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Nicolas Coeurdacier, Hélène Rey, Pablo Winant

▶ **To cite this version:**

Nicolas Coeurdacier, Hélène Rey, Pablo Winant. The risky steady state. 2011. hal-00972801

HAL Id: hal-00972801

https://sciencespo.hal.science/hal-00972801

Preprint submitted on 3 Apr 2014

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The Risky Steady-State

By Nicolas Coeurdacier, Hélène Rey and Pablo Winant *

I. Approximation around a risky steady-state

A. *The risky steady-state*

It is common practice in dynamic macroeconomics to consider the limit behavior of the economy when agents do not anticipate the effect of future shocks. This approximation is referred to as the perfect foresight path of the economy.

The corresponding equilibrium is called the deterministic steady-state. To take optimal decisions rational agents observe the gap with the steady-state values and choose a decision rule which maximizes intertemporal utility of returning to the steady-state.

By contrast, risk-averse agents are aware of the existence of future shocks hitting the economy. As a result, they anticipate the convergence of economic variables to some stochastic steady-state, which is defined as the ergodic distribution of these variables. The properties of this ergodic distribution are mathematically much more challenging than the deterministic steady-state of the perfect foresight case.

In order to avoid these difficulties and to restore some intuition about the convergence behavior of the economy, we propose to define a risky-steady state as follows. The risky steady-state is the point where agents choose to stay at a given date if they expect future risk and if the realization of shocks is 0 at this date.[1] More formally, given a decision rule $Y_t = g^r\left(Y_{t-1}, \epsilon_t\right)$ defining optimal decisions for states Y_{t-1} and shocks ϵ_t, the risky steady-state satisfies:

$$\bar{Y} = g^r\left(\bar{Y}, 0\right)$$

Throughout the paper, for any variable u, we denote by $\bar{u}$ its value at the risky steady-state. As its name suggests, the risky steady-state incorporates information about future expected risk and corresponding optimal decisions. Consider for instance a standard stochastic neoclassical growth model where anticipated volatility leads to precautionary capital accumulation. The level of the stock of capital is higher in the risky steady-state than in the deterministic steady-state, as shown in figure 1.

B. *Linear approximation*

Most standard dynamic macroeconomics problems can be summarized by a function f and a process of random innovations (ϵ_t) with covariance matrix Σ. The solution is a process (Y_t) such that

$$E_t\left[f\left(Y_{t+1}, Y_t, Y_{t-1}, \epsilon_t\right)\right] = 0$$

The local behavior of an economic model around the deterministic steady-state is well known (see Henry Kim, Jinill Kim, Ernst Schaumburg and Christopher Sims (2008)). Under the assumption that shocks are small enough, the perturbation approach consists in finding the deterministic steady-state Y^* such that $f\left(Y^*, Y^*, Y^*\right) = 0$, then to compute a Taylor expansion of the perfect-foresight path, and finally, to correct for the presence of expected risk.

Nevertheless, if the deterministic steady-state, or the perfect foresight path is not properly defined, this method will fail. As we show below, it is the case in a small open economy model where equilibrium wealth is not defined (see Stephanie Schmitt-Grohé and Martin Uribe (2003)), or in portfolio choice problems for which portfolios are

* Coeurdacier: SciencesPo and C.E.P.R., nicolas.coeurdacier@sciences-po.fr. Rey: London Business School, C.E.P.R. and N.B.E.R., hrey@london.edu. Winant: Paris School of Economics, pablo.winant@gmail.com.

[1] A similar concept has been introduced by Michel Juillard and Ondra Kamenik (2005).

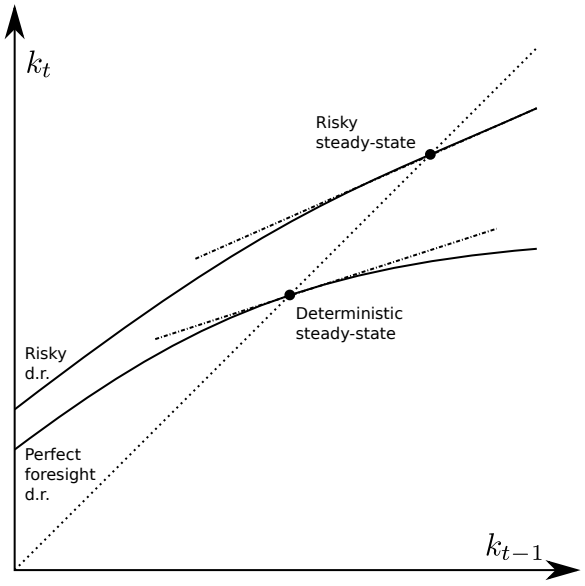


FIGURE 1. DECISION RULES FOR CAPITAL ACCUMULATION WITH AND WITHOUT EXPECTED SHOCKS.

indeterminate in the deterministic steady-state and along the perfect foresight path (see Michael Devereux and Alan Sutherland (2010) or Cédric Tille and Eric van Wincoop (2010)).

For this reason, we are interested in characterizing directly the local behavior around the risky steady-state. As it implies a joint approximation of the steady-state and of the dynamic properties, it can be referred to as an approximation around the risky steady-state. We propose in this sub-section a simple way to build an approximation. In the next section we study some properties of this simple solution.

Let us assume for simplicity that some exogenous variables X_t follow an AR(1) process $X_t = \rho_X \left(X_{t-1} - \bar{X}\right) + \epsilon_t$.
The endogenous ones Y_t are chosen using a decision rule $Y_t = g\left(Y_{t-1}, X_t\right) = g\left(S_t\right)$ where the state-space is $S_t = \left(Y_{t-1}, X_t\right)$. Denoting $\left(X_{t+1}, Y_{t+1}, X_t, Y_t, X_{t-1}, Y_{t-1}\right)$ by V_{t+1} we need to solve the optimality conditions

$$E_t\left[f\left(V_{t+1}\right)\right] = 0$$

which has the same dimension as vector Y_t. In order to take risk into account we replace this original equation by its second order-expansion Φ around expected future variables:

$$\begin{aligned} \Phi \quad = 0 = \quad & f\left(E_t V_{t+1}\right) \\ & + E_t\left[f''.\left[V_{t+1} - E_t V_{t+1}\right]^2\right] \end{aligned} \tag{1}$$

where second order derivatives are taken at point $E_t V_{t+1}$. Our strategy consists in postulating a linear decision rule for Y_t around the unknown risky steady-state $\bar{Y}$:

$$Y_t = \bar{Y} + R_Y\left(S_t - \bar{S}\right)$$

and to identify the risky steady-state $\bar{Y}$ and the coefficients R_Y jointly by solving numerically using indeterminate coefficients the two following local conditions:

$$\begin{aligned} \Phi\left(\bar{S}\right) &= 0 \\ \frac{\partial \Phi}{\partial S_t} &= 0 \end{aligned}$$

The intuition on these conditions will be more easily understood in the next section example. The condition $\Phi\left(\bar{S}\right) = 0$ characterizes the risky-steady state. It is analogous to the condition $f\left(Y^*, Y^*, Y^*\right) = 0$ defining the deterministic steady-state.

II. Intertemporal consumption decisions in a small open economy

Consider a representative agent maximizing the following life-time utility function: $U = \sum_0^\infty \frac{c_t^{1-\gamma}}{1-\gamma}$ with $\gamma > 0$.
We assume that this agent receives an endowment process y_t and can save an amount w_t at an exogenous world interest rate r_t according to the budget constraint:

$$c_t = y_t + w_{t-1} r_t - w_t$$

The exogenous variables (y_t) and (r_t) are two autocorrelated lognormally distributed processes with mean $\overline{y}$ and $\overline{r}$, autocorrelation ρ_y and ρ_r and with standard deviations σ_y and σ_r. Conditional correlation between y_t and r_t is ζ, set to zero for simplicity.

From the maximization program, we can derive the usual Euler equation:

$$\beta E_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} r_{t+1} \right] = 1$$

The deterministic steady-state (c^*, w^*) would be defined by:

$$\begin{aligned} c^* &= \bar{y} + w^* (\bar{r} - 1) \\ \beta \bar{r} &= 1 \end{aligned}$$

These two equation do not define equilibrium values c^* and w^* uniquely. Instead they imply a counterfactual relation between two independent structural parameters β and $\overline{r}$. This does not imply that the original model is not valid[2] but it indicates a limitation of the usual perturbation approach.

As we will show, it is still possible to get an approximation of the solution if we compute the risky steady-state and the coefficients for the dynamics at the same time. The state space being reduced to (w_{t-1}, y_t, r_t), let us postulate a linear solution for w:

$$w_t = \bar{w} + W_w \hat{w}_{t-1} + W_y \hat{y}_t + W_r \hat{r}_t$$

[2]Theoretical results by Gary Chamberlain and Charles Wilson (2000) state the existence of a solution if $\beta \overline{r} < 1$ and if (r_t) is stochastic enough.

In this equation $\bar{w}$ is the unknown risky steady-state value for net foreign assets holdings, $(\hat{w}_{t-1}, \hat{y}_t, \hat{r}_t)$ the deviations from this value and (W_w, W_y, W_r) three coefficients to be determined. The Euler equation equivalent of equation (1) above can be approximated as follows:

$$\begin{aligned} (2) \quad & \frac{1}{\beta} \left(\frac{E_t [c_{t+1}]}{c_t} \right)^\gamma = \\ & E_t [r_{t+1}] \left(1 + \gamma (\gamma + 1) \frac{Var_t (c_{t+1})}{E_t [c_{t+1}]^2} \right) \\ & -\gamma \frac{Cov_t (c_{t+1}, r_{t+1})}{E_t [c_{t+1}]} \end{aligned}$$

At the risky steady state, it becomes:

$$\begin{aligned} \frac{1}{\beta} &= \bar{r} \left(1 + \gamma (\gamma + 1) \frac{\overline{var_t} (c_{t+1})}{\bar{c}^2} \right) \\ & -\gamma \frac{\overline{cov_t} (c_{t+1}, r_{t+1})}{\bar{c}} \end{aligned}$$

where $\overline{cov_t}(.)$ and $\overline{var_t}(.)$ denote second order moments evaluated at the risky steady-state.

In the absence of risk the return on investment must be equal to the inverse of time preference. But a foreign asset whose returns are positively correlated with consumption is less able to provide consumption smoothing which is reflected in the risk-premium term $\gamma \frac{\overline{cov_t}(c_{t+1}, r_{t+1})}{\bar{c}}$. The second term $\gamma (\gamma + 1) \frac{\overline{var_t}(c_{t+1})}{\bar{c}^2}$ comes from precautionary savings. It denotes the desire to save more when the variance of consumption growth is higher.

Table 1 shows the decision rules for various levels of income risk. It stresses that in this model riskier countries will tend to accumulate more wealth than safer ones. Also the evolution law of the state space (w_{t-1}, y_t, r_t) has only stable eigenvalues. Note that this result contradicts the common belief in small open economy applications that consumption follows a unit-root and net foreign asset positions are non-stationary. Various tools have been used in this literature to make the problem stationary (Schmitt-Grohé and Uribe (2003)). We show here that the non-stationarity is an artefact of the approximation around a de-

σ_y	$\bar{w}$	W_w	W_y	$\bar{c}$	$\sigma(c)$
0.01	-0.161	0.944	0.516	0.995	0.032
0.025	0.0	0.945	0.52	1.0	0.032
0.05	0.607	0.95	0.537	1.017	0.034

TABLE 1—DECISION RULES FOR DIFFERENT LEVELS OF INCOME RISK

Solution is computed with $\beta = 0.96$, $\gamma = 2.0$, $\bar{y} = 1.0$, $\rho_y = \rho_r = 0.9$, $\sigma_r = 0.025$, $\zeta = 0$ and $\bar{r} = \frac{1}{\beta} - 0.014$ (such that $\bar{w} = 0$ with $\sigma_y = 0.025$)

terministic steady-state instead of the risky one.

The stabilizing effect of the precautionary term has already been highlighted by Richard Clarida (1987) and Christopher Carroll (2001). The fact that foreign assets are risky implies an additional stabilizing force on the consumption path: following positive income shocks, agents will increase their stock of foreign assets. This will increase the covariance of their consumption with the world stochastic interest rate (term $\gamma \frac{Cov_t(c_{t+1}, r_{t+1})}{E_t[c_{t+1}]}$) in equation (2) and reduce the demand for foreign assets.

III. Conclusion

We develop a new way of approximating standard dynamic stochastic macroeconomics models by solving simultaneously for a linear dynamics of state variables **and** the risky steady-state. The risky steady-state is the equilibrium at which state variables stay constant in presence of expected future shocks but when the innovations for these shocks turn out to be zero.

We study the properties of this approximation in a small open economy model of intertemporal consumption decisions with stochastic incomes and stochastic world interest rates. Contrary to standard approximation around the deterministic steady-state, net foreign assets are well defined at the risky steady-state and are stationary.

We believe that such a method can be applied more broadly to models involving portfolio decisions where standard perturbation methods have shown some limitations. Moreover, the welfare implications for risk-sharing can be quite different in these types of models since uncertainty directly affects steady-state variables.

REFERENCES

Carroll, Christopher. 2001. "A theory of the consumption function, with and without liquidity constraints." *Journal of Economic Perspectives*, 15(3): 23–45.

Chamberlain, Gary, and Charles Wilson. 2000. "Optimal intertemporal consumption under uncertainty." *Review of Economic Dynamics*, 3(3): 365–395.

Clarida, Richard. 1987. "Consumption, liquidity constraints and asset accumulation in the presence of random income fluctuations." *International Economic Review*, 28(2): 339–351.

Devereux, Michael, and Alan Sutherland. 2010. "Country portfolios in open economy models." *forthcoming in Journal of European Economic association.*

Juillard, Michel, and Ondra Kamenik. 2005. "Solving SDGE models: approximation about the stochastic steady state." *Working Paper.*

Kim, Henry, Jinill Kim, Ernst Schaumburg, and Christopher Sims. 2008. "Calculating and using second-order accurate solutions of discrete time dynamic equilibrium models." *Journal of Economic Dynamics and Control*, 32(11): 3397 – 3414.

Schmitt-Grohé, Stephanie, and Martin Uribe. 2003. "Closing small open economy models." *Journal of International Economics*, 61(1): 163–185.

Tille, Cédric, and Eric van Wincoop. 2010. "International capital flows." *Journal of International Economics*, 80(2): 157–175.