thus the approach in the present paper is in line with the approach used in endogenous growth.

Traditionally in general equilibrium direct externalities between firms are described by correspondences $Y_j : (\mathbb{R}^\ell)^{n-1} \to \mathbb{R}^\ell$ such that if $y_{-j} = (y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n)$ is a list of individual production plans for all firms but firm j, then the production set of firm j is $Y_j(y_{-j})$. In the traditional approach, a list of production plans y is an equilibrium for the production sector if $y_j \in Y_j(y_{-j})$ for all j and no $y'_j \in Y_j(y_{-j})$ is more attractive than y_j for any j. Hence, in the traditional approach it is not taken into account that if the production plan of firm j is changed, then the production set of firm j' is changed too. Consequently the production plan of firm j' can become impossible or more attractive production plans can become possible.

2.2 Demand, supply and equilibrium

The demand of every consumer is assumed to be small relative to the rest of the economy and the portfolio of every consumer is assumed to be small relative to the market portfolio e^n. Therefore consumers consider prices to be fixed and suppose their votes have no impact on the decisions in the firms. For a price vector p and a list of individual actions a the problem of consumer i is

$$\begin{array}{ll} \max\limits_{x_i} & u_i(x_i) \\ \text{s.t.} & p \cdot x_i \ \leq \ p \cdot \omega_i + \sum_j \delta_{ij} p \cdot f_j(a). \end{array}$$

Firms are assumed to be small relative to the rest of the economy. Hence consumers consider the decisions in the firms to have no impact on prices. Describing the decision process in firm j takes a few steps. For a price vector p and a list of individual actions a let $P_{ij}(p, a) \subset A_j$ be the set of actions in firm j that make consumer i better off or equivalently wealthier

$$P_{ij}(p,a) \;=\; \{\, a'_j \in A_j \,|\, \textstyle\sum_{j'} \delta_{ij'} p \cdot f_{j'}(a'_j, a_{-j}) > \sum_{j'} \delta_{ij'} p \cdot f_{j'}(a) \,\}.$$

For a price vector p, a list of individual actions a and another action a'_j for firm j let $M_j(p,a,a'_j) \subset \{1,\ldots,m\}$ be the set of consumers who are better off with a'_j than with a_j

$$M_j(p,a,a'_j) \;=\; \{\, i \in \{1,\ldots,m\} \,|\, a'_j \in P_{ij}(p,a) \,\}.$$

Let $\rho \in [0,1[$ be the rate of majority needed to change actions in firms and let $\theta = (\theta_1,\ldots,\theta_n)$, where $\theta_j = (\theta_{1j},\ldots,\theta_{mj})$, $\theta_{ij} \geq 0$ and $\sum_i \theta_{ij} = 1$, be the voting weights. For a price vector p, a list of actions a and another action a'_j for firm j, a change of actions from a_j to a'_j in firm j is adopted if and only if

$$\sum_{i \in M_j(p,a,a'_j)} \theta_{ij} \;>\; \rho.$$

Two cases of voting weights are considered: the shareholder governance where $\theta_{ij} = \delta_{ij}$ (one share, one vote); and, the stakeholder democracy where $\theta_{ij} = 1/m$ (one stakeholder, one vote). For the shareholder governance consumer i can vote in firm j if and only if $\delta_{ij} > 0$ and for the stakeholder democracy all consumers can vote in all firms. However consumers without shares have no interest in voting because they receive no dividends.

For a price vector p and a list of individual actions a, let $Q_j^\rho(p,a) \subset A_j$ be the set of actions preferred to a_j in firm j in the sense that every action in $Q_j^\rho(p,a)$ is preferred to a_j by some majority in firm j

$$Q_j^\rho(p,a) \;=\; \{\, a'_j \in A_j \,|\, \textstyle\sum_{i \in M_j(p,a,a'_j)} \theta_{ij} > \rho \,\}.$$

For a price vector p and a list of individual actions a_{-j} for all firms but firm j the problem of firm j is to find stable actions, i.e., actions a_j such that $Q_j^\rho(p,a_j,a_{-j}) = \emptyset$.

In a ρ-majority stable equilibrium (or ρ-MSE) consumers maximize utilities, actions in firms are stable for unilateral changes and markets clear.

Definition 1 *A* ρ**-majority stable equilibrium** *is a price vector, a list of individual consumption bundles and a list of individual actions* $(\bar{p}, \bar{x}, \bar{a})$ *such that:*

(C) $\bar{x}_i$ *is a solution to the problem of consumer* i *given* $\bar{p}$ *and* $\bar{a}$ *for all* i*.*

(F) $\bar{a}_j$ *is a solution to the problem of firm* j *given* $\bar{p}$ *and* $\bar{a}_{-j}$ *for all* j *such that* $Q_j^\rho(\bar{p}, \bar{a}_j, \bar{a}_{-j}) = \emptyset$*.*

(E) $\sum_i \bar{x}_i = \sum_i \omega_i + \sum_j f_j(\bar{a})$*.*

Example: For $\ell = 1$, $m = 2$ and $n = 2$ suppose that the portfolios are $\delta_1 = (1, 0)$ and $\delta_2 = (0, 1)$, the set of actions $A_2 = A_1 = [0, 1]$, and the production functions $f_1(a_1, a_2) = 1 - a_1 + 2a_2$ and $f_2(a_1, a_2) = 1 - a_2 + 2a_1$.

Let $\rho \in [0, 1[$. Then for the shareholder governance there is a unique ρ-MSE with $\bar{a} = (\bar{a}_1, \bar{a}_2) = (0, 0)$. Indeed $f_1(a'_1, a_2) > f_1(a)$ for all a with $a_1 > 0$ and $a'_1 < a_1$ so consumer 1 votes for a'_1.

Let $\rho \in [1/2, 1[$. Then for the stakeholder democracy there is a continuum of ρ-MSEs because for every pair of actions $a = (a_1, a_2)$ there exists a ρ-MSE with $\bar{a} = a$. Indeed for all a and a'_1 with $a'_1 \neq a_1$, if $a'_1 < a_1$, then $f_2(a') < f_2(a)$ so consumer 2 votes against a'_1 and if $a'_1 > a_1$, then $f_1(a') < f_1(a)$ so consumer 1 votes against a'_1.

2.3 Multilateral changes of actions in firms

Consumers can typically vote in several firms: for the shareholder governance in all firms they have shares in; and, for the stakeholder governance in all firms. Therefore multilateral changes of actions in firms, where voters coordinate changes of actions in several firms, are considered. Perfectly competitive markets make sense for multilateral changes of actions in case firms can be partitioned into groups of firms such that externalities are restricted to firms in the same groups and groups are small relative to the rest of the economy. The partition of firms could reflect location, technology or other characteristics.

For multilateral changes of actions in firms the problem of the firm has to be changed into a problem of the production sector. For a price vector p and a list of individual actions a, let $P_i(p, a) \subset A$ be the set of actions that make consumer i better off or equivalently wealthier

$$P_i(p, a) \;=\; \{\, a' \in A \,|\, \textstyle\sum_j \delta_{ij} p \cdot f_j(a') > \sum_{j'} \delta_{ij} p \cdot f_j(a) \,\}.$$

For a price vector p and a pair of lists of individual actions a and a', let $M(p, a, a') \subset \{1, \ldots, m\}$ be the set of consumers who are better off with a' than with a

$$M(p, a, a') \;=\; \{\, i \in \{1, \ldots, m\} \,|\, a' \in P_i(p, a) \,\}.$$

For a change of actions from a to a' the change is adopted if and only if

$$\sum_{i \in M(p,a,a')} \theta_{ij} \;>\; \rho$$

for all j with $a'_j \neq a_j$. For a price vector p and a list of individual actions a, let $Q^\rho(p, a) \subset A$ be the set of actions preferred to a in the sense that every list of individual actions in $Q^\rho(p, a)$ is preferred to a by majorities in all firms where actions are changed

$$Q^\rho(p, a) \;=\; \{\, a' \in A \,|\, \textstyle\sum_{i \in M(p,a,a')} \theta_{ij} > \rho \text{ for all } j \text{ with } a'_j \neq a_j \,\}.$$

For a price vector p the problem of the production sector is to find stable actions a, i.e., actions a such that $Q^\rho(p, a) = \emptyset$.

In a strong ρ-MSE consumers maximize utilities, actions are stable for multilateral changes and markets clear.

Definition 2 *A* **strong ρ-majority stable equilibrium** *is a price vector, a list of individual consumption bundles and a list of individual actions $(\bar{p}, \bar{x}, \bar{a})$ such that:*

(C) $\bar{x}_i$ is a solution to the problem of consumer i given $\bar{p}$ and $\bar{a}$ for all i.

(P) $\bar{a}$ *is a solution to the problem of the production sector given* $\bar{p}$ *such that* $Q^\rho(\bar{p}, \bar{a}) = \emptyset$*.*

(E) $\sum_i \bar{x}_i = \sum_i \omega_i + \sum_j f_j(\bar{a})$*.*

For the shareholder governance economies need not have strong ρ-MSEs because having a majority of the votes in one firm does not imply having a majority of the votes in another firm as the following example shows.

Example: For $\ell = 1$, $m = 2$ and $n = 2$ suppose the portfolios are $\delta_1 = (1, 0)$ and $\delta_2 = (0, 1)$, the set of actions $A_2 = A_1 = [0, 1]$, and the production functions $f_1(a_1, a_2) = 1 - a_1 + 2a_2$ and $f_2(a_1, a_2) = 1 - a_2 + 2a_1$.

Let $\rho \in [0, 1[$. Then for the shareholder governance there is no strong ρ-MSE: $a = (a_1, a_2)$ with $a_1, a_2 < 1$ is not part of a strong ρ-MSE, because $f_1(a') > f_1(a)$ and $f_2(a') > f_2(a)$ for $a' = a + (1 - \max\{a_1, a_2\})(1, 1)$ so both consumers vote against a; and, $a = (1, a_2)$ is not part of a strong ρ-MSE, because $f_1(a') > f_1(a)$ for $a' = (0, a_2)$ so consumer 1 votes against a.

Let $\rho \in [1/2, 1[$. Then for the stakeholder democracy there is a continuum of strong ρ-MSEs: $a = (a_1, a_2)$ is part of a strong ρ-MSE if and only if $a_1 = 1$ or $a_2 = 1$. Indeed if $a_1 = 1$ or $a_2 = 1$, then $f_1(a') < f_1(a)$ or $f_2(a') < f_2(a)$ for every $a' \neq a$ so one of the consumers votes against a' and if both $a_1 < 1$ and $a_2 < 1$, then $f_1(a') > f_1(a)$ and $f_2(a') > f_2(a)$ for $a' = a + (1 - \max\{a_1, a_2\})(1, 1)$, so both consumers vote against a.

2.4 Assumptions

Consumers are supposed to satisfy the following assumptions:

(A.1) u_i is continuous.

(A.2) u_i is strongly monotone, so $z_i^k \geq x_i^k$ for all k and $z_i \neq x_i$ imply $u_i(z_i) > u_i(x_i)$, and quasi-concave, so $u_i((1-\tau)x_i + \tau z_i) \geq \min\{\, u_i(x_i), u_i(z_i)\,\}$ for all $\tau \in [0, 1]$.

(A.3) The set $u_i^{-1}(r) = \{ x_i \in X \,|\, u_i(x) = r \}$ is bounded from below for all $r \in \mathbb{R}$.

All assumptions are standard.

Firms are supposed to satisfy the following assumptions:

(A.4) A_j is convex and compact.

(A.5) f_j is continuous.

(A.6) f_j is concave, so $f_j^k((1-\tau)a + \tau a') \geq (1-\tau) f_j^k(a) + \tau f_j^k(a')$ for all k and $\tau \in [0, 1]$.

The assumptions ensure that the free disposal hull of the production set

$$\{ y \in \mathbb{R}^{\ell n} \,|\, \exists a : y_j^k \leq f_j^k(a) \text{ for all } j \text{ and } k \}$$

is convex.

3 Results

In the present section results are presented and discussed.

3.1 Objectives of firms

Below it is shown that the outcome of voting over actions in firms can be viewed as solutions to firms maximizing weighted sums of profits. The issue is identification of the weights. Clearly, coalitions of voters with majorities of votes are decisive in the process of choosing weights. Therefore the weights are related to the portfolios of voters in these coalitions. Before stating the results some notation is needed.

Let Λ_+^{n-1} be the unit simplex in $\mathbb{R}_+^n$

$$\Lambda_+^{n-1} \;=\; \{ \lambda \in \mathbb{R}_+^n \,|\, \textstyle\sum_j \lambda_j = 1 \}.$$

For $I = \{1, \ldots, m\}$ let $\Gamma : 2^I \to \Lambda_+^{n-1}$, where 2^I is the set of all subsets of I, be a correspondence that maps coalitions of consumers to convex hulls of normalized portfolios

$$\Gamma(M) \;=\; \begin{cases} \{\, \lambda \in \Lambda_+^{n-1} \,|\, \exists \alpha \in \mathbb{R}_+^{|M|} : \sum_{i \in M} \alpha_i \delta_i = \lambda \} & \text{for } \sum_{i \in M} \delta_i \neq 0 \\ \Lambda_+^{n-1} & \text{for } \sum_{i \in M} \delta_i = 0. \end{cases}$$

For the shareholder governance every coalition M with $\sum_{i \in M} \delta_{ij} > \rho$ is decisive in firm j. Let $I_j^\rho \subset 2^I$ be the set of decisive coalitions so $M \in I_j^\rho$ if and only if $\sum_{i \in M} \delta_{ij} > \rho$. For the stakeholder democracy every coalition M with $|M| > \rho m$ is decisive in every firm. Let $I^\rho \subset 2^I$ be the set of decisive coalitions so $M \in I^\rho$ if and only if $|M| > \rho m$.

For a finite set of vectors $v_1, \ldots, v_k$ let $\mathrm{co}\,\{v_1, \ldots, v_k\}$ be the convex hull of the vectors $v_1, \ldots, v_k$.

The outcome of voting is equivalent to maximizing a weighted sum of profits. Indeed for every decisive coalition there exist weights λ_M in the convex hull of portfolios $\Gamma(M)$ such that the outcome is equivalent to maximizing a weighted sum of profits for all weights in the convex hull of the weights of the decisive coalitions $\mathrm{co}\,\{\lambda_M\}_M$.

Theorem 1 *Consider an economy.*

- *Suppose $(\bar{p}, \bar{x}, \bar{a})$ is a ρ-majority stable equilibrium for the shareholder governance. Then for all j and $M \in I_j^\rho$ there exists $\lambda_M^j \in \Gamma(M)$ such that $\bar{a}_j$ maximizes $\sum_{j'} \lambda_{j'}^j \bar{p} \cdot f_{j'}(a_j, \bar{a}_{-j})$ for all $\lambda^j \in co\,\{\lambda_M^j\}_{M \in I_j^\rho}$.*

- *Suppose $(\bar{p}, \bar{x}, \bar{a})$ is a ρ-majority stable equilibrium for the stakeholder democracy. Then for all j and $M \in I^\rho$ there exists $\lambda_M^j \in \Gamma(M)$ such that $\bar{a}_j$ maximizes $\sum_{j'} \lambda_{j'}^j \bar{p} \cdot f_{j'}(a_j, \bar{a}_{-j})$ for all $\lambda^j \in co\,\{\lambda_M^j\}_{M \in I^\rho}$.*

- *Suppose $(\bar{p}, \bar{x}, \bar{a})$ is a strong ρ-majority stable equilibrium for the stakeholder democracy. Then for all $M \in I^\rho$ there exists $\lambda_M^j \in \Gamma(M)$ such that $\bar{a}$ maximizes $\sum_{j'} \lambda_{j'} \bar{p} \cdot f_{j'}(a)$ for all $\lambda \in co\,\{\lambda_M^j\}_{M \in I^\rho}$.*

Proof: The proof is by contradiction. For the shareholder governance suppose $(\bar{p}, \bar{x}, \bar{a})$ is a ρ-MSE and there exist j and $M \in I_j^\rho$ such that $\bar{a}_j$ does not maximize $\sum_{j'} \lambda j' \bar{p} \cdot f_{j'}(a_j, \bar{a}_{-j})$ for any $\lambda \in \Gamma(M)$.

According to Theorem 21.1 in Rockafellar (1970) either: (1) there exists $a_j \in A_j$ such that

$$\sum_{j'} \delta_{ij'} \bar{p} \cdot (f_{j'}(\bar{a}_j, \bar{a}_{-j}) - f_{j'}(a_j, \bar{a}_{-j})) \ < \ 0$$

for all $i \in M$; or alternatively, (2) there exists $(\alpha_i)_i$, where $\alpha_i \geq 0$ for all $i \in M$ and $\sum_i \alpha_i > 0$, such that

$$\sum_i \alpha_i \sum_{j'} \delta_{ij'} \bar{p} \cdot (f_{j'}(\bar{a}_j, \bar{a}_{-j}) - f_{j'}(a_j, \bar{a}_{-j})) \ \geq \ 0$$

for all $a_j \in A_j$.

Suppose $a_j \in A_j$ is a solution to (1), then $a_j \in Q_j^\rho(\bar{p}, \bar{a})$. This contradicts that $(\bar{p}, \bar{x}, \bar{a})$ is a ρ-MSE.

Suppose $(\alpha_i)_i$, where $\alpha_i \geq 0$ for all i and $\sum_i \alpha_i > 0$, is a solution to (2), then $\bar{a}_j$ maximizes $\sum_{j'} \alpha'_{j'} \bar{p} \cdot f_{j'}(a_j, \bar{a}_{-j})$ for $\alpha' = \sum_i \alpha_i \delta_i$, but this contradicts that $\bar{a}_j$ does not maximize $\sum_{j'} \lambda j' \bar{p} \cdot f_{j'}(a_j, \bar{a}_{-j})$ for any $\lambda \in \Gamma(M)$.

Suppose for all j and $M \in I_j^\rho$ there exists λ_M^j such that $\bar{a}_j$ maximizes $\sum_{j'} \lambda_{Mj'}^j \bar{p} \cdot f_{j'}(a_j, \bar{a}_{-j})$. Then clearly $\bar{a}_j$ maximizes $\sum_{j'} \lambda_{j'}^j \bar{p} \cdot f_{j'}(a_j, \bar{a}_{-j})$ for all $\lambda^j \in \mathrm{co}\,\{\lambda_M^j\}_{M \in I_j^\rho}$.

The proofs for the stakeholder democracy are identical to the proof for the shareholder governance.

Q.E.D.

For a ρ-MSE or a strong ρ-MSE $(\bar{p}, \bar{x}, \bar{a})$ assume the sets of actions are smooth manifolds with boundary, the production functions are differentiable and the $n \times q$-matrix of derivatives of profits

$$\begin{pmatrix} \bar{p}^T D_{a_j} f_1(\bar{a}) \\ \vdots \\ \bar{p}^T D_{a_j} f_n(\bar{a}) \end{pmatrix}$$

has rank n for all j. Then the outcome of voting is equivalent to maximizing a weighted sum of profits for weights being in the intersection of the convex hulls of portfolios of decisive coalitions $\cap_M \Gamma(M)$.

Corollary 1 *Assume A_j is a smooth q-dimensional manifold with boundary and $f_j : A \to \mathbb{R}^\ell$ is a differentiable function for all j. For $(\bar{p}, \bar{a})$ assume the $n \times q$-matrix*

$$\begin{pmatrix} \bar{p}^T D_{a_j} f_1(\bar{a}) \\ \vdots \\ \bar{p}^T D_{a_j} f_n(\bar{a}) \end{pmatrix}$$

has rank n for every j.

- *Suppose $(\bar{p}, \bar{x}, \bar{a})$ is a ρ-majority stable equilibrium for the shareholder governance. Then for all j there exists $\lambda^j \in \cap_{M \in I_j^\rho} \Gamma(M)$ such that $\bar{a}_j$ maximizes $\sum_{j'} \lambda^j_{j'} \bar{p} \cdot f_{j'}(a_j, \bar{a}_{-j})$.*

- *Suppose $(\bar{p}, \bar{x}, \bar{a})$ is a ρ-majority stable equilibrium for the stakeholder democracy. Then for all j there exists $\lambda^j \in \cap_{M \in I^\rho} \Gamma(M)$ such that $\bar{a}_j$ maximizes $\sum_{j'} \lambda^j_{j'} \bar{p} \cdot f_{j'}(a_j, \bar{a}_{-j})$.*

- *Suppose $(\bar{p}, \bar{x}, \bar{a})$ is a strong ρ-majority stable equilibrium for the stakeholder democracy. Then there exists $\lambda \in \cap_{M \in I^\rho} \Gamma(M)$ such that $\bar{a}$ maximizes $\sum_{j'} \lambda_{j'} \bar{p} \cdot f_{j'}(a)$.*

Proof: The proof is by contradiction. For the shareholder governance suppose $(\bar{p}, \bar{x}, \bar{q})$ is a ρ-MSE and there exists j such that $\cap_{M \in I_j^\rho} \Gamma(M) = \emptyset$ or $\bar{a}_j$ does not maximize $\sum_{j'} \lambda_{j'} \bar{p} \cdot f_{j'}(a_j, \bar{a}_{-j})$ for any $\lambda \in \cap_{M \in I_j^\rho} \Gamma(M)$.

It follows from Theorem 1 that if $(\bar{p}, \bar{x}, \bar{a})$ is a ρ-MSE, then for all j and $M \in I_j^\rho$, there exists $\lambda^j_M \in \Gamma(M)$ such that $\bar{a}_j$ maximizes $\sum_{j'} \lambda_{Mj'} \bar{p} \cdot f_{j'}(a_j, \bar{a}_{-j})$. Therefore suppose there exist $M, M' \in I_j^\rho$ and $\lambda \in \Gamma(M)$ and $\lambda' \in \Gamma(M')$, where $\lambda' \neq \lambda$, such that $\bar{a}_j$ maximizes both $\sum_{j'} \lambda_{j'} p \cdot f_{j'}(a_j, \bar{a}_{-j})$ and $\sum_{j'} \lambda'_{j'} p \cdot f_{j'}(a_j, \bar{a}_{-j})$.

By assumption if $\bar{a}_j$ is in the interior of A_j, then $\bar{a}_j + \Delta a_j \in A_j$ provided Δa_j is sufficiently small. If $\bar{a}_j$ is in the boundary of A_j, then there exists

$v \neq 0$ such that for all $\Delta a_j \in \mathbb{R}^q$ with $\Delta a_j \neq 0$, if $v \cdot \Delta a_j \leq 0$, then there exists a sequence $(\Delta a_j^k)_{k \in \mathbb{N}}$, where $\lim_{k\to\infty} \|\Delta a_j^k\| = 0$ and

$$\lim_{k\to\infty} \left\| \frac{1}{\|\Delta a_j^k\|} \Delta a_j^k - \frac{1}{\|\Delta a_j\|} \Delta a_j \right\| \;=\; 0,$$

such that $\bar{a}_j + \Delta a_j^k \in A_j$ for all k.

Let $v = 0$ in case $\bar{a}_j$ is in the interior of A_j. Then $\bar{a}_j$ maximizing $\sum_{j'} \lambda_{j'} p^T f_{j'}(a_j, \bar{a}_{-j})$ implies there is no solution to

$$\begin{aligned} (\textstyle\sum_{j'} \lambda_{j'} p^T D_{a_j} f_{j'}(\bar{a}_j, \bar{a}_{-j})) \cdot \Delta a_j \;&>\; 0 \\ v \cdot \Delta a_j \;&<\; 0 \end{aligned} \tag{1}$$

and $\bar{a}_j$ maximizing $\sum_{j'} \lambda'_{j'} p^T f_{j'}(a_j, \bar{a}_{-j})$ implies there is no solution to

$$\begin{aligned} (\textstyle\sum_{j'} \lambda'_{j'} p^T D_{a_j} f_{j'}(\bar{a}_j, \bar{a}_{-j})) \cdot \Delta a_j \;&>\; 0 \\ v \cdot \Delta a_j \;&<\; 0. \end{aligned} \tag{2}$$

The two vectors $\sum_{j'} \lambda_{j'} p^T D_{a_j} f_{j'}(\bar{a}_j, \bar{a}_{-j})$ and $\sum_{j'} \lambda'_{j'} p^T D_{a_j} f_{j'}(\bar{a}_j, \bar{a}_{-j})$ are not collinear because by assumption the $n \times q$-matrix

$$\begin{pmatrix} \bar{p}^T D_{a_j} f_1(\bar{a}) \\ \vdots \\ \bar{p}^T D_{a_j} f_n(\bar{a}) \end{pmatrix}$$

has rank n for every j and λ and λ' are not collinear. Hence there exists a solution to (1) or (2), but this contradicts that $\bar{a}_j$ maximizes both $\sum_{j'} \lambda_{j'} p^T f_{j'}(a_j, \bar{a}_{-j})$ and $\sum_{j'} \lambda'_{j'} p^T f_{j'}(a_j, \bar{a}_{-j})$. Thus $(\bar{p}, \bar{x}, \bar{a})$ is not a ρ-MSE.

The proofs for the stakeholder democracy are identical to the proof for the shareholder governance.

Q.E.D.

A list of individual actions a is productively efficient if and only if there does not exist another list of individual actions a' such that $f_j^k(a') \geq f_j^k(a)$ for all j and k and $f(a') \neq f(a)$. As far as productive efficiency of actions is concerned, the stakeholder democracy is likely to perform better than the shareholder governance. For the shareholder governance, suppose that there exist firms j and j' such that $(\cap_{M \in I_j^\rho} \Gamma(M)) \cap (\cap_{M' \in I_{j'}^\rho} \Gamma(M')) = \emptyset$. Then the outcome of the production sector is productively inefficient because firms are maximizing weighted sums of profits using different weights. For the stakeholder democracy, the outcome of the production sector can be productively efficient because firms can be maximizing weighted sums of profits using identical weights (assuming $\cap_{M \in I^\rho} \Gamma(M) \cap \mathbb{R}^n_{++} \neq \emptyset$). Moreover the outcome of the production sector is productively efficient in strong ρ-MSEs (assuming $\cap_{M \in I^\rho} \Gamma(M) \subset \mathbb{R}^n_{++}$) because firms are using identical weights.

3.2 Existence of equilibrium

Consider a society consisting of some individuals who must choose an alternative from a set of alternatives. Suppose the set of alternatives is compact and convex and has dimension k and individuals have convex and continuous preferences. Then, as shown in Greenberg (1979), there exists a majority stable equilibrium provided the rate of majority is at least $k/(k+1)$.

In the present paper the conflicts between voters on the objectives of firms can be formulated as conflicts over actions or conflicts over relative weights on profits in firms. For ρ-MSEs the dimension of the set of actions A_j is q and the dimension of the set of relative weights Λ_+^{n-1} is $n-1$. For strong ρ-MSEs the dimension of the set of actions A is qn and the dimension of the set of relative weights Λ_+^{n-1} is $n-1$.

Theorem 2 *Consider an economy.*

- *Suppose*

$$\rho \ \geq \ \min\left\{\frac{q}{q+1}, \frac{n-1}{n}\right\}.$$

  Then there exist ρ-majority stable equilibria for both governances.

- *Suppose*

$$\rho \ \geq \ \frac{n-1}{n}.$$

Then there exist strong ρ-majority stable equilibria for the stakeholder democracy.

The bound on the rate of majority in Theorem 2 is very high even for economies with few firms, but the bound is binding. Indeed as the following example shows, there exists an economy such that if the rate of majority is lower than the bound, then the economy does not have a ρ-MSE. However in Section 4 it is discussed how the bound can be lowered to $1 - 1/e \approx 0.64$ or even 0.5 for some classes of distributions of portfolios.

Example: For $\ell = 1$ and $n = m$ suppose consumer i is the sole owner of firm $j = i$ and has $1/m$ of the shares in firm $j = m$ for $i < m$ and consumer i has $1/m$ of the shares in firm $j = m$ for $i = m$ and the rate of majority is $\rho < (n-1)/n$. Suppose $A_j = \Lambda_+^{n-1}$ for all j and the production functions are $f_j(a_m, a_{-m}) = a_{mj}$ for all j so output in every firm only depends on the action of firm m.

Let $\rho < q/(q+1) = (n-1)/n$. If $a_{mj} > 0$, then all consumers but consumer $i = j$ are better off with a'_m where $a'_{mj'} = a_{mj'} + a_{mj}/(m-1)$ for all $j' \neq j$ and $a'_{mj'} = 0$ for $j' = j$. Therefore for all $a_m \in A_m$ it is possible to make at least $n-1$ out of n consumers better off. Hence there is no ρ-MSE for neither the shareholder governance nor the stakeholder democracy.

3.3 Proof of Theorem 2

For ρ-MSEs under the shareholder governance the proof consists of two parts: $\min\{q/(q+1), (n-1)/n\} = q/(q+1)$; and, $\min\{q/(q+1), (n-1)/n\} = (n-1)/n$. In the first part the conflict between shareholders in firm j is over actions. The first part is a straightforward application of the theorem in Shafer & Sonnenschein (1975) or Corollary A.1 in Won & Yannelis (2008). Alternatively In the second part the conflict between shareholders in firm j is over weights on profits. The second part consists of two steps: first artificial equilibria are introduced and shown to be equivalent to ρ-MSEs; and, second existence of artificial equilibria is established.

The proof for ρ-MSEs under the stakeholder democracy is identical to the proof for the shareholder governance. The proof for strong ρ-MSEs under the stakeholder democracy is identical to the second part of the proof for the shareholder governance.

The $\min\{q/(q+1),(n-1)/n\} = q/(q+1)$ **part**

Assume $\rho \geq q/(q+1)$. Then Q_j^ρ has open graph and $a_j \notin \operatorname{co} Q_j^\rho(p,a)$ for all (p,a) according to the proof of Theorem 2 in Greenberg (1979).

Let $v_L, v_U \in \mathbb{R}^\ell$ be such that if $a \in A$, then $v_L^k < f_j^k(a) < v_U^k$ for all j and k. Let $w_L \in \mathbb{R}^\ell$ be such that if $u_i(x_i) \geq u_i(\omega_i + \sum_j \delta_{ij} v_L)$, then $x_i^k > w_L^k$ for all i and k. Let the truncated consumption set $X^T \subset \mathbb{R}^\ell$ be defined by

$$X^T \;=\; \{\, x \in X \,|\, w_L^k \leq x^k \leq \textstyle\sum_i \omega_i^k + n v_U^k - (m-1) w_L^k \text{ for all } k \,\}.$$

Let the budget correspondence $C_i : S \times A \to X^T$ be defined by

$$C_i(p,a) \;=\; \{\, x_i' \in X^T \,|\, p \cdot x_i' \leq p \cdot \omega_i + \textstyle\sum_j \delta_{ij} f_j(a) \,\}.$$

Then C_i is continuous with non-empty, convex and compact values. Let the preference correspondence $Q_i : X^T \to X^T$ be defined by

$$Q_i(x_i) \;=\; \{\, x_i' \in X^T \,|\, u_i(x_i') > u_i(x_i) \,\}.$$

Then Q_i has open graph with convex values and $x_i \notin Q_i(x_i)$ for all x_i.

Let the preference correspondence of the auctioneer $Q_0 : S \times (X^T)^m \times A \to S$ be defined by

$$\begin{aligned} Q_0(p,x,a) \;=\; \{\, p' \in S \,|\, & p' \cdot (\textstyle\sum_i x_i - \sum_i \omega_i - \sum_j f_j(a)) \\ & > p \cdot (\textstyle\sum_i x_i - \sum_i \omega_i - \sum_j f_j(a)) \,\}. \end{aligned}$$

Then Q_0 has open graph with convex values and $p \notin Q_0(p,x,a)$ for all (p,x,a).

According to the theorem in Shafer & Sonnenschein (1975) or Corollary A.1 in Won & Yannelis (2008) there exists $(\bar{p},\bar{x},\bar{a}) \in S \times (X^T)^m \times A$ such that $\bar{x}_i \in C_i(\bar{p}, \bar{a})$ and $Q_i(\bar{x}_i) \cap C_i(\bar{p}, \bar{a}) = \emptyset$ for all i, $Q_j^\rho(\bar{p}, \bar{a}) = \emptyset$ for all j and $Q_0(\bar{p}, \bar{x}, \bar{a}) = \emptyset$.

Next it is shown that an equilibrium of the truncated economy is a ρ-MSE of the original economy. Suppose $\bar{p}_k = 0$ for some k, then $\bar{x}_i^k = \sum_{i'} \omega_{i'}^k + nv_U^k - (m-1)w_L^k$ for all i because u_i is strongly monotone for all i. Therefore $\sum_i \bar{x}_i^k > \sum_i \omega_i^k + \sum_j f_j^k(\bar{a})$ because

$$\begin{aligned} &\sum_i \bar{x}_i^k - \sum_i \omega_i^k - \sum_j f_j^k(\bar{a}) \\ &\quad = (m-1)\sum_i(\omega_i + \sum_j \delta_{ij} v_U^k - w_L^k) + \sum_j (v_U^k - f_j^k(\bar{a})) \\ &\quad > (m-1)\sum_i(\omega_i + \sum_j \delta_{ij} v_L^k - w_L^k) + \sum_j (v_U^k - f_j^k(\bar{a})) \end{aligned}$$

Hence $\sum_i \bar{x}_i^{k'} > \sum_i \omega_i^{k'} + \sum_j f_j^{k'}(\bar{a})$ for all k' because $Q_0(\bar{p}, \bar{x}, \bar{a}) = \emptyset$. Thus there exists i such that $\bar{p} \cdot \bar{x}_i > \bar{p} \cdot \omega_i + \sum_j \delta_{ij} \bar{p} \cdot f_j(\bar{a})$, but this contradicts that $\bar{x}_i \in C_i(\bar{p}, \bar{a})$. All in all, $\bar{p}_k > 0$ for all k.

Condition (E) in Definition 1 is satisfied because $Q_0(\bar{p}, \bar{x}, \bar{a}) = \emptyset$ and $\bar{p}^k > 0$ for all k imply $\sum_i \bar{x}_i^k = \sum_i \omega_i^k + \sum_j f_j^k(\bar{a})$ for all k. Condition (F) in Definition 1 is satisfied because $Q_j^\rho(\bar{p}, \bar{a}) = \emptyset$ for all j. It takes a few steps to show that condition (C) in Definition 1 is satisfied. Firstly $\bar{x}_i^k > w_L^k$ because $\omega_i^k + \sum_j \delta_{ij} f_j^k(\bar{a}) > \omega_i^k + nv_L^k$ for all k. Secondly $\bar{x}_i^k < \sum_{i'} \omega_{i'}^k + nv_U^k - (m-1)w_L^k$ for all i and k because $\bar{x}_{i'}^k > w_L^k$ for all i' and k, $f_j(\bar{a}) < v_U^k$ for all j and k and $\sum_i \bar{x}_i = \sum_i \omega_i + \sum_j f_j(\bar{a})$. Thirdly if $u_i(x_i) > u_i(\bar{x}_i)$, then $u_i((1-\tau)x_i + \tau\bar{x}_i) > u_i(\bar{x}_i)$ for all $\tau \in]0,1]$. Indeed if there exists $\bar{\tau} \in]0,1[$ such that $u_i((1-\bar{\tau})x_i + \bar{\tau}\bar{x}_i) = u_i(\bar{x}_i)$, then for $e^\ell \in \mathbb{R}^\ell$ being the vector with all coordinates equal to one there exists $\varepsilon > 0$ such that $u_i(x_i - \varepsilon e^\ell) > u_i(\bar{x}_i)$, because u_i is continuous, and $u_i((1-\bar{\tau})(x_i + \varepsilon e^\ell) + \bar{\tau}\bar{x}_i) < u_i(\bar{x}_i)$, because u_i is strongly monotone, but this contradicts that u_i is quasi-concave. Therefore if there exists $x_i \in X$ such that $u_i(x_i) > u_i(\bar{x}_i)$ and $\bar{p} \cdot x_i \leq \bar{p} \cdot \omega_i + \sum_j \delta_{ij} \bar{p} \cdot f_j(\bar{a})$, then there exists $x_i' \in X^T$ such that $u_i(x_i') > u_i(\bar{x}_i)$ and $\bar{p} \cdot x_i' \leq \bar{p} \cdot \omega_i + \sum_j \delta_{ij} \bar{p} \cdot f_j(\bar{a})$. Hence $\bar{x}_i \in C_i(\bar{p}, \bar{a})$ and $Q_i(\bar{x}_i) \cap C_i(\bar{p}, \bar{a}) = \emptyset$ for all i imply that condition (C) in Definition 1 is satisfied. All in all, $(\bar{p}, \bar{x}, \bar{a})$ is a ρ-MSE.

Artificial economies

Following Tvede & Crès (2005) an artificial economy is used. The problem of a firm is decomposed into a problem of selecting a price vector for profit maximization and a problem of maximizing profit.

Describing the problem of selecting a price vector takes a few steps. Let $M_{\delta \neq 0} \subset \{1, \ldots, m\}$ be the set of consumers with shares in some firm

$$M_{\delta \neq 0} \;=\; \{\, i \in \{1, \ldots, m\} \,|\, \delta_i \neq 0 \,\}.$$

Then $\Gamma(\{i\}) \in \Lambda^n_+$ is the normalized portfolio of consumer i. For $i \in M_{\delta \neq 0}$ let the function $\phi^i : S \to \Lambda^{\ell n - 1}$ be defined by

$$\phi^i(p) \;=\; \begin{pmatrix} \phi^i_1(p) \\ \vdots \\ \phi^i_n(p) \end{pmatrix} \;=\; \begin{pmatrix} \Gamma_1(\{i\})p \\ \vdots \\ \Gamma_n(\{i\})p \end{pmatrix}.$$

Then $\phi^i(p)$ is the ideal point of consumer i in the sense that consumer i wants firm j to maximize $\sum_{j'} \phi^i_{j'}(p) \cdot f_{j'}(a_j, a_{-j})$ for every j.

For every $i \in M_{\delta \neq 0}$ let the correspondence $V_i : S \times \Lambda^{\ell n - 1} \to \Lambda^{\ell n - 1}$ associate every price vector p in S and vector μ in $\Lambda^{\ell n - 1}$ with the set of vectors μ' in $\Lambda^{\ell n - 1}$ closer to $\phi^i(p)$ than μ

$$V_i(p, \mu) \;=\; \{\, \mu' \in \Lambda^{\ell n - 1} \,|\, \textstyle\sum_j \|\mu'_j - \phi^i_j(p)\|^2 < \sum_j \|\mu_j - \phi^i_j(p)\|^2 \,\}.$$

Let the correspondence $N : S \times \Lambda^{\ell n - 1} \times \Lambda^{\ell n - 1} \to M_{\delta \neq 0}$ associate every price vector p in S and pair of vectors μ and μ' in $\Lambda^{\ell n - 1}$ with the set of consumers with $\mu' \in V_i(p, \mu)$

$$N(p, \mu, \mu') \;=\; \{\, i \in M_{\delta \neq 0} \,|\, \mu' \in V_i(p, \mu) \,\}.$$

Let the correspondence $W^\rho_j : S \times \Lambda^{\ell n - 1} \to \Lambda^{\ell n - 1}$ associate every price vector p and vector μ in $\Lambda^{\ell n - 1}$ with the set of vectors μ' in $\Lambda^{\ell n - 1}$ closer to $\phi^i(p)$ than μ for a majority of consumers

$$W^\rho_j(p, \mu) \;=\; \{\mu' \in \Lambda^{\ell n - 1} \,|\, \textstyle\sum_{i \in N(p, \mu, \mu')} \theta_{ij} > \rho \,\}.$$

Definition 3 *An* **artificial equilibrium** *is a list of individual vectors, a price vector, a list of individual consumption bundles and a list of individual actions* $(\bar{\mu}, \bar{p}, \bar{x}, \bar{a})$*, where* $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_n)$ *and* $\bar{\mu}_j \in \Lambda^{\ell n-1}$ *for all* j*, such that:*

(C) $\bar{x}_i$ *is a solution to the problem of consumer* i *given* $\bar{p}$ *and* $\bar{a}$ *for all* i*.*

(F') $\bar{a}_j$ *maximizes the profit of firm* j *given* $\bar{\mu}_j$ *and* $\bar{a}_{-j}$*, so* $\bar{a}_j$ *is a solution to*

$$\begin{array}{ll} \max\limits_{a_j} & \sum\limits_{j'} \bar{\mu}_j^{j'} \cdot f_{j'}(a_j, \bar{a}_{-j}) \\ s.t. & a_j \in A_j. \end{array}$$

(F") $W_j^\rho(\bar{p}, \bar{\mu}_j) = \emptyset$*.*

(E) $\sum_i \bar{x}_i = \sum_i \omega_i + \sum_j f_j(\bar{a})$*.*

The problems of the firms (F') and (F") in Definition 3 are artificial in the sense that they are not related to the preferences of the consumers. However as shown in Lemma 1 if $(\bar{\mu}, \bar{p}, \bar{x}, \bar{a})$ is an artificial equilibrium, then $(\bar{p}, \bar{x}, \bar{a})$ is a ρ-MSE.

Lemma 1 *Suppose* $(\bar{\mu}, \bar{p}, \bar{x}, \bar{a})$ *is an artificial equilibrium, then* $(\bar{p}, \bar{x}, \bar{a})$ *is a* ρ*-majority stable equilibrium.*

Proof: Suppose $(\bar{\mu}, \bar{p}, \bar{x}, \bar{a})$ is an artificial equilibrium. Then (C) and (E) are satisfied. Therefore it suffice to show that (F) is satisfied. The strategy of the proof is to show that if $Q_j^\rho(\bar{p}, \bar{a}) \neq \emptyset$, then $W_j^\rho(\bar{p}, \bar{\mu}) \neq \emptyset$.

Suppose $Q_j^\rho(\bar{p}, \bar{a}) \neq \emptyset$, then there exists $a_j \in A_j$ such that

$$\sum_{i \in M_j(\bar{p}, \bar{a}, a_j)} \theta_{ij} > \rho.$$

Let $\bar{y}_{j'} = f_{j'}(\bar{a})$ and $y_{j'} = f_{j'}(a_j, \bar{a}_{-j})$ for all j', then

$$\sum_{j'} \bar{\mu}_j^{j'} \cdot (y_{j'} - \bar{y}_{j'}) \ \leq \ 0$$

and

$$\sum_{j'} \phi^i_{j'}(\bar{p}) \cdot (y_{j'} - \bar{y}_{j'}) \ > \ 0$$

for all $i \in M_j(\bar{p}, \bar{a}, a_j)$.

Let $\mu_j \in \Lambda^{\ell n - 1}$ be defined by

$$\mu_j \ = \ \bar{\mu}_j + \tau \left(\begin{pmatrix} y_1 - \bar{y}_1 \\ \vdots \\ y_n - \bar{y}_n \end{pmatrix} - \frac{\sum_{j'} (y^k_{j'} - \bar{y}^k_{j'}) \cdot e}{\ell n} \begin{pmatrix} e \\ \vdots \\ e \end{pmatrix} \right).$$

Then tedious and straightforward calculations show that $\mu_j \in V_i(\bar{p}, \bar{\mu}_j)$ if and only if

$$\sum_{j'} (2\phi^i_{j'}(\bar{p}) - (\mu^{j'}_j + \bar{\mu}^{j'}_j)) \cdot (\mu^{j'}_j - \bar{\mu}^{j'}_j) \ > \ 0.$$

Therefore there exists $\tau > 0$ such that $\mu_j \in V_i(\bar{p}, \bar{\mu}_j)$ for all $j \in M_j(\bar{p}, \bar{a}, a_j)$ because

$$\sum_{j'} (2\phi^i_{j'}(\bar{p}) - (\mu^{j'}_j + \bar{\mu}^{j'}_j)) \cdot (\mu^{j'}_j - \bar{\mu}^{j'}_j) \ = \ 2\tau \sum_{j'} (\phi^i_{j'}(\bar{p}) - \bar{\mu}^{j'}_j) \cdot (y_{j'} - \bar{y}_{j'})$$

$$-\tau^2 \sum_{j'} \left\| (y_{j'} - \bar{y}_{j'}) - \frac{\sum_{j''} (y_{j''} - \bar{y}_{j''}) \cdot e}{\ell n} e \right\|^2.$$

Hence if $Q^\rho_j(\bar{p}, \bar{a}) \neq \emptyset$, then $W^\rho_j(\bar{p}, \bar{\mu}) \neq \emptyset$. Thus if $(\bar{\mu}, \bar{p}, \bar{x}, \bar{a})$ is an artificial equilibrium, then $(\bar{p}, \bar{x}, \bar{a})$ is a ρ-MSE.

Q.E.D.

The $\min\{q/(q+1), (n-1)/n\} = (n-1)/n$ **part**

If $(\bar{\mu}, \bar{p}, \bar{x}, \bar{a})$ is an artificial equilibrium, then $\bar{\mu}_j$ is in the set $\text{co}\,\{\phi^i(\bar{p})\}_{i \in M_{\delta \neq 0}}$ for all j. The set $\text{co}\,\{\phi^i(\bar{p})\}_{i \in M_{\delta \neq 0}}$ has dimension $n-1$ or less. Indeed $\mu \in \text{co}\,\{\phi^i(p)\}_{i \in M_{\delta \neq 0}}$ if and only if there exists $\lambda \in \Gamma(M_{\delta \neq 0})$ such that

$$\mu \ = \ \begin{pmatrix} \lambda_1 p \\ \vdots \\ \lambda_n p \end{pmatrix}.$$

Assume $\rho \geq (n-1)/n$. Then $\cap_{M \in I_j^\rho} \Gamma(M)$ is non-empty according to the proof of Theorem 2 in Greenberg (1979). For any $\lambda^j \in \cap_{M \in I_j^\rho} \Gamma(M)$, let the function $\psi^j : S \to \Lambda^{\ell n - 1}$ be defined by

$$\psi^j(p) \;=\; \begin{pmatrix} \lambda_1^j p \\ \vdots \\ \lambda_n^j p \end{pmatrix}.$$

Then $\psi^j(p) \in \cap_{M \in I_j^\rho} \mathrm{co}\,\{\phi^i(p)\}_{i \in M}$ for all p. Therefore $W_j^\rho(p, \psi^j(p)) = \emptyset$ for all p. Hence (F”) is satisfied for all $p \in S$.

For p and a_{-j} the problem of firm j is

$$\begin{aligned} \max_{a_j} \quad & \sum_{j'} \psi_{j'}^j(p) \cdot f_{j'}(a_j, a_{-j}) \\ \text{s.t.} \quad & a_j \in A_j. \end{aligned}$$

It follows from Berge’s maximum theorem that the solution correspondence $\alpha_j : S \times A_{-j} \to A_j$ of firm j is upper hemi-continuous and convex valued.

For p and a the truncated problem of consumer i is

$$\begin{aligned} \max_{x_i} \quad & u_i(x_i) \\ \text{s.t.} \quad & \begin{cases} p \cdot x_i \leq p \cdot \omega_i + \sum_j \delta_{ij} f_j(a) \\ x_i \in X^T. \end{cases} \end{aligned}$$

It follows from Berge’s maximum theorem that the demand correspondence $\beta_i : S \times A \to X^T$ is upper hemi-continuous and convex valued.

For x and a the price problem is

$$\begin{aligned} \max_{p} \quad & p \cdot \left(\sum_i x_i - \sum_i \omega_i - \sum_j f_j(a) \right) \\ \text{s.t.} \quad & p \in S. \end{aligned}$$

It follows from Berge’s maximum theorem that the price correspondence $\gamma : (X^T)^m \times A \to S$ is upper hemi-continuous and convex valued.

Let the correspondence $g : S \times (X^T)^m \times A \to S \times (X^T)^m \times A$ be defined by

$$g(p, x, a) \;=\; (\gamma(x, a), \beta_1(p, a), \ldots, \beta_m(p, a), \alpha_1(p, a_{-1}), \ldots, \alpha_n(p, a_{-n})).$$

It follows from Kakutani's fixed point theorem that the correspondence has a fixed point $(\bar{p}, \bar{x}, \bar{a})$. Clearly, $(\bar{\mu}, \bar{p}, \bar{x}, \bar{a})$, where $\bar{\mu} = (\psi^1(\bar{p}), \ldots, \psi^n(\bar{p}))$, is an artificial equilibrium so $(\bar{p}, \bar{x}, \bar{a})$ is a ρ-MSE according to Lemma 1.

3.4 Indeterminacy of equilibria

From the proof of Theorem 2 it follows that the outcome of maximizing a weighted sum of profits for weights in the intersection of the convex hulls of portfolios of decisive coalitions $\cap_M \Gamma(M)$ corresponds to voting.

Corollary 2 *Consider an economy.*

- *Suppose* $\cap_{M \in I^\rho_j} \Gamma(M) \neq \emptyset$ *for all* j*. Then for every* $(\lambda^1, \ldots, \lambda^n)$ *with* $\lambda^j \in \cap_{M \in I^\rho_j} \Gamma(M)$ *for all* j *there exists a* ρ*-majority stable equilibrium* $(\bar{p}, \bar{x}, \bar{a})$*, where* $\bar{a}_j$ *maximizes* $\sum_{j'} \lambda^j_{j'} \bar{p} \cdot f_{j'}(a_j, \bar{a}_{-j})$ *for all* j*, for the shareholder governance.*

- *Suppose* $\cap_{M \in I^\rho} \Gamma(M) \neq \emptyset$*. Then for every* $(\lambda^1, \ldots, \lambda^n)$ *with* $\lambda^j \in \cap_{M \in I^\rho} \Gamma(M)$ *for all* j *there exists a* ρ*-majority stable equilibrium* $(\bar{p}, \bar{x}, \bar{a})$*, where* $\bar{a}_j$ *maximizes* $\sum_{j'} \lambda^j_{j'} \bar{p} \cdot f_{j'}(a_j, \bar{a}_{-j})$ *for all* j*, for the stakeholder democracy.*

- *Suppose* $\cap_{M \in I^\rho} \Gamma(M) \neq \emptyset$*. Then for every* $\lambda \in \cap_{M \in I^\rho} \Gamma(M)$ *there exists a strong* ρ*-majority stable equilibrium* $(\bar{p}, \bar{x}, \bar{a})$*, where* $\bar{a}$ *maximizes* $\sum_{j'} \lambda_{j'} \bar{p} \cdot f_{j'}(a)$*, for the stakeholder democracy.*

According to Corollary 2 for all weights in the intersections of the convex hulls of normalized portfolios of decisive coalitions there exist equilibria where firms maximize weighted sums of profits for these weights. Suppose that for some firm j the intersection of the convex hulls of normalized portfolios of decisive coalitions contains more than one weight. Then there is a continuum of weights in the intersection of the convex hulls of normalized portfolios of decisive coalitions. Moreover, for the same firm j suppose that the actions maximizing weighted sums of profits vary with the weights. Then there is indeterminacy of equilibria because there is a continuum of equilibria.

As shown in the proof of Theorem 2 if $\rho \geq (n-1)/n$, then $\cap_{M \in I_j^\rho} \Gamma(M) \neq \emptyset$ for all j and $\cap_{M \in I^\rho} \Gamma(M) \neq \emptyset$. However depending on the distribution of portfolios $\cap_{M \in I_j^\rho} \Gamma(M) \neq \emptyset$ for all j and $\cap_{M \in I^\rho} \Gamma(M) \neq \emptyset$ is possible for lower values of ρ. Therefore Corollary 2 rests on assumptions on $\cap_{M \in I_j^\rho} \Gamma(M)$ and $\cap_{M \in I^\rho} \Gamma(M)$ rather than assumptions on ρ.

Corollary 2 is a kind of converse to Theorem 1 and Corollary 1. On the one hand the results in Corollary 2 and Corollary 1 are converses, but Corollary 1 rests on stronger assumptions than Corollary 2. On the other hand the result in Theorem 1 is weaker than the converse of Corollary 2, but Corollary 2 and Theorem 1 rest on identical assumptions.

3.5 Internalization in equilibrium

Consider a ρ-MSE or a strong ρ-MSE $(\bar{p}, \bar{x}, \bar{a})$. Then actions are outcomes of voting in firms, but they can be viewed as solutions to firms maximizing weighted sums of profits as shown in Theorem 1 and Corollary 1. Therefore it is natural to consider the extreme cases of no internalization and perfect internalization as solutions to problems of maximizing weighted sums of profits for different weights.

There is *no internalization* in case actions in firms corresponds to firms not taking externalities into account. Hence the action of every firm is a solution to the problem of maximzing its profit

$$\begin{aligned} \max_{a_j} \quad & \bar{p} \cdot f_j(a_j, \bar{a}_{-j}) \\ \text{s.t.} \quad & a_j \in A_j. \end{aligned}$$

Typically the equilibrium allocation is not Pareto optimal.

There is *perfect internalization* in case actions in firms corresponds to firms taking externalities into account. Hence the action of every firm is a solution to the problem of maximizing aggregate profit

$$\max_{a_j} \quad \sum_{j'} \bar{p} \cdot f_{j'}(a_j, \bar{a}_{-j})$$
$$\text{s.t.} \quad a_j \in A_j.$$

If the sets of actions are convex and the production functions are concave, then the equilibrium allocation is Pareto optimal.

Perfect internalization is possible in equilibrium if the normalized market portfolio $(1/n)e^n$ is in the convex hull of the normalized portfolios of every decisive coalition.

Theorem 3 *Consider an economy.*

- *Suppose*
$$\frac{1}{n}e^n \in \bigcap_j \bigcap_{M \in I_j^\rho} \Gamma(M).$$
Then for the shareholder governance there exist ρ-majority stable equilibria with perfect internalization.

- *Suppose*
$$\frac{1}{n}e^n \in \bigcap_{M \in I^\rho} \Gamma(M).$$
Then for the stakeholder democracy there exist both ρ-majority stable equilibria and strong ρ-majority stable equilibria with perfect internalization.

Proof: For the shareholder governance assume

$$\frac{1}{n}e^n \in \bigcap_j \bigcap_{M \in I_j^\rho} \Gamma(M).$$

Then

$$\frac{1}{n}\begin{pmatrix} p \\ \vdots \\ p \end{pmatrix} \in \bigcap_{j} \bigcap_{M \in I_j^\rho} \mathrm{co}\,\{\phi^i(p)\}_{i \in M}$$

for all p. Therefore it follows from second part of proof of Theorem 2 that there exists a ρ-MSE with perfect internalization.

The proofs for the stakeholder democracy are identical to the proof for the shareholder governance.

Q.E.D.

4 Concluding remarks

In the present paper we have studied general equilibrium economies with perfectly competitive markets and direct externalities between firms. Actions in firms are decided by majority voting. Since there are externalities between firms, shareholders typically do not agree on objectives of firms: they want firms to maximize dividends of portfolios rather than profits.

We found that: (1) voting is equivalent to maximizing weigted sums of profits for weights in the intersection of the convex hulls of portfolios of majorities of voters; (2) ρ-majority stable equilibria exist in case the rate of majority is at least $\min\{q/(q+1), (n-1)/n\}$, where n is the number of firms and q is the dimension of the set of actions; and, (3) an efficient outcome can be the outcome of voting in case the market portfolio is in the convex hull of portfolios of majorities of voters. Moreover two governances, namely the shareholder governance (one share, one vote) and the stakeholder democracy (one stakeholder, one vote) were compared. The outcome of the production sector is more likely to be productively efficient for the stakeholder democracy than for shareholder governance.

In relation to (2), it is possible to lower the rate of majority needed to ensure existence of ρ-MSEs. Assume the distribution of portfolios is symmetric around the diagonal (the line going through the zero portfolio and the market portfolio). Symmetric distributions of portfolios correspond to no wealth effects in portfolios: the relative distribution of shares does not depend on the amount of shares. Then for the stakeholder democracy there exist ρ-MSEs and strong ρ-MSEs for the rate of simple majority $\rho = 0.5$. Indeed according to Grandmont (1978) and the second part of the proof of Theorem 2 the mean portfolio is stable in the sense that actions, which maximize a weighted sum of profits with the mean portfolio as weights, are stable for the rate of simple majority $\rho = 0.5$.

In relation to (3), since the mean portfolio is the market portfolio for the stakeholder democracy, there exist ρ-MSEs with perfect internalization for symmetric distributions of portfolios for $\rho = 0.5$. However for the shareholder governance the mean portfolio in firm j is $\sum_i \delta_{ij}\delta_i$ which typically is not the market portfolio. Therefore ρ-MSEs with perfect internalization need not exist. Actually, since the distribution of portfolios for firm j need not be symmetric for voting weights $\theta_{ij} = \delta_{ij}$, ρ-MSEs need not exist for the shareholder governance, unless the rate of majority is increased to $(n-1)/n$.

In relation to (2) it is shown in Caplin & Nalebuff (1991) that ρ-MSEs exist for rates of majority lower than 64 percent. It would be a useful exercise to substitute our framework with a finite number of agents with their framework with a continuum of agents. Very interestingly, in Caplin & Nalebuff (1991) it is shown that the mean portfolio is a ρ-MSE. In relation to (3), once again, for the stakeholder democracy the mean portfolio is the market portfolio so there exist ρ-MSEs with perfect internalization. However, for the shareholder governance the mean portfolio is typically not the market portfolio so ρ-MSEs with perfect internalization need not exist.

We therefore conjecture that internalization is more likely to be the outcome of voting for the stakeholder democracy than for the shareholder governance.

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